Doctoral Thesis

A moment problem of Wiener space and stochastic numerical analysis

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(Wiener空間のモーメント問題と確率数値解析)

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Abstract

The present thesis consists of four parts. The first part, Chapter 2, is on representing Brownian martingales. We gave a necessary and sufficient condition for a Brownian martingale to be expanded to an integration of more simple processes in a certain setting. This can be new tool to analyze Brownian martingales. The second part, Chapter 3, is on derivatives of expectations of some kind of affine processes with respect to the initial value of the process. We showed that the derivative can be written by the expectations of the process and another affine process whose parameter is different from the original one. This result is obtained by solving a certain ordinal differential equation associated with the process. The third part, Chapter 4, is on stochastic equation of negative integer-indexed process. We define a new class of such equation, Tanaka–Yor equation, and showed the existence of its weak solution. This can be applied to justify the "symmetrization" of stochastic differential equation (SDE) with C^2-class boundary. The fourth part, Chapter 5, is on periods of quantum walks on cycle graphs and relation between absolute zeta functions and them. We focused on Hadamard walks and Grover walks with 3 states on cycle graphs. We gave a unified proof for their periods. This method can be applied to other cases. Moreover, we calculated absolute zeta functions of zeta functions of them explicitly when the walk has finite period.

Contents

Chapter 1

Introduction

First of all, we use the following notations throughout this thesis: $(\Omega, \mathcal{F}, P)$ is a fixed probability space equipped with standard Brownian motion W. Depending on the section, it may be 1-dimensional or may be multidimensional.

1.1 On a certain martingale representation and the related infinite dimensional moment problem

On this topic, W is considered to be 1-dimensional. Let $\mathcal{F}^W = (\mathcal{F}_t^W)_{t\in[0,1]}$ be the filtration generated by W. That is,

$$\mathcal{F}_t^W := \sigma(W_s; 0 \le s \le t), \quad t \in [0,1].$$

Furthermore, let $\mathcal{M}_1^2$ be defined as

$$\mathcal{M}_1^2 := \{(X_t)_{t\in[0,1]} \mid X_0 = 1,\ X_t \text{ is in } L^2 \text{ for all } t \in [0,1],\ \text{and } X \text{ is } \mathcal{F}^W\text{-martingale}\}.$$

In this case, for $g \in L^2[0,1]$, we can say that

$$S_t^g := \exp\left(\int_0^t g(s)\, dW_s - \frac{1}{2}\int_0^t g^2(s)\, ds\right)$$

is a typical element of $\mathcal{M}_1^2$. In this thesis, we refer to such a martingale as the *exponential martingale with respect to g*.

The celebrated Itô's martingale representation theorem states that for every $X \in \mathcal{M}_1^2$ there exists an $\mathcal{F}^W$-adapted process f with $\|f\|_{L^2[0,1]\times\Omega)} < \infty$ such that

$$X_t = 1 + \int_0^t f(s)\, dW_s, \quad t \in [0,1].$$

The proof of the martingale representation theorem often uses the fact that the set $\{S^g \mid g \in L^2[0,1]\}$ is total in $\mathcal{M}_1^2$ (see e.g. [1]); for a given $X \in \mathcal{M}_1^2$ and $\varepsilon > 0$, we can choose $g_1, \cdots, g_n$ and $c_1, \cdots, c_n \in \mathbf{R}$ such that

$$\left\| X - \sum_{i=1}^{n} c_i S^{g_i} \right\|_{L^2(\Omega \times [0,1])} < \varepsilon$$

Now, the question arises whether it is possible to express any element of $\mathcal{M}_1^2$ as an equality using an infinite number of exponential martingales with respect to g, and when such expression is possible. Precisely, the problem setting could be given as follows:

For an element X of $\mathcal{M}_1^2$, find a necessary and sufficient condition for the existence of a probability space $(\Theta, \mathcal{G}, Q)$ and a family of $L^2[0,1]$-functions $(g(\cdot, \theta))_{\theta \in \Theta}$ such that the following equation hold:

$$X_t = \int_\Theta S_t^{g(\cdot,\theta)} Q(d\theta), \quad t \in [0,1]. \tag{1.1}$$

This thesis partially answers to this question using results from an infinite-dimensional version of the so-called *moment problem*.

Now, let us briefly review the history of the moment problem. The most classical moment problem is the one concerning measures on the real line, which was proposed by Stieltjes in 1894–1895: For a given sequence of real numbers $(\mu_i)_{i=0}^{\infty}$, find an necessary and sufficient condition for the existence of a Borel measure λ such that for all i,

$$\int_0^\infty x^i \lambda(dx) = \mu_i.$$

This problem was posed and simultaneously answered by Stieltjes [2, 3].

The case where we replace $[0, \infty)$ with $(-\infty, \infty)$ is known as Hamburger's moment problem, and the case where we replace it with $[0, 1]$ is known as Hausdorff's moment problem. These problems were solved by Hamburger and Hausdorff, respectively [4, 5, 6]. Then, in 1935–1936, Haviland considered the problem in a multidimensional setting and gave a necessary and sufficient condition that can be applied for the problem in which the support of the measure is restricted to any closed set [7, 8]. Especially, this is a representation that can also describe the solution of the three aforementioned problems in a unified manner.

The above history is described, for example, in the book [9].

Subsequently, in 2015, the problem was extended to countably infinite dimensions by Alpay et al. [10]. The result is that Haviland's conditions can be directly applied to the countably infinite dimensional case. This will be discussed in detail in Section 2.1, where we also explain the necessary details, including ω-dimensional moment problems, for the proof of our results. In Section 2.2, we present the main result and its proof, which is based on the results by Alpay et al. In Section 2.3, we provide some illustrative examples. Here, ω represents the smallest limit ordinal, not an element of the sample space.

This chapter is based on the author's paper [11].

1.2 Derivatives of expectations of diffusion affine processes

Affine process is a class of continuous-time stochastic processes. This is an important class and includes the Cox–Ingersoll–Ross (CIR) model, which is a cerebrated model of interest rate. In this thesis, we derived some formulas concerning such processes. For example, for a certain type of affine processes whose initial value is x X^x, $t > 0$, and a function $f\colon [0,\infty) \to \mathbf{R}$ with suitable integrability, we represent $\partial_x E[f(X_t^x)]$ by $E[f(X_t^x)]$ and the expectation of another affine process whose parameter is different from that of the original process.

Here, "the derivative of an expectation of stochastic process with respect to its initial value" is called *delta* in mathematical finance, and it plays an important role in both theory and practice. In such area, our formulas could help to speed up numerical computations. (Although this is not discussed in this thesis.)

Furthermore, although our results are mainly for 1-dimensional processes, we introduce an extension for a very simple multidimensional case.

1.3 Tanaka–Yor equation

Let $\mathbf{Z}_{\leq 0}$ be the set of all nonpositive integers. This study is on an equation of stochastic process indexed by $\mathbf{Z}_{\leq 0}$. In 1975, Tsirelson introduced a special case of such processes to show that an example of SDE which has no strong solution [12]. Then, Yor generalized this equation and classify them by existence and uniqueness of its solution [13]. Moreover, Akahori et al. generalized this equation more and showed that it can be classified in the same way in the generalized setting [14]. We studied a modified version of the equation in [14] motivated to prove the law-uniqueness of a type of SDE. This is related to a problem in mathematical finance area, and the connection is explained in subsubsection 4.1.1.

1.4 Absolute zeta functions and periodicity of quantum walks on cycles

This subsection is partly taken from a previous research paper [15] by Konno.

This work is a continuation of [15, 16]. Quantum walks are considered to be the corresponding model for random walks in quantum systems. Quantum walks play important roles in various fields such as mathematics, quantum physics, and quantum information processing. Concerning Quantum walks, see [17, 18, 19, 20, 21], and as for RW, see [22, 23], for instance. On the other hand, absolute zeta functions are zeta functions over $\mathbf{F}_1$, where $\mathbf{F}_1$ can be viewed as a kind of limit of $\mathbf{F}_p$ as $p \to 1$. Here $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ stands for the field of p elements for a prime number p. This thesis presents a connection between quantum walks and absolute zeta functions. Concerning absolute zeta functions, see [24, 25, 26, 27, 28, 29, 30, 31].

In this thesis, first we deal with periods of quantum walks on cycles, especially Hadamard walks and Grover walks with 3 state, which are well-studied models among quantum walks. Afterwards, we consider a zeta function $\zeta_{U_G}(u)$ determined by U_G which is a time evolution matrix of such a quantum walk on G, where G. Then we prove that $\zeta_{\mathbf{U}_G}(u)$ is an absolute automorphic form of weight $-2m$. Also, we consider an absolute zeta function $\zeta_{\zeta_{U_G}}(s)$ for our zeta function $\zeta_{U_G}(u)$. As an example, we calculate $\zeta_{\zeta_{U_G}}(s)$ for the cycle graph C_n with n vertices and n edges, and show that it is expressed as the multiple gamma function of order 2 via the multiple Hurwitz zeta function of order 2. Finally, we obtained the functional equation for $\zeta_{\zeta_{U_{C_n}}}(s)$ with the multiple sine function of order 2. The present thesis is the first step of the study on a relation between quantum walks and absolute zeta functions.

Chapter 2

On a certain martingale representation and the related infinite dimensional moment problem

2.1 Preliminaries

Consider countably many indeterminate $X_0, X_1, \ldots$. We call a product of a real number and these indeterminates

$$c\prod_{k=0}^{N} X_k^{m_k} \qquad (c \in \mathbf{R}, N \in \mathbf{Z}_{\geq 0}, m_k \in \mathbf{Z}_{\geq 0}, k = 0, \ldots, N)$$

an ω-variate monomial. Furthermore, we call a sum of finitely many ω-variate monomials *an ω-variate polynomial.* Here, ω represents the smallest limit ordinal, not an element of the sample space.

Let $(\mu_{\boldsymbol{i}} \in \mathbf{R} \mid \boldsymbol{i} \colon \omega \to \mathbf{Z}_{\geq 0}, \sum \boldsymbol{i}(j) < \infty)$ be a family of real numbers. Here, we call such $\boldsymbol{i}$ *an index* and define $\operatorname{len} \boldsymbol{i} := \max\{k \in \mathbf{Z}_{\geq 0} \mid \boldsymbol{i}(k) \neq 0\} < \infty$ for $\boldsymbol{i} \neq (0, 0, \ldots)$ and $\operatorname{len}(0, 0, \ldots) := -1$. For such a family, we can consider a mapping μ whose domain is all ω-variate polynomials. Firstly, define the value for ω-variate monomials by the following:

$$\mu\left(c\prod_{k=0}^{N} X_k^{m_k}\right) := c\mu_{(m_0, m_1, \ldots, m_N, 0, 0, \ldots)}$$

and then the domain is extended to the entire ω-variate polynomials so that the mapping is additive.

Next, we define the property *K-positivity* for such a family. Let K be a subset of $\mathbf{R}^\omega$, the whole of real sequences. We call such a family of numbers is *K-positive* if $\mu(p) \geq 0$ as long as ω-variate polynomial p satisfies for all $\theta \in K, p(\theta) \geq 0$.

Theorem 2.1 (ω-dimensional moment problem [10])**.** *Let $K \subset \mathbf{R}^\omega$ be closed and $(\mu_{\boldsymbol{i}} \in \mathbf{R} \mid \boldsymbol{i}\colon \omega \to \mathbf{Z}_{\geq 0}, \sum \boldsymbol{i}(j) < \infty)$ be a given family of real numbers so that $\mu_{(0,0,\dots)} = 1$. There exists a Borel measure λ on $\mathbf{R}^\omega$ such that, for all $\boldsymbol{i}$,*

$$\int_{\mathbf{R}^\omega} \prod_{j=0}^{\operatorname{len} \boldsymbol{i}} x_j^{\boldsymbol{i}(j)} \lambda(d\boldsymbol{x}) = \mu_{\boldsymbol{i}}$$

holds and $\operatorname{supp}(\lambda) \subset K$ *is satisfied if and only if the family is K-positive.*

2.2 Main theorem

To state the main theorem, we explain its settings in order.

First of all, we use λ instead of Q for the measure we seek just for visual convenience.

For each natural number n, let $\Delta_n := \{(s_1, \dots, s_n) \in [0,1]^n \mid s_1 \leq \cdots \leq s_n\}$. For simplicity, we consider only a measure λ which satisfies

$$\lambda\left(\left\{\theta \in \mathbf{R}^\omega \;\middle|\; \sum_{j=0}^{\infty} \theta_j^2 = \infty\right\}\right) = 0.$$

Consider a one-dimensional L^2-martingale X. We can find a sequence of functions (f_n) for X, $f_n\colon \Delta_n \to \mathbf{R}$ for each n, such that

$$X_t = 1 + \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{s_{n-1}} f_n(s_1, \dots, s_n)\, dW_{s_n} \cdots dW_{s_1}, \quad t \in [0,1]$$

(see e.g. [1]). This kind of expression is what is known as *chaos expansion.* Moreover, the process S^g for $g \in L^2[0,1]$, which is defined in section 1 can be expressed as bellow:

$$S_t^g = 1 + \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{s_2} g(s_1) \cdots g(s_n) dW_{s_1} \cdots dW_{s_n}, \quad t \in [0,1].$$

Now, let $(e_j)_{j \in \mathbf{Z}_{\geq 0}}$ be a Complete Orthonormal System (CONS) of $L^2[0,1]$. Then $((s_1, \dots, s_n) \mapsto e_{\boldsymbol{j}(1)}(s_1) \dots e_{\boldsymbol{j}(n)}(s_n) \mid \boldsymbol{j}\colon \{1, \dots, n\} \to \mathbf{Z}_{\geq 0})$ is a CONS of $L^2([0,1]^n)$ for each n. To expand f_n with respect to this CONS, we extend the domain of f_n to $[0,1]^n$ by symmetric way. Then the extended function $\hat{f}_n$ is expanded into a sum:

$$\hat{f}_n(s_1, \dots, s_n) = \sum_{\boldsymbol{j}\colon \{1,\dots,n\} \to \mathbf{Z}_{\geq 0}} f_{\boldsymbol{j}}\; e_{\boldsymbol{j}(1)}(s_1) \cdots e_{\boldsymbol{j}(n)}(s_n).$$

Here, thanks to the symmetry, $f_{\boldsymbol{j}} = f_{\boldsymbol{j} \circ \sigma}$ for all $\sigma \in \mathfrak{S}_n$ (symmetric group of degree n).

Take $\mathbf{R}^\omega$ as Θ and assume that the form of g is

$$g(s,\theta)=\sum_{j=0}^{\infty}\theta_j e_j(s).$$

Then it holds that

$$\begin{aligned}
&\int_\Theta\int_0^t\cdots\int_0^{s_{n-1}} g(s_1,\theta)\dots g(s_n,\theta)\,dW_{s_n}\cdots dW_{s_1}\lambda(d\theta)\\
&=\int_\Theta\int_0^t\cdots\int_0^{s_{n-1}}\sum_{j=0}^{\infty}\theta_j e_j(s_1)\cdots\sum_{j=0}^{\infty}\theta_j e_j(s_n)\,dW_{s_n}\cdots dW_{s_1}\lambda(d\theta)\\
&=\int_\Theta\int_0^t\cdots\int_0^{s_{n-1}}\sum_{\boldsymbol{j}\,:\,\{1,\dots,n\}\to\mathbf{Z}_{\geq 0}}\theta_{\boldsymbol{j}(1)}\cdots\theta_{\boldsymbol{j}(n)}e_{\boldsymbol{j}(1)}(s_1)\cdots e_{\boldsymbol{j}(n)}(s_n)\,dW_{s_n}\cdots dW_{s_1}\lambda(d\theta)\\
&=\int_0^t\cdots\int_0^{s_{n-1}}\sum_{\boldsymbol{j}\,:\,\{1,\dots,n\}\to\mathbf{Z}_{\geq 0}}\int_\Theta\theta_{\boldsymbol{j}(1)}\cdots\theta_{\boldsymbol{j}(n)}\lambda(d\theta)\;e_{\boldsymbol{j}(1)}(s_1)\cdots e_{\boldsymbol{j}(n)}(s_n)\,dW_{s_n}\cdots dW_{s_1}.
\end{aligned}$$

Now, in Theorem 2.1, take $f_{\boldsymbol{j}}$ as $\mu_{\boldsymbol{n}(\boldsymbol{j})}$ and we get a measure λ which satisfies

$$\int_\Theta\theta_{\boldsymbol{j}(1)}\cdots\theta_{\boldsymbol{j}(n)}\lambda(d\theta)=\int_\Theta\prod_{k=0}^{\operatorname{len}\boldsymbol{n}(\boldsymbol{j})}\theta_k^{\boldsymbol{n}(\boldsymbol{j})(k)}\lambda(d\theta)=\mu_{\boldsymbol{n}(\boldsymbol{j})}=f_{\boldsymbol{j}}.$$

Thus, we got necessary and sufficient condition for the moment problem for the Wiener spaces in the case $\Theta=\mathbf{R}^\omega$ and $g(s,\theta)=\sum\theta_j e_j(s)$ for some CONS of $L^2[0,1]$ $(e_j)_j$.

Theorem 2.2. *Let $(X_t)_{t\in[0,1]}$ be a square integrable martingale with $X_0=1$ and $(e_j)_{j=1}^\infty$ be a CONS of $L^2[0,1]$. Then the necessary and sufficient condition to exist a Borel measure λ on $\mathbf{R}^\omega$ which satisfies*

$$\lambda\left(\left\{\theta\in\mathbf{R}^\omega\;\middle|\;\sum_{j=0}^{\infty}\theta_j^2=\infty\right\}\right)=0$$

such that

$$X_t=\int_{\mathbf{R}^\omega}S_t^{g(\cdot,\theta)}\lambda(d\theta),\quad t\in[0,1],$$

where $g(s,\theta)=\sum_{j=0}^\infty\theta_j e_j(s)$ is that the map μ defined as above is $\mathbf{R}^\omega$-positive.

2.3 Example

Example 2.3. If the number of terms in the chaos expansion is finite except 0 and 1, then there is no λ as stated in Theorem 2.2.

Proof. Suppose there exists a measure λ as described in Theorem 2.2 for X with a finite number of chaos expansion terms. We aim to show that in such a case, λ would be equal to $\delta_{(0,0,\dots)}$ (where δ_a denotes the Dirac measure with mass at a).

Firstly, if the number of chaos expansion terms of X is finite, then there exists a sufficiently large N such that $f_{2N} \equiv 0$. Consequently, all coefficients obtained from the expansion, such as $f_{(0,0,\dots,0)}$ (with 0 appearing $2N$ times), $f_{(1,1,\dots,1)}$ (with 1 appearing $2N$ times), and so on, are all equal to zero. Therefore, the desired measure λ must satisfy

$$\int_{\mathbf{R}^\omega} \theta_j^{2N} \lambda(d\theta) = 0$$

for all $j \in \mathbf{Z}_{\geq 0}$.

Now, let us assume $\lambda \neq \delta_{(0,0,\dots)}$ and seek a contradiction. First, we divide $\mathbf{R}^\omega$ into subsets as follows:

$$\left\{ \prod_{j=0}^{\infty} [a_j/2^n, (a_j+1)/2^n] \mid a_j \in \mathbf{Z}, j \in \mathbf{Z}_{\geq 0} \right\}$$

If $\lambda \neq \delta_{(0,0,\dots)}$, then for at least one $\prod_{j=0}^{\infty}[a_j/2^n, (a_j+1)/2^n]$, the measure of $\prod_{j=0}^{\infty}[a_j/2^n, (a_j+1)/2^n] \setminus \{(0,0,\dots)\}$ should not be zero. Let us denote this set as A. Furthermore, by choosing a sufficiently large n, we can ensure that A does not contain $(0,0,\dots)$. Also, for each j, of $a_j/2^n$ and $(a_j+1)/2^n$ of A, let m_j be the one whose absolute value is smaller. Then, with such a choice of A, at least one m_j must be nonzero. Let $\mathcal{B}(A)$ denote the collection of all Borel sets contained in A, and define $m := (m_0, m_1, \dots)$.

Now, consider the measure

$$\lambda' := \lambda - \lambda(A)\lambda\restriction_{\mathcal{B}(A)} + \lambda(A)\delta_m$$

For j' such that $m_{j'} \neq 0$, we have

$$\int_{\mathbf{R}^\omega} \theta_{j'}^{2N} \lambda(d\theta) \geq \int_{\mathbf{R}^\omega} \theta_{j'}^{2N} \lambda'(d\theta) \geq \lambda(A) m_{j'}^{2N} > 0$$

and this leads to a contradiction. $\square$

Chapter 3

Derivatives of expectations of diffusion affine processes

3.1 Preliminaries and previous results

First of all, *the Bessel processes of dimension n* is a 1-dimensional process of distance between the origin and an n-dimensional Brownian motion. Formally, the process can be defined as follows:

$$R := \sqrt{\sum_{k=1}^{n} (B^{(k)})^2}$$

where $B = (B^{(1)}, \dots, B^{(n)})$ is an n-dimensional Brownian motion. Then, R^2 satisfies the following SDE:

$$d(R_t^2) = 2R_t\, dW_t + n\, dt,$$

Where $dW_t := (1/R_t) \sum_{k=1}^{n} B^{(k)}\, dB^{(k)}$ is a 1-dimensional Brownian motion. Inspired this equation, consider the following SDE for real parameter $\delta \geq 0$:

$$dX_t = 2\sqrt{|X_t|}\, dW_t + \delta\, dt. \tag{3.1}$$

By a well-known theorem, it is easily seen that this SDE has unique strong solution for each initial value $x \in [0, \infty)$ and the solution satisfies $X \geq 0$ a.s. Thus, hereafter we write

$$dX_t = 2\sqrt{X_t}\, dW_t + \delta\, dt$$

instead of (3.1). This solution is called *the squared Bessel process of dimension δ* and we write $X^{\delta,x}$ for such a process with initial value x.

Then, we obtained the following formulas on the derivative of its expectation with respect to the initial value and a kind of integration by parts. Before to state the results, we introduce a condition for decreasing property of functions.

Condition 3.1. For $t > 0$ and measurable $f\colon [0,\infty) \to \mathbf{R}$, we write "$f$ satisfies DC(t)" if there exist positive real numbers M and C and $\varepsilon \in (0, 1/(2t))$ such that if $y > M$, then $|f(y)| < Ce^{\varepsilon y}$.

Remark 3.2. If f satisfies DC(t), $f(X_t^{\delta,x})$ is integrable for any $\delta > 0$ and $x > 0$. This can be checked easily by considering the explicit form of the transition density of $X_t^{\delta,x}$.

Theorem 3.3. *Let $\delta > 0$ and $t > 0$. For a continuous function f, suppose f satisfies* DC(t). *Then the following formula holds:*

$$\partial_x E\big[f(X_t^{\delta,x})\big] = \frac{1}{2t}\big(E\big[f(X_t^{\delta+2,x})\big] - E\big[f(X_t^{\delta,x})\big]\big).$$

Theorem 3.4. *Let $\delta > 2$, $x \geq 0$, and $t > 0$. For $f \in C^1$, suppose f and f' satisfy* DC(t). *Then the following formula holds:*

$$E\big[f'(X_t^{\delta,x})\big] = \frac{1}{2t}\big(E\big[f(X_t^{\delta,x})\big] - E\big[f(X_t^{\delta-2,x})\big]\big).$$

Furthermore, combining these two formulas, we got the following interesting corollary.

Corollary 3.5. *Let $\delta > 0$ and $t > 0$. For $f \in C^1$, suppose f and f' satisfy* DC(t). *Then the following formula holds:*

$$\partial_x E\big[f(X_t^{\delta,x})\big] = E\big[f'(X_t^{\delta+2,x})\big].$$

Note that Altman obtained a formula similar to the one in Theorem 3.3 in 2018 [33].

Here, squared Bessel processes can be considered as a special kind of *affine processes*. It can be defined as follows:

Definition 3.6. A d-dimensional Markov process $(X, (P_x)_x)$ is called affine, if for evry $t > 0$, the characteristic function $E_x[\exp(i\langle \cdot, X_t\rangle)]\colon \mathbf{R}^d \to \mathbf{C}$ has exponential-affine dependence on x. That is, there exist functions $g, h\colon [0,\infty) \times \mathbf{R}^d \to \mathbf{C}$ such that $E_x[\exp(i\langle \cdot, X_t\rangle)]$ has the following form:

$$E_x[\exp(i\langle \theta, X_t\rangle)] = \exp(g(t,\theta) + xh(t,\theta)).$$

Here, $\langle \theta, x\rangle := \sum_{k=1}^d \theta_k x_k$.

Note that this definition can be more generalized. See [34] for example.

Now, for parameters $\alpha > 0$ and $\beta, b \in \mathbf{R}$, consider the following 1-dimensional SDE:

$$X_t^x = \sqrt{\alpha|X_t^x|}\, dW_t + (\beta X_t^x + b)dt, \quad X_0^x = x \tag{3.2}$$

This has unique strong solution for each $x > 0$ and by the same argument as in the case of squared Bessel process, the solution is nonnegative almost surely. So we write

$$X_t^x = \sqrt{\alpha X_t^x}\, dW_t + (\beta X_t^x + b)dt, \quad X_0^x = x \tag{3.3}$$

instead of (3.2). It is known that the family $(X^x)_x$ is affine process (See e.g. [34]). Thus we call the solutions of SDE (3.3) *diffusion affine process with parameter* (α, β, b). Also, this is an extension of squared Bessel processes, that is, squared Bessel process of dimension δ is the diffusion affine process with parameter $(4, 0, \delta)$. In the following sections, we extend the formulas such as in Theorem 3.3 and so on for diffusion affine processes.

3.2 Derivatives of expectations of diffusion affine processes

In this section, we consider 1-dimensional diffusion affine processes. It is known that the function h in the characteristic function $\exp(g(t,\theta) + xh(t,\theta))$ corresponding to the SDE (3.3) satisfies the following ordinary differential equation (See [34] for example):

$$\partial_t h(t,\theta) = \alpha h(t,\theta)^2 + \beta h(t,\theta), \qquad h(0,\theta) = i\theta, \tag{3.4}$$

and g is given by the integral of h:

$$g(t,\theta) = \int_0^t bh(s,\theta)\, ds.$$

In this case, we can solve the differential equation (3.4) explicitly, and the solution is

$$h(t,\theta) = \frac{2i\theta}{2 - \alpha i\theta t}$$

for $\beta = 0$ and

$$h(t,\theta) = \frac{2\beta e^{\beta t} i\theta}{2\beta - (e^{\beta t} - 1)\alpha i\theta}$$

for $\beta \neq 0$. Moreover,

$$g(t,\theta) = \frac{2b}{\alpha} \operatorname{Log} \frac{2}{2 - \alpha i\theta t}$$

for $\beta = 0$ and

$$g(t,\theta) = \frac{2b}{\alpha} \operatorname{Log} \frac{2\beta}{2\beta - (e^{\beta t} - 1)\alpha i\theta}$$

for $\beta \neq 0$.

Now, our first theorem is on the derivative of $E[f(X_t^x)]$ with respect to the initial value x.

Theorem 3.7. *Suppose $2b > \alpha$. Let $t > 0$ be a fixed real number and let X^x and $\tilde{X}^x$ be diffusion affine processes with parameters (α, β, b) and $(\alpha, \beta, b + \alpha/2)$ respectively, and their initial values are $x > 0$. Then, for $f \in L^1[0, \infty)$ such that $f(X_t^x)$ and $f(\tilde{X}_t^x)$ are integrable, it holds that*

$$\partial_x E[f(X_t^x)] = \begin{cases} \dfrac{2}{\alpha t}\big(E[f(\tilde{X}_t^x)] - E[f(X_t^x)]\big), & \beta = 0, \\ \dfrac{2\beta e^{\beta t}}{\alpha(e^{\beta t} - 1)}\big(E[f(\tilde{X}_t^x)] - E[f(X_t^x)]\big), & \beta \neq 0. \end{cases}$$

Proof. First, consider the case where $\beta = 0$. We will use Lévy's inversion formula, so we check the integrability of the characteristic functions. Now $\varphi_{x,t}$ and $\tilde{\varphi}_{x,t}$ denote the characteristic functions of X_t^x and $\tilde{X}_t^x$ respectively. Then we have the following inequality:

$$\begin{aligned}
|\varphi_{x,t}(\theta)| &= |\exp(g(t,\theta) + xh(t,\theta))| \\
&= \left|\exp\left(\frac{2b}{\alpha}\operatorname{Log}\frac{2}{2-\alpha i\theta t} + x\frac{2i\theta}{2-\alpha i\theta t}\right)\right| \\
&\le \left(\frac{2}{|2-\alpha i\theta t|}\right)^{2b/\alpha}\exp\left(\frac{2|\theta|x}{|2-\alpha i\theta t|}\right) \\
&= \left(\frac{2}{\sqrt{4+(\alpha t\theta)^2}}\right)^{2b/\alpha}\exp\left(\frac{2|\theta|x}{\sqrt{4+(\alpha t\theta)^2}}\right) \\
&\le \frac{2^{2b/\alpha}}{(\alpha t|\theta|)^{2b/\alpha}}\exp\left(\frac{2x}{\alpha t}\right)
\end{aligned}$$

In addition to this, the integral of $|\varphi_{x,t}|$ on any compact subset is finite since $|\varphi_{x,t}|$ is $\mathbf{R}$-valued continuous function. Therefore, if the parameter (α, b) satisfies $2b > \alpha$, we have $\int_{\mathbf{R}} |\varphi_{x,t}(\theta)| d\theta < \infty$. Now, we are assuming $2b > \alpha$ and $(\alpha, b+\alpha/2)$ satisfies $2(b+\alpha/2) > \alpha$ since b, which is grater than $\alpha/2$, is positive. Thus $\varphi_{x,t}$ and $\tilde{\varphi}_{x,t}$ are both integrable, and by Lévy's inversion formula, X_t^x and $\tilde{X}_t^x$ has continuous density function $q_{x,t}$ and $\tilde{q}_{x,t}$ (say). Now, we can start calculation. One hand, for the left hand side of the formula, rewrite the expectation using the densitiy function:

$$\begin{aligned}
\partial_x E[f(X_t^x)] &= \partial_x \int_{\mathbf{R}} f(y) q_{x,t}(y)\, dy \\
&= \partial_x \int_{\mathbf{R}} f(y)\left(\frac{1}{2\pi}\int_{\mathbf{R}} \varphi_{x,t}(\theta) e^{-i\theta y}\, d\theta\right) dy.
\end{aligned}$$

Here, we can focus on the differential coefficient at $x = \xi$. In this case, we have

$$\left| f(y)\left(\frac{1}{2\pi}\int_{\mathbf{R}} \varphi_{x,t}(\theta) e^{-i\theta y}\, d\theta\right)\right| \le \frac{1}{2\pi}|f(y)|\int_{\mathbf{R}} \frac{2^{2b/\alpha}}{(\alpha t\theta)^{2b/\alpha}} d\theta \exp\left(\frac{2}{\alpha t}(\xi+1)\right)$$

on a certain neighborhood of ξ and f is integrable, we can interchange the derivative and the integral. Then we can continue to calculate as follows:

$$\partial_x \int_{\mathbf{R}} f(y)\left(\frac{1}{2\pi}\int_{\mathbf{R}} \varphi_{x,t}(\theta) e^{-i\theta y}\, d\theta\right) dy = \frac{1}{2\pi}\int_{\mathbf{R}} f(y)\left(\partial_x \int_{\mathbf{R}} \varphi_{x,t}(\theta) e^{-i\theta y}\, d\theta\right) dy$$

Again, we know

$$|\varphi_{x,t}(\theta) e^{-i\theta y}| \le \frac{2^{2b/\alpha}}{(\alpha t\theta)^{2b/\alpha}}\exp\left(\frac{2}{\alpha t}(\xi+1)\right)$$

and we can interchange the derivative and the integral. Therefore, we have

$$
\begin{aligned}
\frac{1}{2\pi}\int_{\mathbf{R}} f(y)\Big(\partial_x \int_{\mathbf{R}} \varphi_{x,t}(\theta)e^{-i\theta y}\, d\theta\Big)\, dy& \\
&= \frac{1}{2\pi}\int_{\mathbf{R}} f(y)\Big(\int_{\mathbf{R}} \partial_x\varphi_{x,t}(\theta)e^{-i\theta y}\, d\theta\Big)\, dy \\
&= \frac{1}{2\pi}\int_{\mathbf{R}} f(y)\int_{\mathbf{R}} \partial_x \exp(g(t,\theta)+xh(t,\theta))e^{-i\theta y}\, d\theta dy \\
&= \frac{1}{2\pi}\int_{\mathbf{R}} f(y)\int_{\mathbf{R}} h(t,\theta)\exp(g(t,\theta)+xh(t,\theta))e^{-i\theta y}\, d\theta dy.
\end{aligned}
$$

On the other hand, we can calculate the right hand side of the formula as follows:

$$
\begin{aligned}
&E[f(\tilde{X}_t^x)] - E[f(X_t^x)] = \int_{\mathbf{R}} f(y)\tilde{q}_{x,t}(y)\, dy - \int_{\mathbf{R}} f(y)q_{x,t}(y)\, dy \\
&= \int_{\mathbf{R}} f(y)\Big(\frac{1}{2\pi}\int_{\mathbf{R}} \tilde{\varphi}_{x,t}(\theta)e^{-i\theta y}\, d\theta\Big)dy - \int_{\mathbf{R}} f(y)\Big(\frac{1}{2\pi}\int_{\mathbf{R}} \varphi_{x,t}(\theta)\Big)e^{-i\theta y}\, d\theta dy \\
&= \frac{1}{2\pi}\int_{\mathbf{R}} f(y)\int_{\mathbf{R}} (\tilde{\varphi}_{x,t}(\theta) - \varphi_{x,t}(\theta))e^{-i\theta y}\, d\theta dy \\
&= \frac{1}{2\pi}\int_{\mathbf{R}} f(y)\int_{\mathbf{R}} (\exp\Big(\tilde{g}(t,\theta)+x\tilde{h}(t,\theta)\Big) - \exp(g(t,\theta)+xh(t,\theta)))e^{-i\theta y}\, d\theta dy
\end{aligned}
$$

Thus, our aim is to show the following identity:

$$
h(t,\theta)\exp(g(t,\theta)+xh(t,\theta)) = \frac{2}{\alpha t}(\exp\Big(\tilde{g}(t,\theta)+x\tilde{h}(t,\theta)\Big) - \exp(g(t,\theta)+xh(t,\theta))). \tag{$\star$}
$$

Now, since we have the explicit form of g and h, we know the left hand side is equal to

$$
\frac{2i\theta}{2-\alpha i\theta t}\Big(\frac{2}{2-\alpha i\theta t}\Big)^{2b/\alpha}\exp\Big(x\frac{2i\theta}{2-\alpha i\theta t}\Big),
$$

and the right hand side can be deform as follows:

$$
\begin{aligned}
&\frac{2}{\alpha t}(\exp\Big(\tilde{g}(t,\theta)+x\tilde{h}(t,\theta)\Big) - \exp(g(t,\theta)+xh(t,\theta))) \\
&\quad= \frac{2}{\alpha t}\Bigg[\Big(\frac{2}{2-\alpha i\theta t}\Big)^{2(b+\alpha/2)/\alpha}\exp\Big(x\frac{2i\theta}{2-\alpha i\theta t}\Big) - \Big(\frac{2}{2-\alpha i\theta t}\Big)^{2b/\alpha}\exp\Big(x\frac{2i\theta}{2-\alpha i\theta t}\Big)\Bigg] \\
&\quad= \frac{2}{\alpha t}\Bigg[\Big(\frac{2}{2-\alpha i\theta t}\Big)^{2(b+\alpha/2)/\alpha} - \Big(\frac{2}{2-\alpha i\theta t}\Big)^{2b/\alpha}\Bigg]\exp\Big(x\frac{2i\theta}{2-\alpha i\theta t}\Big).
\end{aligned}
$$

Also, we see that

$$\begin{aligned}
\left(\frac{2}{2-\alpha i\theta t}\right)^{2(b+\alpha/2)/\alpha} - \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha} &= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha+1} - \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha} \\
&= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha}\left(\frac{2}{2-\alpha i\theta t} - 1\right) \\
&= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha}\frac{\alpha i\theta t}{2-\alpha i\theta t}
\end{aligned}$$

Combining these calculation, we have the desired identity ($\star$).

For the case where $\beta \neq 0$, we can show that the formula holds by similar way, so we only check the different points. First, the integrability of $\varphi_{x,t}$ can be checked as follows:

$$\begin{aligned}
|\varphi_{x,t}(\theta)| &= |\exp(g(t,\theta) + xh(t,\theta))| \\
&= \left|\exp\left(\frac{2b}{\alpha}\operatorname{Log}\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta} + x\frac{2\beta e^{\beta t} i\theta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)\right| \\
&\leq \left(\frac{2\beta}{|2\beta-(e^{\beta t}-1)\alpha i\theta|}\right)^{2b/\alpha}\exp\left(\frac{2\beta e^{\beta t}|\theta| x}{|2\beta-(e^{\beta t}-1)\alpha i\theta|}\right) \\
&= \left(\frac{2}{\sqrt{4\beta^2+((e^{\beta t}-1)\alpha\theta)^2}}\right)^{2b/\alpha}\exp\left(\frac{2\beta e^{\beta t}|\theta| x}{\sqrt{4\beta^2+((e^{\beta t}-1)\alpha\theta)^2}}\right) \\
&\leq \frac{2^{2b/\alpha}}{(|e^{\beta t}-1|\alpha|\theta|)^{2b/\alpha}}\exp\left(\frac{2x}{|e^{\beta t}-1|\alpha}\right).
\end{aligned}$$

Therefore, also in this case, $|\varphi_{x,t}|$ and $|\tilde{\varphi}_{x,t}|$ are integrable on $\mathbf{R}$. Moreover, what we have to check is that the following identity holds:

$$\begin{aligned}
&\left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2(b+\alpha/2)/\alpha} - \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha} \\
&\qquad = \frac{\alpha(e^{\beta t}-1)}{2\beta e^{\beta t}}\frac{2\beta e^{\beta t} i\theta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha}.
\end{aligned}$$

This can be checked as follows:

$$\begin{aligned}
&\left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2(b+\alpha/2)/\alpha} - \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha} \\
&\quad = \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha+1} - \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha} \\
&\quad = \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha}\left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta} - 1\right) \\
&\quad = \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha}\frac{(e^{\beta t}-1)\alpha i\theta}{2\beta-(e^{\beta t}-1)\alpha i\theta}.
\end{aligned}$$

This completes all of the proof. $\square$

Moreover, corresponding to Theorem 3.4 can also be obtained.

Theorem 3.8. *Suppose $b > \alpha$. Let $t > 0$ be a fixed real number and let X^x and $\tilde{X}^x$ be diffusion affine processes with parameters (α, β, b) and $(\alpha, \beta, b - \alpha/2)$ respectively, and their initial values are $x > 0$. Then, for $f \in C^1[0, \infty)$ such that f and f' are in $L^1[0, \infty)$ and $f'(X_t^x)$, $f(X_t^x)$, and $f(\tilde{X}_t^x)$ are integrable, it holds that*

$$E[f'(X_t^x)] = \begin{cases} \dfrac{2}{\alpha t}\big(E[f(X_t^x)] - E[f(\tilde{X}_t^x)]\big), & \beta = 0, \\ \dfrac{2\beta}{\alpha(e^{\beta t} - 1)}\big(E[f(X_t^x)] - E[f(\tilde{X}_t^x)]\big), & \beta \neq 0. \end{cases}$$

Proof. We checked integrability of some functions, so we can interchange the order of integration in the following deformation. Also, we use the following property of Fourier transform:

$$\int_{\mathbf{R}} f'(y) e^{-i\theta y} dy = i\theta \int_{\mathbf{R}} f(y) e^{-i\theta y} dy.$$

$$\begin{aligned} E[f'(X_t)] &= \int_{\mathbf{R}} f'(y) q_{x,t}(y)\, dy \\ &= \int_{\mathbf{R}} f'(y) \frac{1}{2\pi} \left(\int_{\mathbf{R}} \varphi_{x,t}(\theta) e^{-i\theta y}\, d\theta \right) dy \\ &= \frac{1}{2\pi} \int_{\mathbf{R}} f'(y) \int_{\mathbf{R}} \exp(g(t,\theta) + xh(t,\theta)) e^{-i\theta y}\, d\theta dy \\ &= \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} f'(y) e^{-i\theta y}\, dy\, \exp(g(t,\theta) + xh(t,\theta))\, d\theta \\ &= \frac{1}{2\pi} \int_{\mathbf{R}} i\theta \int_{\mathbf{R}} f(y) e^{-i\theta y}\, dy\, \exp(g(t,\theta) + xh(t,\theta))\, d\theta \\ &= \frac{1}{2\pi} \int_{\mathbf{R}} f(y) \int_{\mathbf{R}} i\theta \exp(g(t,\theta) + xh(t,\theta)) e^{-i\theta y}\, d\theta dy \end{aligned}$$

In addition to this, we have already obtained the following identity:

$$\begin{aligned} &E[f(X_t^x)] - E[f(\tilde{X}_t^x)] \\ &\quad = \frac{1}{2\pi} \int_{\mathbf{R}} f(y) \int_{\mathbf{R}} (\exp(g(t,\theta) + xh(t,\theta)) - \exp\Big(\tilde{g}(t,\theta) + x\tilde{h}(t,\theta)\Big)) e^{-i\theta y}\, d\theta dy \end{aligned}$$

Note that the parameters should satisfy the conditon $2(b - \alpha/2)/\alpha > 1$. However this is equivalent to $b > \alpha$, so this condition holds under our assumption. Then, combining the above two deformation, the remainder we have to show is that the following identities hold:

$$i\theta \exp(g(t,\theta) + xh(t,\theta)) = \frac{2}{\alpha t}(\exp(g(t,\theta) + xh(t,\theta)) - \exp\Big(\tilde{g}(t,\theta) + x\tilde{h}(t,\theta)\Big))$$

for $\beta = 0$, and

$$i\theta \exp(g(t,\theta) + xh(t,\theta)) = \frac{2\beta}{\alpha(e^{\beta t}-1)}(\exp(g(t,\theta) + xh(t,\theta)) - \exp\Big(\tilde{g}(t,\theta) + x\tilde{h}(t,\theta)\Big))$$

for $\beta \neq 0$.

Moreover, since $\tilde{h}$ is equal to h in our assumption, the identities we have to derive is the following:

$$i\theta \exp(g(t,\theta)) = \frac{2}{\alpha t}(\exp(g(t,\theta)) - \exp(\tilde{g}(t,\theta)))$$

for $\beta = 0$, and

$$i\theta \exp(g(t,\theta)) = \frac{2\beta}{\alpha(e^{\beta t}-1)}(\exp(g(t,\theta)) - \exp(\tilde{g}(t,\theta)))$$

For the case where $\beta = 0$, we have the following deformation:

$$\begin{aligned}
\exp(g(t,\theta)) - \exp(\tilde{g}(t,\theta)) &= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha} - \left(\frac{2}{2-\alpha i\theta t}\right)^{2(b-\alpha/2)/\alpha} \\
&= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha} - \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha-1} \\
&= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha}\left(1 - \left(\frac{2}{2-\alpha i\theta t}\right)^{-1}\right) \\
&= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha}\left(1 - \frac{2-\alpha i\theta t}{2}\right) \\
&= \left(\frac{2}{2-\alpha i\theta t}\right)^{2b/\alpha}\frac{\alpha i\theta t}{2}.
\end{aligned}$$

For the case where $\beta = 0$, we have the following deformation:

$$\begin{aligned}
\exp(g(t,\theta)) - \exp(\tilde{g}(t,\theta)) &= \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha} - \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2(b-\alpha/2)/\alpha} \\
&= \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha} - \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha-1} \\
&= \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha}\left(1 - \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{-1}\right) \\
&= \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha}\left(1 - \frac{2\beta-(e^{\beta t}-1)\alpha i\theta}{2\beta}\right) \\
&= \left(\frac{2\beta}{2\beta-(e^{\beta t}-1)\alpha i\theta}\right)^{2b/\alpha}\frac{(e^{\beta t}-1)\alpha i\theta}{2\beta}.
\end{aligned}$$

This completes all of the proof. $\square$

Also, combining these two theorems, we have corresponding to Corollary 3.5.

Corollary 3.9. *Suppose $2b > \alpha$. Let $t > 0$ be a fixed real number and let X^x and $\tilde{X}^x$ be diffusion affine processes with parameters (α, β, b) and $(\alpha, \beta, b + \alpha/2)$ respectively, and their initial values are $x > 0$. Then, for $f \in C^1[0, \infty)$ such that f and f' are in $L^1[0, \infty)$ and $f'(\tilde{X}_t^x)$, $f(X_t^x)$, and $f(\tilde{X}_t^x)$ are integrable, it holds that*

$$\partial_x E[f(X_t^x)] = e^{\beta t} E[f'(\tilde{X}_t^x)].$$

Proof. Apply Theorem 3.3 directly, and Theorem 3.4 for processes with parameters $(\alpha, \beta, b + \alpha/2)$ and (α, β, b). Here, we have to check the condition for the parameters, $2(b + \alpha/2) > \alpha$, but this is equivalent to $2b > \alpha$ and this is exactly our current assumption. □

Remark 3.10. The class Diffusion affine process contains CIR model, which is a cereblated model of interest rate. The model is given by the following SDE:

$$dr_t = k(\theta - r_t)\, dt + \sigma \sqrt{r_t}\, dW_t.$$

Here, $k, \theta \geq 0$, and $\sigma > 0$ are parameters, which correspond to the speed of adjustment to the mean, the mean, and the volatility. In view of its importance in application, we rewrite the above results using the symbols of the CIR model.

Corollary 3.11. *Suppose $2k\theta > \sigma^2$. Let $t > 0$ be a fixed real number and let r and $\tilde{r}$ be CIR processes whose parameters are $(\sigma^2, -k, \theta)$ and $(\sigma^2, -k, \theta + \sigma^2/(2k))$ respectively, and their initial values are the same positive number. Then, for $f \in L^1[0, \infty)$ such that $f(r_t)$ and $f(\tilde{r}_t)$ are integrable, it holds that*

$$\partial_{r_0} E[f(r_t)] = \frac{-2ke^{-kt}}{\sigma^2(e^{-kt} - 1)} \big(E[f(\tilde{r}_t)] - E[f(r_t)] \big).$$

Corollary 3.12. *Suppose $k\theta > \sigma^2$. Let $t > 0$ be a fixed real number and let r and $\tilde{r}$ be CIR whose parameters are $(\sigma^2, -k, \theta)$ and $(\sigma^2, -k, k\theta - \sigma^2/(2k))$ respectively, and their initial values are the same positive number. Then, for $f \in C^1[0, \infty)$ such that f and f' are in $L^1[0, \infty)$ and $f'(r_t)$, $f(r_t)$, and $f(\tilde{r}_t)$ are integrable, it holds that*

$$E[f'(r_t)] = \frac{-2ke^{-kt}}{\sigma^2(e^{-kt} - 1)} \big(E[f(r_t)] - E[f(\tilde{r}_t)] \big).$$

Corollary 3.13. *Suppose $2k\theta > \sigma^2$. Let $t > 0$ be a fixed real number and let r and $\tilde{r}$ be CIR processes whose parameters are $(\sigma^2, -k, \theta)$ and $(\sigma^2, -k, \theta + \sigma^2/(2k))$ respectively, and their initial values are the same positive number. Then, for $f \in C^1[0, \infty)$ such that f and f' are in $L^1[0, \infty)$ and $f'(\tilde{r}_t)$, $f(r_t)$, and $f(\tilde{r}_t)$ are integrable, it holds that*

$$\partial_x E[f(r_t)] = e^{-kt} E[f'(\tilde{r}_t)].$$

3.3 An extension to very easy multidimensional case

Consider the following d-dimensional SDE:

$$dX_t^x = \alpha(X_t^x)dW_t + (\beta X_t^x + b)dt, \qquad X_0^x = x,$$

where W is a d-dimensional standard Brownian motion and $\alpha\colon \mathbf{R}^d \to \mathbf{R}^d \times \mathbf{R}^d$ is defined by

$$\alpha((x_k)_{k=1,\dots,d})_{i,j} = \delta_{i,j}\sqrt{\alpha_i|x_i|}, \qquad 1 \le i,j \le d, \tag{$*$}$$

with $\alpha_i > 0$, $i = 1,\dots,d$, β is a diagonal matrix of order d, and $b = (b_i)_{i=1,\dots,d} \in \mathbf{R}^d$. Also, we assume the initial value $x = (x_i)_{i=1,\dots,d}$ satisfies $x_i \ge 0$, $i = 1,\dots,d$. In this case, all of components $((X^x)_1,\dots,(X^x)_d)$ are independent, so X^x is essentially a combination of 1-dimensional diffusion affine processes. Therefore, this SDE admits a strong solution whose conponents are all nonnegative almost surely, and we obtain the following theorems and corollary as a consequence of the theorem in the previous section. Note that we write e_i for the i-th vector of the standard basis of $\mathbf{R}^n$.

Theorem 3.14. *Let k be in $\{1,\dots,d\}$. Suppose $2b_k > \alpha_k$. Let $t > 0$ be a fixed real number and let X^x and $\tilde{X}^x$ are the solutions of SDE $(*)$ whose parameters are $((\alpha_i)_{i=1,\dots,d}, \beta, b)$ and $((\alpha_i)_{i=1,\dots,d}, \beta, b + (\alpha_k/2)e_k)$ respectively, and their initial values are $x \in (0,\infty)^d$. Then, for $f \in L^1([0,\infty)^d)$ such that $f(X_t^x)$ and $f(\tilde{X}_t^x)$ are integrable, it holds that*

$$\partial_{x_k} E[f(X_t^x)] = \begin{cases} \dfrac{2}{\alpha_k t}\big(E[f(\tilde{X}_t^x)] - E[f(X_t^x)]\big), & \beta_{k,k} = 0, \\ \dfrac{2\beta_{k,k}}{\alpha_k(e^{\beta_{k,k}t} - 1)}\big(E[f(\tilde{X}_t^x)] - E[f(X_t^x)]\big), & \beta_{k,k} \ne 0. \end{cases}$$

Theorem 3.15. *Let k be in $\{1,\dots,d\}$. Suppose $b_k > \alpha_k$. Let $t > 0$ be a fixed real number and let X^x and $\tilde{X}^x$ are the solutions of SDE $(*)$ whose parameters are $((\alpha_i)_{i=1,\dots,d}, \beta, b)$ and $((\alpha_i)_{i=1,\dots,d}, \beta, b - (\alpha_k/2)e_k)$ respectively, and their initial values are $x \in (0,\infty)^d$. Then, for $f \in L^1([0,\infty)^d)$ such that $f(y_1,\dots,y_{k-1},\cdot,y_{k+1},\dots,y_d) \in C^1[0,\infty)$ for each $y_1,\dots,y_{k-1},y_{k+1},\dots,y_d \in [0,\infty)$ and $\partial_{y_k} f \in L^1([0,\infty)^d)$, and $\partial_{y_k} f(X_t^x)$, $f(X_t^x)$, and $f(\tilde{X}_t^x)$ are integrable, it holds that*

$$E[(\partial_{y_k} f)(X_t^x)] = \begin{cases} \dfrac{2}{\alpha_k t}\big(E[f(X_t^x)] - E[f(\tilde{X}_t^x)]\big), & \beta_{k,k} = 0, \\ \dfrac{2\beta_{k,k}e^{\beta_{k,k}t}}{\alpha_k(e^{\beta_{k,k}t} - 1)}\big(E[f(X_t^x)] - E[f(\tilde{X}_t^x)]\big), & \beta_{k,k} \ne 0. \end{cases}$$

Corollary 3.16. *Let k be in $\{1,\dots,d\}$. Suppose $2b_i > \alpha_i$, $i = 1,\dots,d$. Let $t > 0$ be a fixed real number and let X^x and $\tilde{X}^x$ are the solutions of SDE $(*)$ whose parameters are $((\alpha_i)_{i=1,\dots,d}, \beta, b)$ and $((\alpha_i)_{i=1,\dots,d}, \beta, b - (\alpha_k/2)e_k)$ respectively, and their initial values are $x \in (0,\infty)^d$. Then, for $f \in L^1([0,\infty)^d)$ such* *that* $f(y_1,\dots,y_{k-1},\cdot,y_{k+1},\dots,y_d)\in C^1[0,\infty)$ *for each* $y_1,\dots,y_{k-1},y_{k+1},\dots,y_d\in[0,\infty)$ *and* $\partial_{y_k}f\in L^1([0,\infty)^d)$, *and* $\partial_{y_k}f(X_t^x)$, $f(X_t^x)$, *and* $f(\tilde{X}_t^x)$ *are integrable, it holds that*

$$E[(\partial_{y_k}f)(X_t^x)]=\begin{cases}\dfrac{2}{\alpha_k t}\big(E[f(X_t^x)]-E[f(\tilde{X}_t^x)]\big), & \beta_{k,k}=0,\\ \dfrac{2\beta_{k,k}e^{\beta_{k,k}t}}{\alpha_k(e^{\beta_{k,k}t}-1)}\big(E[f(X_t^x)]-E[f(\tilde{X}_t^x)]\big), & \beta_{k,k}\neq 0.\end{cases}$$

Proof of Theorem 3.14. We will prove for the case where $\beta_{k,k}=0$. The proof for the other case is almost the same so we omit it.

For simplicity, consider only the case where $k=1$. First of all, writing $\mathcal{L}_Z$ for the law of random variable Z, we can deform the left hand side as follows because $(X_t^x)_1$ and $((X_t^x)_2,\dots,(X_t^x)_d)$ are independent:

$$\begin{aligned}\partial_{x_1}E[f(X_t^x)]&=\partial_{x_1}\int_{\mathbf{R}^d}f(y)\mathcal{L}_{X_t^x}(dy)\\&=\partial_{x_1}\int_{\mathbf{R}^{d-1}}\int_{\mathbf{R}}f(y)\mathcal{L}_{(X_t^x)_1}(dy_1)\mathcal{L}_{((X_t^x)_2,\dots,(X_t^x)_d)}(d(y_2,\dots,y_d))\\&=\partial_{x_1}\int_{\mathbf{R}^{d-1}}\int_{\mathbf{R}}f(y)q_{x_1,t}(y_1)dy_1\mathcal{L}_{((X_t^x)_2,\dots,(X_t^x)_d)}(d(y_2,\dots,y_d))\\&=\partial_{x_1}\int_{\mathbf{R}^{d-1}}\int_{\mathbf{R}}f(y)\left(\frac{1}{2\pi}\int_{\mathbf{R}}e^{-i\theta y_1}\varphi_{x_1,t}(\theta)d\theta\right)dy_1\mathcal{L}_{((X_t^x)_2,\dots,(X_t^x)_d)}(d(y_2,\dots,y_d))\end{aligned}$$

Here, considering the differential coefficient at $x_1=\xi_1$, it holds that

$$\left|\int_{\mathbf{R}}f(y)\int_{\mathbf{R}}e^{-i\theta y_1}\varphi_{x_1,t}(\theta)d\theta dy_1\right|\leq\int_{\mathbf{R}}|f(y)|dy_1\int_{\mathbf{R}}\frac{2^{2b/\alpha}}{(\alpha t\theta)^{2b/\alpha}}d\theta\exp\left(\frac{2}{\alpha t}(\xi_1+1)\right)$$

on a certain neighborhood of x_1 and the right hand side

$$(y_2,\dots,y_d)\mapsto\int_{\mathbf{R}}|f(y)|dy_1\int_{\mathbf{R}}\frac{2^{2b/\alpha}}{(\alpha t\theta)^{2b/\alpha}}d\theta\exp\left(\frac{2}{\alpha t}(\xi_1+1)\right)$$

is integrable. Thus we can interchange the order of derivative and integral. Then we have

$$\begin{aligned}&\partial_{x_1}\int_{\mathbf{R}^{d-1}}\int_{\mathbf{R}}f(y)\left(\frac{1}{2\pi}\int_{\mathbf{R}}e^{-i\theta y_1}\varphi_{x_1,t}(\theta)d\theta\right)dy_1\mathcal{L}_{((X_t^x)_2,\dots,(X_t^x)_d)}(d(y_2,\dots,y_d))\\&=\int_{\mathbf{R}^{d-1}}\partial_{x_1}\int_{\mathbf{R}}f(y)\left(\frac{1}{2\pi}\int_{\mathbf{R}}e^{-i\theta y_1}\varphi_{x_1,t}(\theta)d\theta\right)dy_1\mathcal{L}_{((X_t^x)_2,\dots,(X_t^x)_d)}(d(y_2,\dots,y_d))\\&=\int_{\mathbf{R}^{d-1}}\partial_{x_1}E[f((X_t^x)_1,y_2,\dots,y_d)]\mathcal{L}_{((X_t^x)_2,\dots,(X_t^x)_d)}(d(y_2,\dots,y_d))\end{aligned}$$

Here, $(X_t^x)_1$ is a one-dimensional diffusion affine process with parameter $(\alpha_1,\beta_{1,1},b_1)$ and initial value x_1. Therefore, we get

$$\partial_{x_1}E[f((X_t^x)_1,y_2,\dots,y_d)]=\frac{1}{\alpha_1 t}(E[f((X_t^x)_1,y_2,\dots,y_d)]-E[f((\tilde{X}_t^x)_1,y_2,\dots,y_d)])$$

by Theorem 3.7, where $(\tilde{X}_t^x)_1$ is a diffusion affine process with parameter $(\alpha_1, \beta_{1,1}, b_1+\alpha_1/2)$. Furthermore, combining the fact that

$$\mathcal{L}_{((\tilde{X}_t^x)_2,\ldots,(\tilde{X}_t^x)_d)} = \mathcal{L}_{((X_t^x)_2,\ldots,(X_t^x)_d)}$$

(because $((X^x)_2, \ldots, (X^x)_d)$ and $((\tilde{X}^x)_2, \ldots, (\tilde{X}^x)_d)$ are solutions of the same SDE), we obtain the desied result. $\square$

We can prove Theorem 3.15 in the same way as above and can prove Corollary 3.16 in the same way as in the proof of Corollary 3.9, so we omit the proofs.

Chapter 4

Tanaka–Yor equations

4.1 Previous researches

4.1.1 Symmetrization of SDEs

Consider pricing of options. Let X be the price of underlying asset, $T > 0$ be the maturity, and f be the payoff function. Then the price of knock-in barrier option with lower barrier K can be expressed by the following expectation:

$$E[X_T 1_{(\tau > T)}], \qquad \tau := \inf\{t > 0 \colon X_t \leq K\}.$$

This expectation includes the difficulty that it is path-dependent. To address this difficulty, Imamura et al. proposed a method called symmetrization [35]. First of all, they assumed that X satisfies the following SDE:

$$dX_t = \sigma(X_t) dW_t + \mu(X_t) dt, \tag{4.1}$$

where W is one-dimensional Brownian motion and they imposed appropriate conditions on σ and μ. Then they defined the new coefficients $\tilde{\sigma}$ and $\tilde{\mu}$ as follows:

$$\tilde{\sigma}(x) := \begin{cases} \sigma(x), & x > K, \\ \sigma(2K - x), & x \leq K, \end{cases}$$
$$\tilde{\mu}(x) := \begin{cases} \mu(x), & x > K, \\ -\mu(2K - x), & x \leq K. \end{cases}$$

Moreover, they consider the following SDE:

$$d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t) dW_t + \tilde{\mu}(\tilde{X}_t) dt.$$

They named this SDE the symmetrization of the SDE (4.1). In this setting, they proved the following formula [35]:

Theorem 4.1. *Under appropriate conditions regarding to σ and μ, it holds that*

$$E[f(X_T) 1_{(\tau > T)}] = E[f(\tilde{X}_T) 1_{(\tilde{X}_T > K)}] - E[f(2K - X_T) 1_{(\tilde{X_T} < K)}].$$

They pointed out that the right-hand side is a combination of the prices of plain vanilla options and claimed that this formula makes it much easier to calculate the price of barrier options. This is because the expected value on the right-hand side is not path-dependent. In fact, the numerical experiments in their paper suggest that their method can be used to calculate the prices of barrier options very efficiently.

Moreover, Akahori et al. extended this method to the multidimensional case [36]. In this case, the barrier is a hyperplane in the Euclidean space. This is straightforward extension of one-dimensional case.

Furthermore, Hishida et al. extended it one more step in [37]. They considered it in Euclidean space, but assumed the boundary to be a C^2-class surface. We will explain this in a little more detail. Let W be a d'-dimensional Brownian motion and consider a d-dimensional SDE

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt. \tag{4.2}$$

Now, the barrier is $\{x \in \mathbf{R}^d \colon g(x) = 0\}$ where $g \colon \mathbf{R}^d \to \mathbf{R}$ is a C^2-class function. Then the price of the barrier option whose payoff function is f can be expressed as follows:

$$E[f(X_T)1_{(\tau>T)}], \qquad \tau := \inf\{t > 0 \colon g(X_t) < 0\}.$$

Then, they focused on the process $Z = g(X)$. By applying Itô's formula and combine it with the original SDE (4.2), we can get a $(d+1)$-dimensional SDE. If we consider that $(X^{(1)}, \dots, X^{(d)}, g(X))$ is the price of new underlying asset, the barrier is hyperplain $\{x \in \mathbf{R} \colon x_{d+1} = 0\}$. Thus we can apply multidimensional symmetrization of Akahori et al. Here, the symmetrized SDE can be discribed as follows:

$$\begin{aligned} d\tilde{X}_t &= \sigma(\tilde{X}_t)dW_t + \mu(\tilde{X}_t)dt, \\ d\tilde{Z}_t &= \operatorname{sgn}(\tilde{Z}_t)dZ_t. \end{aligned}$$

Therefore, we obtain an expression of the price of the barrier option in terms of expectations that are not path-dependent:

$$E[f(X_t)1_{(\tau>T)}] = E[f(\tilde{X}_T)1_{(\tilde{Z}_T \geq 0)}] - E[f(\tilde{X}_t)1_{(\tilde{Z}_T < 0)}].$$

They also showed the results of their numerical calculations in their paper, and the results suggested that their method was more efficient than the conventional method.

This is a great theorem, but currently we need to assume that $\tilde{Z}$ and $-\tilde{Z}$ have the same distribution for any weak solution $\tilde{Z}$ of the last SDE. We believe that this theorem would be much more useful if we could remove this assumption, and it is our goal to show this. To restate what needs to be shown is as follows.

Question 4.2. Let Z be a continuous semimartingale and consider the following SDE:

$$dY_t = \operatorname{sgn}(Y_t)dZ_t.$$

Prove that Y and $-Y$ have the same distribution for any weak solution Y of the SDE.

Obviously, it is enough to show the law-uniqueness of the SDE. This chapter is the first step toward this goal.

4.1.2 Tsirelson–Yor equations

Yor investigated the following $\mathbf{R}$-valued and $\mathbf{Z}_{\leq 0}$-indexed stochastic equation in [13]:

$$\eta_k = \xi_k + \{\eta_{k-1}\}, \qquad k \in \mathbf{Z}_{\leq 0}, \tag{4.3}$$

where ξ is a known independent noise and $\{x\}$ represents the fractional part of x. We assume that the distribution of the noise ξ_k is μ_k. This equation has it origin in Tsirelson's paper [12]. In the paper, Tsirelson showed an example of SDE which has no strong solution using such a discrete-time stochastic equation. For this equation, the notion of solution is defined as follows:

Definition 4.3. Let Ω be $\mathbf{R}^{\mathbf{Z}_{\leq 0}}$ and P be a Borel probability measure on Ω. If P satisfies the following conditions, P is called a solution of the equation (4.3):
Let $\eta_k((\omega_l)_l) := \omega_k$, $\xi_k := \eta_k - \{\eta_{k-1}\}$, $\mathcal{F}_k^\eta := \sigma(\eta_l, l \leq k)$, $\mathcal{F}_k^\xi := \sigma(\xi_l, l \leq k)$, and write μ_k for the given law (of ξ_k).

- The law of ξ_k is μ_k for all $k \in \mathbf{Z}_{\leq 0}$.
- ξ_k is independent of $\mathcal{F}_{k-1}^\eta$ for all $k \in \mathbf{Z}_{\leq 0}$.

Moreover, if $\mathcal{F}_k^\eta = \mathcal{F}_k^\xi$ up to P-null set for all $k \in \mathbf{Z}_{\leq 0}$, the solution P is called *strong*.

First, Yor proved that equation (4.3) always has at least one solution. Furthermore, they classified the equation depends on $\mu = (\mu_k)_k$ in the view point of its solutions. More precisely, he showed the following trichotomy:

1. The equation has only one solution and it is non-strong.
2. The equation has at least two solutions and has at least one strong solution.
3. The equation has at least two solutions and has no strong solution.

In [14], Akahori et al. generalized Yor's study. Their setting is as follows: $(S, \mathcal{S})$ is a Polish space. $(G, *, \mathcal{G})$ is a compact group acting to S and write $g(\cdot)$ for the action. $\theta \colon S \to G$ is a measurable map. $(\xi_k)_{k \in \mathbf{Z}_{\leq 0}}$ is an independent known noise. Also, they assume two conditions.

Assumption 4.4.

1. $g(\cdot)$ is an automorphism on S for each $g \in G$.
2. $g * \theta(s) = \theta(g(s))$ for all $s \in S$, $g \in G$.

In this setting, they studied the following equation and they called this equation *a Tsirelson–Yor equation*:

$$\eta_k = \theta(\eta_{k-1})(\xi_k), \qquad k \in \mathbf{Z}_{\leq 0}. \tag{4.4}$$

Of courese, this includes Yor's equation (4.3). Then, the notion of a solution to such a equation is defined by the same way as in Yor's case. In this case, Akahori et al. proved that Tsirelson–Yor equation (4.4) always has at least one solution, first. Furthermore, they showed that the same trichotomy as in Yor's study [13] holds in their generalized setting:

1. The equation has only one solution and it is non-strong.

2. The equation has at least two solutions and has at least one strong solution.

3. The equation has at least two solutions and has no strong solution.

We want to mimic this result and to utilize the result to justify the multidimensional symmetrization with C^2-class boundary.

4.2 Main result

In our setting, $(S, \cdot, \mathcal{S})$ is a Polish topological *group*, $(G, *, \mathcal{G})$ is a compact group, $\psi \colon S \times G \to S$ is a group action such that

$$\psi(\cdot, g) \colon S \to S \text{ are group-automorphism for all } g \in G, \tag{4.5}$$

and $\theta \colon S \to G$ is a measurable map. We may write $g(s)$ instead of $\psi(s, g)$ for $g \in G$ and $s \in S$ and also may write ab instead of $a \cdot b$ for $a, b \in S$. Moreover, we assume "g and θ are commute" for all $g \in G$, in the following sense:

$$g * \theta(s) = \theta(g(s))$$

for all $s \in S$ and $g \in G$.

Now $t \mapsto f(s,t) := \psi(t, \theta(s)) \cdot s$ is bijective for each $s \in S$ thanks to (4.5), and we can define a map $f^{-1} \colon S \times S \to S$ by $f^{-1} := \pi_2 \circ (\pi_1, f)^{-1}$, where π_i is the projection to the i-th component of the Cartesian product. Here, $f^{-1}(s, f(s,u)) = u$ holds for all $s, u \in S$ and $g \in G$.

In this thesis, we consider the following equation and we call it *Tanaka–Yor equation*.

$$\eta_k = f(\eta_{k-1}, \xi_k), \quad k \in Z_{\leq 0}. \tag{4.6}$$

We call this equation *Tanaka–Yor equation*. Note that this is not a generalization of Tsirelson–Yor equations but an analogue. That is, the class Tanaka–Yor equations dose not include the class Tsirelson–Yor equations. Then, the solution to the equation (4.6) is defined in the same way as for Tsirelson–Yor equations. Moreover, for consistency, we assume the following two conditions:

Assumption 4.5. 1. For each k, μ_k is "G-invariant":

$$\int_S h(g(s))\mu_k(ds) = \int_S h(s)\mu_k(ds)$$

for all $g \in G$ and arbitrary bounded measurable function $h\colon S \to \mathbf{R}$.

2. If the joint distribution of $(\zeta_k)_{k\in\mathbf{Z}_{k\leq 0}}$ is $\bigotimes_{k\in\mathbf{Z}_{\leq 0}} \mu_k$, then the law of $\zeta_j\zeta_{j-1}\cdots\zeta_N$ converges to some probability law ν_j for each $j \in \mathbf{Z}_{\leq 0}$.

Lemma 4.6. *Let Y be a random variable whose law is G-invariant, $\phi\colon S \to S$ be a map which is "commuting" with ψ in the following sense:*

$$g(\phi(x)) = \phi(g(x))$$

for all $g \in G$, $\mathcal{L}_Y$-a.a. x. Then $\phi(Y)$ is also G-invariant.

Proof. Let $h\colon S \to \mathbf{R}$ be a bounded measurable function. Then

$$E[h(g(\phi(Y)))] = E[h(\phi(g(Y)))] \tag{4.7}$$
$$= E[h(\phi(Y))] \tag{4.8}$$

holds for all $g \in G$. □

Theorem 4.7. *For any Tanaka–Yor equation, at least one solution exists. Moreover, under any solution P, $\theta(\eta_k)$ is independent of $(\xi_j)_{j\in\mathbf{Z}_{\leq 0}}$ for any $k \in \mathbf{Z}_{\leq 0}$. In particular, P is not strong.*

Proof. To begin with, we construct probability measure P_k on $S^{\{k,k+1,\ldots,0\}}$ for each $k \in \mathbf{Z}_{\leq 0}$. To construct the measure P_k, let η_k^k be a random variable valued in S whose law is ν_k and $\xi_k, \xi_{k+1}, \ldots \xi_0$ be random variables valued in S whose joint law is $\mu_k \otimes \mu_{k+1} \otimes \cdots \otimes \mu_0$. Then define a finite sequence of random variables $(\eta_j^k)_{k\leq j\leq 0}$ inductively by

$$\eta_{j+1}^k := f(\eta_j^k, \xi_{j+1}). \tag{4.9}$$

Now set P_k the joint law of $(\eta_k^k, \eta_{k-1}^k, \ldots, \eta_0^k)$.

Then the solution P we want is the Kolmogorov extension of the probability measures $(P_k)_{k\in\mathbf{Z}_{\leq 0}}$. Here, we have to verify the consistency to guarantee that we can extend the measures. That is, we have to verify the equality

$$P_k(S \times B) = P_{k+1}(B)$$

for all $B \in \mathcal{B}(S^{\{k+1,\ldots,0\}})$ for each k. This is equivalent to the equality of the laws of $(\eta_{k+1}^k, \ldots \eta_0^k)$ and $(\eta_{k+1}^{k+1}, \ldots \eta_0^{k+1})$. Furthermore, it is enough to show that the law of η_{k+1}^k is equal to that of η_{k+1}^{k+1}. This is because both sequences $(\eta_j^k)_{k+2\leq j\leq 0}$ and $(\eta_j^{k+1})_{k+2\leq j\leq 0}$ are defined by the same inductive formula (4.9), and η_{k+1}^k and η_{k+1}^{k+1} are independent of $\xi_{k+2}, \ldots, \xi_0$ respectively.

Then let us check the equality of the laws of η^k_{k+1} and η^{k+1}_{k+1}. For arbitrary bounded measurable function $h_i\colon S \to \mathbf{R}; i = 1, 2$, we have

$$\begin{aligned}
E[h_1(\theta(\eta^k_k)(\xi_{k+1}))h_2(\eta^k_k)] &= E[E[h_1(\theta(\eta^k_k)(\xi_{k+1}))|\eta^k_k]h_2(\eta^k_k)] \\
&= E[E[h_1(\xi_{k+1})|\eta^k_k]h_2(\eta^k_k)] \\
&= E[h_1(\xi_{k+1})h_2(\eta^k_k)] \\
&= E[h_1(\xi_{k+1})]E[h_2(\eta^k_k)] \\
&= E[h_1(\theta(\eta^k_k)(\xi_{k+1}))]E[h_2(\eta^k_k)].
\end{aligned}$$

This shows $\theta(\eta^k_k)(\xi_{k+1})$ is independent of η^k_k and distributed as μ_{k+1}. Thus, the law of η^k_{k+1} is ν_{k+1}, which is the same as that of η^{k+1}_{k+1}.

Next, let us confirm the independence of $\theta(\eta_k)$ and $\{\xi_j\}_{j\in\mathbf{Z}_{\leq 0}}$ for any $k \in \mathbf{Z}_{\leq 0}$. First, notice that the map $\phi_x\colon y \mapsto f(y, x)$ is "commuting with ψ " in the sense in Lemma 4.6 not depending on x. In fact, we have

$$\begin{aligned}
g(\phi_x(y)) &= g(f(y, x)) \\
&= g(\theta(y)(x) \cdot y) \\
&= g(\theta(y)(x)) \cdot g(y) \\
&= (g * \theta(y))(x) \cdot g(y) \\
&= \theta(g(y))(x) \cdot g(y) \\
&= f(g(y), x) \\
&= \phi_x(g(y)),
\end{aligned}$$

and if Y is G-invariant, then $\phi_x(Y)$ is also G-invariant by Lemma 4.6. Thus, for arbitrary bounded measurable functions $h_1\colon G \to \mathbf{R}$ and $h_2\colon S \to \mathbf{R}$, letting U be a uniform random variable which is independent of ξ_k,

$$\begin{aligned}
E[h_1(\theta(\eta_k))h_2(\xi_k)] &= E[h_1(\theta(\phi_{\xi_k}(\eta_{k-1})))h_2(\xi_k)] \\
&= E[E[h_1(\theta(\phi_{\xi_k}(\eta_{k-1}))) \mid \xi_k]h_2(\xi_k)] \\
&= E[h_1(U)h_2(\xi_k)] \\
&= E[h_1(U)]E[h_2(\xi_k)] \\
&= E[h_1(\theta(\eta_k))]E[h_2(\xi_k)].
\end{aligned}$$

Therefore, $\theta(\eta_k)$ is independent of ξ_k for all $k \in \mathbf{Z}_{\leq 0}$. This argument can be applied for

$$\phi^2_{x_1,x_2} := \phi_{x_2} \circ \phi_{x_1}$$

and further

$$\phi^l_{x_1,\ldots,x_l} := \phi_{x_l} \circ \cdots \circ \phi_{x_1}$$

for any l as well, since they are all G-invariant. Consequently, $\theta(\eta_k)$ is independent of $\{\xi_l\colon l \leq k\}$. This proves what we desired. $\square$

Remark 4.8. Recall that the SDE in Question 4.2 in subsection 4.1 is as follows:

$$dY_t = \mathrm{sgn}(Y_t)dZ_t.$$

Now, take a sequence of real numbers $(t_k)_{k\in\mathbf{Z}_{\leq 0}}$ which satisfies $0 < \cdots < t_{-1} < t_0$ and $\lim_{k\to-\infty} t_k = 0$, and discretize the above SDE:

$$X_{t_k} = \mathrm{sgn}(X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) + X_{t_{k-1}}, \qquad k \in \mathbf{Z}_{\leq 0}.$$

Then, this is a Tanaka–Yor equation where $S = \mathbf{R}$, $G = \{\pm 1\} \cong \mathbf{Z}/2\mathbf{Z}$, $\theta = \mathrm{sgn}$, $\eta_k = X_{s_k}$, $\xi_k = Y_{t_k} - Y_{t_{k-1}}$, and the action of the group is ordinal multiplication. Although we showed only existence of solutions of Tanaka–Yor equation, if we can show the uniqueness of solutions of the equation, it could be applied to the proof of law-uniqueness of the original SDE.

Chapter 5

Absolute zeta functions and periodicity of quantum walks on cycles

5.1 Preliminaries

5.1.1 Absolute Zeta Functions

This section is taken from a previous research paper [16] by Akahori et al.

In this section, we briefly review the framework on the absolute zeta functions, which can be considered as zeta function over $\mathbf{F}_1$, and absolute automorphic forms (see [26, 27, 28, 29, 30] and references therein, for example).

Let $f(x)$ be a function $f : \mathbf{R} \to \mathbf{C} \cup \{\infty\}$. We say that f is an *absolute automorphic form* of weight D if f satisfies

$$f\left(\frac{1}{x}\right) = Cx^{-D} f(x)$$

with $C \in \{-1, 1\}$ and $D \in \mathbf{Z}$. The *absolute Hurwitz zeta function* $Z_f(w, s)$ is defined by

$$Z_f(w,s) = \frac{1}{\Gamma(w)} \int_1^\infty f(x)\ x^{-s-1} (\log x)^{w-1}\, dx,$$

where $\Gamma(x)$ is the gamma function (see [38], for instance). Then taking $u = e^t$, we see that $Z_f(w, s)$ can be rewritten as the Mellin transform:

$$Z_f(w,s) = \frac{1}{\Gamma(w)} \int_0^\infty f(e^t)\ e^{-st}\ t^{w-1} dt. \tag{5.1}$$

Moreover, the *absolute zeta function* $\zeta_f(s)$ is defined by

$$\zeta_f(s) = \exp\left(\frac{\partial}{\partial w} Z_f(w,s)\Big|_{w=0}\right).$$

Here we introduce the *multiple Hurwitz zeta function of order* r, $\zeta_r(s, x, (\omega_1, \ldots, \omega_r))$, the *multiple gamma function of order* r, $\Gamma_r(x, (\omega_1, \ldots, \omega_r))$, and the *multiple sine function of order* r, $S_r(x, (\omega_1, \ldots, \omega_r))$, respectively (see [26, 27, 29], for example):

$$\zeta_r(s, x, (\omega_1, \ldots, \omega_r)) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + x)^{-s}, \tag{5.2}$$

$$\Gamma_r(x, (\omega_1, \ldots, \omega_r)) = \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x, (\omega_1, \ldots, \omega_r)) \Big|_{s=0} \right), \tag{5.3}$$

$$S_r(x, (\omega_1, \ldots, \omega_r)) = \Gamma_r(x, (\omega_1, \ldots, \omega_r))^{-1} \, \Gamma_r(\omega_1 + \cdots + \omega_r - x, (\omega_1, \ldots, \omega_r))^{(-1)^r}. \tag{5.4}$$

Now we present the following key result derived from Theorem 4.2 and its proof in Korokawa [27] (see also Theorem 1 in Kurokawa and Tanaka [29]):

Theorem 5.1. *If f has the form*

$$f(x) = x^{l/2} \frac{(x^{m(1)} - 1) \cdots (x^{m(a)} - 1)}{(x^{n(1)} - 1) \cdots (x^{n(b)} - 1)} \tag{$\star$}$$

for some $l \in \mathbf{Z}$, $a, b \in \mathbf{Z}_{>0}$, $m(i), n(j) \in \mathbf{Z}_{>0}$ $(i = 1, \ldots, a, j = 1, \ldots, b)$, then the following holds:

$$\begin{aligned} Z_f(w, s) &= \sum_{I \subset \{1,\ldots,a\}} (-1)^{|I|} \zeta_b(w, s - \deg(f) + m(I), \boldsymbol{n}), \\ \zeta_f(s) &= \prod_{I \subset \{1,\ldots,a\}} \Gamma_b(s - \deg(f) + m(I), \boldsymbol{n})^{(-1)^{|I|}}, \\ \zeta_f(D - s)^C &= \varepsilon_f(s) \zeta_f(s), \end{aligned}$$

where

$$\deg(f) = l/2 + \sum_{i=1}^{a} m(i) - \sum_{j=1}^{b} n(j), \quad m(I) = \sum_{i \in I} m(i),$$

$$\boldsymbol{n} = (n(1), \ldots, n(j)), \quad D = l + \sum_{i=1}^{a} m(i) - \sum_{j=1}^{b} n(j),$$

$$C = (-1)^{a-b}, \quad \varepsilon_f = \prod_{I \subset \{1,\ldots,a\}} S_b(s - \deg(f) + m(I), \ \boldsymbol{n} \ )$$

5.1.2 Cycle graphs and Quantum walks on it

For $N \geq 2$, *undirected cycle graph with N vertices* is an undirected graph which have N vertices and every vertices have exactly 2 edges. We write this graph by C_N. Formally, C_N is defined as follows:

Definition 5.2. The set of vertices of C_N is $\{0, 1, \ldots, N-1\}$ and the set of edges of C_N is $\{\{k, k+1\} \mid k = 0, \ldots, N-1\}$ with identifying N and 0.

We write $V(C_N)$ for the set of vertices of C_N.

A quantum walk is the time-evolving sequence of states consisting of position and chirality. Formally, a state is a vector which is an element of a tensor product of two Hilbert spaces over $\mathbf{C}$, $\mathcal{H}_P$ and $\mathcal{H}_C$. In this thesis, $\mathcal{H}_P$ is $\mathrm{span}\{|x\rangle \mid x \in V(C_N)\}$ and $\mathcal{H}_C$ is $\mathrm{span}\{|\leftarrow\rangle, |\rightarrow\rangle\}$ for Hadamard walks, and $\mathrm{span}\{|\leftarrow\rangle, |\cdot\rangle, |\rightarrow\rangle\}$ for Grover walks with 3 states. Note that the elements of "span" are considered to be the orthonormal basis of each space. Then, each state can be represented as follows:

$$\sum_{x \in V(C_n)} |x\rangle \otimes s, \qquad s \in \mathcal{H}_C.$$

Usually, we assume that the initial state Ψ_0 satisfies $\|\Psi_0\| = 1$. Moreover, we consider the case where the time evolution operator U is decomposed as $U = SC$. Here, S is called *shift operator* and defined by the following formulas:

$$\begin{aligned} S(|x\rangle \otimes |\leftarrow\rangle) &:= |x-1\rangle \otimes |\leftarrow\rangle, \\ S(|x\rangle \otimes |\rightarrow\rangle) &:= |x+1\rangle \otimes |\rightarrow\rangle, \\ S(|x\rangle \otimes |\cdot\rangle) &:= |x\rangle \otimes |\cdot\rangle. \end{aligned}$$

Furthermore, C is called *coin operator* and defined by the following:

$$C := \sum_{x \in V(C_N)} |x\rangle\langle x| \otimes A$$

for some unitary operator A on $\mathcal{H}_C$. We call this operator A *the local coin operator*. In this case, S and C are both unitary, and then U is also unitary. Now, the time evolution is defined as usual:

$$\Psi_{n+1} := U\Psi_n.$$

Of course, we have $\Psi_n = U^n\Psi_0$. We are interested in this time-evolution operator U. In each of the following subsections, matrix representations of U are shown.

Moreover, we introduce the term period of quantum walk.

Definition 5.3. For a quantum walk whose time-evolution operater is U, *the period of the quantum walk* is defined the following minimum:

$$\min\{n \geq 1 \mid U^n = 1\}.$$

If the set in the above formula is empty, then the period is defined to be ∞.

Of course, if T is the period of a quantum walk, it hold that

$$\Psi_T = \Psi_0$$

for any initial state Ψ_0.

Also, we define the zeta function of a quantum walk on a cycle graph:

Definition 5.4. For a quantum walk on a cycle graph whose time-evolution operator is U, *the zeta function of the quantum walk* ζ is defined as follows:

$$\zeta(u) := \det(I - uU)^{-1},$$

where I is the identity operator.

This definition can be seen in Konno–Sato theorem and this is suggested by graph zeta functions. [39]

5.1.3 Cyclotomic polynomials

Also, we treat polynomial rings and cyclotomic polynomials in this chapter.

Definition 5.5. $\mathbf{Z}[x], \mathbf{Q}[x], \mathbf{R}[x], \mathbf{C}[x]$ denote the polynomial rings with integer coefficients, rational coefficients, real coefficients, and complex coefficients respectively.

Then *cyclotomic polynomials* are defined as follows:

Definition 5.6. For $n \in \mathbf{Z}_{>0}$, *cyclotonomic polynomial* $\Phi_n(x)$ is defined as follows:

$$\Phi_n(x) := \prod_{\substack{1 \leq k \leq n-1 \\ \gcd(k,n)=1}} \left(x - \exp\left(\frac{2\pi i k}{n} \right) \right).$$

Note that $\Phi_n(x) \in \mathbf{Z}[x]$ for all n.

Now, the following proposition is the key of this chapter.

Proposition 5.7 (See e.g. [40])**.** *If all of the roots of a monic polynomial with rational coefficients $f(x)$ are roots of unity, then $f(x) \in \mathbf{Z}[x]$.*

Here, *monic* means "the nonzero coefficient of highest degree is equal to 1," and *root of unity* means a complex number z which satisfies

$$z^n = 1$$

for some $n \in \mathbf{Z}_{>0}$. This proposition is a consequence of the fact that the minimal polynomial over $\mathbf{Q}$ of any root of unity is cyclotonomic polynomial, in particular, integer coefficient polynomial.

In this thesis, this proposition is used as follows. First, note that for any square matrix A and positive integer n, if λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n. Therefore, if A has a complex number which is not root of unity as its eigenvalue, then every A^n has a complex number which is not equal to 1 as its eigenvalue. This implies that A^n is not the identity matrix.

5.2 Hadmard walks

Hadamard walks are a well-studied class of quantum walks. There are two types of Hadamard walks: M type and F type. They are characterized by local coin operators as usual. The definition of these walks by coin operators are as follows:

Definition 5.8. *An M-type Hadamard walk* and *an F-type Hadamard walk* on C_N is an quantum walk whose local coin operators are as follows respectively:

$$A^{H,M} = \frac{1}{\sqrt{2}} \begin{array}{cc} & \begin{array}{cc} \langle\leftarrow| & \langle\rightarrow| \end{array} \\ \begin{array}{c} \langle\leftarrow| \\ \langle\rightarrow| \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{array}, \qquad A^{H,F} = \frac{1}{\sqrt{2}} \begin{array}{cc} & \begin{array}{cc} \langle\leftarrow| & \langle\rightarrow| \end{array} \\ \begin{array}{c} \langle\leftarrow| \\ \langle\rightarrow| \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{array},$$

In this thesis, we focus on Hadamard walks of F type. $U_N^{H,F}$ denotes the time-evolution operator of an F-type Hadamard walk on C_N. With respect to the ordered basis of $\mathcal{H}_P \otimes \mathcal{H}_C$ $(\langle 0| \otimes \langle\leftarrow|, \langle 0| \otimes \langle\rightarrow|, \langle 1| \otimes \langle\leftarrow|, \ldots, \langle N-1| \otimes \langle\rightarrow|)$, the matrix representation of $U_N^{H,F}$ is as follows:

$$U_2^{H,F} = \begin{bmatrix} O & A^{H,F} \\ A^{H,F} & O \end{bmatrix}$$

and

$$U_N^{H,F} = \begin{bmatrix} O & L & O & \cdots & O & R \\ R & O & L & O & O & O \\ O & R & O & \ddots & O & O \\ \vdots & O & \ddots & \ddots & \ddots & O \\ O & O & O & \ddots & O & L \\ L & O & O & O & R & O \end{bmatrix}$$

for $N \geq 3$, where O represents the zero matrix and R and L are defined as follows:

$$L := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad R := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Note that L and R can be represented as

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A^{H,F} \qquad \text{and} \qquad R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A^{H,F}.$$

Furthermore, let $f_N^{H,F}$ be the characteristic polynomial of $U_N^{H,F}$, that is,

$$f_N^{H,F}(x) := \det\left(xI_{2N} - U_N^{H,F}\right).$$

By definition, the factorization of $f_N^{H,F}(x)$ is obtained as follows.

Proposition 5.9. *For $N \geq 2$, it holds that*

$$f_N^{H,F}(x) = \prod_{k=0}^{N-1} (x^2 - \sqrt{2}\cos(2\pi k/N)\, x + 1).$$

5.2.1 Periods of F-type Hadamard walks

The periods of M-type Hadamard walks are already known by Dukes and Konno et al. [41, 42] The result is as follows:

Theorem 5.10. *Let $T_N^{H,M}$ be the period of M-type Hadamard walk on $C_N (N \geq 2)$. Then the following holds:*

$$T_N^{H,M} = \begin{cases} 2, & (N = 2), \\ 8, & (N = 4), \\ 24, & (N = 8), \\ \infty, & (otherwise). \end{cases}$$

In this thesis, we point out that the same approach as for M type is effective for F type and give the proof below. Now, let $T_N^{H,F}$ be the period of F-type Hadamard walk on C_N $(N \geq 2)$.

Theorem 5.11. *The periods of F-type Hadamard walk on C_N $(N \geq 2)$ are as follows:*

$$T_N^{H,F} = \begin{cases} 8, & (N = 2), \\ 8, & (N = 4), \\ 24, & (N = 8), \\ \infty, & (otherwise). \end{cases}$$

Proof. First, by Proposition 5.9, we know that

$$f_2^{H,F}(x) = \Phi_8(x), \tag{5.5}$$
$$f_4^{H,F}(x) = \Phi_4(x)^2\Phi_8(x), \tag{5.6}$$
$$f_8^{H,F}(x) = \Phi_3(x)^2\Phi_4(x)^2\Phi_6(2)^2\Phi_8(x) \tag{5.7}$$

hold. Thus, $T_2^{H,F} = 8$, $T_4^{H,F} = 8$, and $T_8^{H,F} = 24$ hold.

Then, if N has odd prime as its factor, the same argument in [42] can be applied. Therefore, the remainder is the case where N is a power of 2 which is greater than 2^4.

Let $N = 2^n$. First, we consider the case where $n = 4$. We know

$$f_{2^4}^{H,F}(x) = x^{32} + 8x^{30} + 34x^{28} + 100x^{26} + \frac{901}{4}x^{24} + 409x^{22} + \frac{2465}{4}x^{20} + \frac{1567}{2}x^{18} + 848x^{16} + \frac{1567}{2}x^{14} + \frac{2465}{4}x^{12} + 409x^{10} + \frac{901}{4}x^8 + 100x^6 + 34x^4 + 8x^2 + 1$$

holds. Thus $f_{2^4}^{H,F}(x)$ is monic and in $\mathbf{Q}[x]$ but not in $\mathbf{Z}[x]$. Then, by Proposition 5.7, $f_{2^4}^{H,F}(x)$ have a root which is not a root of unity. Therefore, we get $T_{2^4}^{H,F} = \infty$.

For $n > 4$, $f^{H,F}_{2^n}(x)$ has $f^{H,F}_{2^4}(x)$ as a factor. More explicitly, by Proposition 5.9,

$$\begin{aligned}
f^{H,F}_{2^n}(x) &= \prod_{k=0}^{2^n-1} (x^2 - \sqrt{2}\cos(2\pi k/2^n)\, x + 1) \\
&= \prod_{\substack{0\le k\le 2^n-1 \\ 2^{n-4}|k}} (x^2 - \sqrt{2}\cos(2\pi k/2^n)\, x + 1) \prod_{\substack{0\le k\le 2^n-1 \\ 2^{n-4}\nmid k}} (x^2 - \sqrt{2}\cos(2\pi k/2^n)\, x + 1) \\
&= \prod_{l=0}^{2^4-1} (x^2 - \sqrt{2}\cos\bigl(2\pi \cdot 2^{n-4} l/2^n\bigr)\, x + 1) \prod_{\substack{0\le k\le 2^n-1 \\ 2^{n-4}\nmid k}} (x^2 - \sqrt{2}\cos(2\pi k/2^n)\, x + 1) \\
&= f^{H,F}_{2^4}(x) \prod_{\substack{0\le k\le 2^n-1 \\ 2^{n-4}\nmid k}} (x^2 - \sqrt{2}\cos(2\pi k/2^n)\, x + 1).
\end{aligned}$$

Therefore, also $f^{H,F}_{2^n}(x)$ has a root which is not a root of unity, and then, we know that $T^{H,F}_{2^n} = \infty$ holds. $\square$

5.2.2 Absolute zeta functions of zeta functions of Hadamard walks

Denote the zata functions of M-type and F-type Hadamard walk on C_N by $\zeta^{H,M}_{C_N}, \zeta^{H,F}_{C_N}$ respectively ($N \geq 2$). From the discussion so far, we can conclude the following about these functions. Especially, we can get an explicit expression of absolute zeta functions of these zeta functions.

Theorem 5.12. *We have the following explicit expression of absolute zeta functions and their functional equation.*

M type: $N = 2$

$$\begin{aligned}
Z_{\zeta^{H,M}_{C_2}}(w, s) &= \zeta_2\left(w, s+4, (2,2)\right), \\
\zeta_{\zeta^{H,M}_{C_2}}(s) &= \Gamma_2\left(s+4, (2,2)\right), \\
\zeta_{\zeta^{H,M}_{C_2}}(-4-s) &= S_2\left(s+4, (2,2)\right)\ \zeta_{\zeta^{H,M}_{C_2}}(s).
\end{aligned}$$

$N = 4$

$$\begin{aligned}
Z_{\zeta^{H,M}_{C_4}}(w, s) &= \sum_{I\subset\{1\}} (-1)^{|I|}\ \zeta_3\left(w, s+8+4|I|), (2,2,8)\right), \\
\zeta_{\zeta^{H,M}_{C_4}}(s) &= \prod_{I\subset\{1\}} \Gamma_3\left(s+8+4|I|, (2,2,8)\right)^{(-1)^{|I|}}, \\
\zeta_{\zeta^{H,M}_{C_4}}(-8-s) &= \Bigl(\prod_{I\subset\{1\}} S_3\left(s+8+4|I|, (2,2,8)\right)^{(-1)^{|I|}}\Bigr)\zeta_{\zeta^{H,M}_{C_4}}(s).
\end{aligned}$$

$N=8$

$$Z_{\zeta_{C_8}^{H,M}}(w,s)=\sum_{I\subset\{1,\dots,5\}}(-1)^{|I|}\ \zeta_7\left(w,s+16+m(I),(2,2,2,2,8,12,12)\right),$$
$$\zeta_{\zeta_{C_8}^{H,M}}(s)=\prod_{I\subset\{1,\dots,5\}}\Gamma_7\left(s+16+m(I),(2,2,2,2,8,12,12)\right)^{(-1)^{|I|}},$$
$$\zeta_{\zeta_{C_8}^{H,M}}(-16-s)=\Big(\prod_{I\subset\{1,\dots,5\}}S_7\left(s+16+m(I),(2,2,2,2,8,12,12)\right)^{(-1)^{|I|}}\Big)\zeta_{\zeta_{C_8}^{H,M}}(s).$$

F type: $N=2$

$$Z_{\zeta_{C_2}^{H,F}}(w,s)=\zeta_1\left(w,s+4,(8)\right)-\zeta_1\left(w,s+8,(8)\right),$$
$$\zeta_{\zeta_{C_2}^{H,F}}(s)=\frac{\Gamma\left(\frac{s+4}{8}\right)}{\Gamma\left(\frac{s+8}{8}\right)}\cdot n^{-\frac{1}{2}},$$
$$\zeta_{\zeta_{C_2}^{H,F}}(-4-s)=-\cot\left(\frac{s\pi}{8}\right)\zeta_{\zeta_{C_2}^{H,F}}(s).$$

$N=4$

$$Z_{\zeta_{C_4}^{H,F}}(w,s)=\sum_{I\subset\{1,2\}}(-1)^{|I|}\ \zeta_2\left(w,s+8+2|I|),(4,8)\right),$$
$$\zeta_{\zeta_{C_4}^{H,F}}(s)=\prod_{I\subset\{1,2\}}\Gamma_2\left(s+8+2|I|,(4,8)\right)^{(-1)^{|I|}},$$
$$\zeta_{\zeta_{C_4}^{H,F}}(-8-s)=\Big(\prod_{I\subset\{1,2\}}S_2\left(s+8+2|I|,(4,8)\right)^{(-1)^{|I|}}\Big)\zeta_{\zeta_{C_4}^{H,F}}(s).$$

$N=8$

$$Z_{\zeta_{C_8}^{H,F}}(w,s)=\sum_{I\subset\{1,\dots,4\}}(-1)^{|I|}\ \zeta_4\left(w,s+16+m(I),(4,6,6,8)\right),$$
$$\zeta_{\zeta_{C_8}^{H,F}}(s)=\prod_{I\subset\{1,\dots,4\}}\Gamma_4\left(s+16+m(I),(4,6,6,8)\right)^{(-1)^{|I|}},$$
$$\zeta_{\zeta_{C_8}^{H,F}}(-16-s)=\prod_{I\subset\{1,\dots,4\}}S_4\left(s+16+m(I),(4,6,6,8)\right)^{(-1)^{|I|}}\ \zeta_{\zeta_{C_8}^{H,F}(s)}.$$

Proof. The proofs are almost the same for each $\zeta_{C_N}^{H,M}$ and $\zeta_{C_N}^{H,F}$. Thus we check only for $\zeta_{C_4}^{H,M}$. First, by the definition,

$$\zeta_{C_4}^{H,M}(u)=\det\Big(I_8-uU_4^{H,M}\Big)^{-1}.$$

Now, by Proposition 5.9, we know

$$\det\Big(xI_8-U_4^{H,M}\Big)=\frac{(x^2-1)^2(x^8-1)}{x^4-1}$$

By substituting $x = 1/u$, we get

$$\det\Big(\frac{1}{u}I_8 - U_4^{H,M}\Big) = \frac{((1/u)^2-1)^2((1/u)^8-1)}{(1/u)^4-1},$$
$$\det\Big(I_8 - uU_4^{H,M}\Big) = \frac{(u^2-1)^2(u^8-1)}{u^4-1}.$$

Therefore, we conclude

$$\zeta_{C_4}^{H,M}(u) = \frac{u^4-1}{(u^2-1)^2(u^8-1)}.$$

Here, we know that $\zeta_{C_4}^{H,M}$ is an absolute automorphic form of weight -8. Then, by Theorem 5.1, we obtain the desired result.

Note that the explicit forms of the zeta functions are as follows:

$$\begin{aligned}
\zeta_{C_2}^{H,M}(u) &= \frac{1}{(1-u^2)^2},\\
\zeta_{C_4}^{H,M}(u) &= \frac{u^4-1}{(u^2-1)^2(u^8-1)},\\
\zeta_{C_8}^{H,M}(u) &= \frac{(u^4-1)^3(u^6-1)^2}{(u^2-1)^4(u^8-1)(u^{12}-1)^2},\\
\zeta_{C_2}^{H,F}(u) &= \frac{u^4-1}{u^8-1},\\
\zeta_{C_4}^{H,F}(u) &= \frac{(u^2-1)^2}{(u^4-1)(u^8-1)},\\
\zeta_{C_8}^{H,F}(u) &= \frac{(u^2-1)^4}{(u^4-1)(u^6-1)^2(u^8-1)}.
\end{aligned}$$

Moreover, all of them are absolute automorphic forms of weight -4, -8, -16, -4, -8, and -16 respectively. □

5.3 Grover walks

Grover walks are another well-studied class of quantum walks. There are two types of Grover walks as in the case of Hadamard walks: M type and F type. They are characterized by local coin operators as usual. In this thesis we focus on the 3-states model. The definition of these walks by local coin operators are as follows:

Definition 5.13. *An M-type Grover walk with* 3 *states* and *an F-type Grover walk with* 3 *states* on C_N are quantum walks whose coin operators are as follows respectively:

$$A^{G_3,M} = \frac{1}{3}\begin{array}{c} \\ \langle\leftarrow| \\ \langle\cdot| \\ \langle\rightarrow| \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \langle\leftarrow| & \langle\cdot| & \langle\rightarrow| \end{array} \\ \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \end{array}, \qquad A^{G_3,F} = \frac{1}{3}\begin{array}{c} \\ \langle\leftarrow| \\ \langle\cdot| \\ \langle\rightarrow| \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \langle\leftarrow| & \langle\cdot| & \langle\rightarrow| \end{array} \\ \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \end{array}.$$

$U_N^{G_3,M}$ and $U_N^{G_3,F}$ denote the time-evolution opeartors of M-type and F-type Grover walk on C_N, respectively. With respect to the ordered basis of $\mathcal{H}$ ($\langle 0| \otimes \langle \leftarrow|$, $\langle 0| \otimes \langle \cdot|$, $\langle 0| \otimes \langle \rightarrow|$, $\langle 1| \otimes \langle \leftarrow|, \ldots, \langle N-1| \otimes \langle \rightarrow|$), the matrix representation of $U_N^{G_3,M}$ and $U_N^{G_3,F}$ is as follows:

$$U_2^{G_3,X} = \begin{bmatrix} S & L^{(X)} + R^{(X)} \\ L^{(X)} + R^{(X)} & S \end{bmatrix}, \qquad X = M, F$$

and

$$U_N^{G_3,X} = \begin{bmatrix} S & L^{(X)} & O & \cdots & O & R^{(X)} \\ R^{(X)} & S & L^{(X)} & O & O & O \\ O & R^{(X)} & S & \ddots & O & O \\ \vdots & O & \ddots & \ddots & \ddots & O \\ O & O & O & \ddots & S & L^{(X)} \\ L^{(X)} & O & O & O & R^{(X)} & S \end{bmatrix}, \qquad X = M, F$$

for $N \geq 3$, where O represents the zero matrix and $S, L^{(X)}$, and $R^{(X)}$ are defined as follows:

$$S := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} A^{G_3,M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} A^{G_3,F},$$

$$L^{(X)} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A^{G_3,X}, \qquad R^{(X)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{G_3,X}, \qquad X = M, F.$$

Furthermore, let $f_N^{G_3,M}$ and $f_N^{G_3,F}$ be the characteristic polynomials of $U_N^{G_3,M}$ and $U_N^{G_3,F}$, that is,

$$f_N^{G_3,M}(x) := \det\Big(xI_{3N} - U_N^{G_3,M}\Big), \qquad f_N^{G_3,F}(x) := \det\Big(xI_{3N} - U_N^{G_3,F}\Big).$$

Here, the following factorization is known. See [43] for example.

Proposition 5.14. *For $N \geq 2$, it holds that*

$$f_N^{G_3,M}(x) = (x-1)^N \prod_{k=0}^{N-1} \left(x^2 + \frac{2}{3}\left(2 + \cos\left(\frac{2\pi k}{N}\right)\right) x + 1 \right),$$

$$f_N^{G_3,F}(x) = (x-1)^N \prod_{k=0}^{N-1} \left(x^2 - \frac{2}{3}\left(2 + \cos\left(\frac{2\pi k}{N}\right)\right) x + 1 \right).$$

5.3.1 Periods of Grover walks with 3 states

The periods of Grover walks with 3 states ware clarified by Kajiwara et al. in [44] as follows:

Theorem 5.15. *Let $T_N^{G_3,M}$ be the period of M-type Grover walk with 3 states on C_N ($N \geq 2$). Then*

$$T_N^{G_3,M} = \begin{cases} 6, & (N=3), \\ \infty, & (otherwise) \end{cases}$$

holds.

Furthermore, Let $T_N^{G_3,F}$ be the period of F-type Grover walk with 3 states on C_N ($N \geq 2$). Then

$$T_N^{G_3,F} = \begin{cases} 4, & (N=3), \\ \infty, & (otherwise) \end{cases}$$

holds.

In this thesis, we point out that the same approach as in the proof in the previous subsection works for this theorem. The proof of this approach is given below.

Proof. First, we show it for M type.

For $N = 3$, we know it holds that

$$f_3^{G_3,M}(x) = \Phi_1(x)^3\Phi_2(x)^2\Phi_3(x)^2$$

by Proposition 5.14. The result is a direct consequence of this factorization.

For $N = 2$, by Proposition 5.14 again, we know it holds that

$$f_2^{G_3,M}(x) = x^6 + \frac{2}{3}x^5 - x^4 - \frac{4}{3} - x^2 + \frac{2}{3} + 1.$$

Thus, $f_2^{G_3,M}(x)$ is monic and in $\mathbf{Q}[x]$ but not in $\mathbf{Z}[x]$. Therefore, by Proposition 5.7, $f_N^{G_3,M}(x)$ has a root which is not a root of unity, and this implies $T_2^{G_3,M} = \infty$ holds.

For the case where $N \geq 4$ is not a multiple of 3, we know it holds that

$$f_N^{G_3,M}(x) = \prod_{k=0}^{N-1}\left(x^3 + \frac{1}{3}\left(1 + 2\cos\left(\frac{2\pi k}{N}\right)\right)x^2 - \frac{1}{3}\left(1 + 2\cos\left(\frac{2\pi k}{N}\right)\right)x - 1\right)$$

by Proposition 5.14 again. Then the coefficient of x^{3N-1} can be calculated as follows:

$$\begin{aligned}\sum_{k=0}^{N-1}\frac{1}{3}\left(1 + 2\cos\left(\frac{2\pi k}{N}\right)\right) &= \frac{N}{3} + \sum_{k=0}^{N-1}\cos\left(\frac{2\pi k}{N}\right) \\ &= \frac{N}{3}.\end{aligned}$$

Therefore, $f_N^{G_3,M}(x)$ is not in $\mathbf{Z}[x]$, and $f_N^{G_3,M}(x)$ is monic and in $\mathbf{Q}[x]$ by definition. Thus, by Proposition 5.7, we know $T_N^{G_3,M} = \infty$.

For the case where $N \geq 4$ is a multiple of 3 but is not a power of 3, we can take a prime factor p of N such that $p \neq 3$. Then like the above-mentioned proof in the previous subsection, we can show that $f_N^{G_3,M}(x)$ has $f_p^{G_3,M}(x)$ as its factor. Here, we already know that $f_p^{G_3,M}(x)$ has a root which is not a root of unity. Thus, also $f_N^{G_3,M}(x)$ has a root which is not a root of unity, and we get $T_N^{G_3,M} = \infty$.

Finally, for the case where $N \geq 4$ is a power of 3, it is enough to examine $f_9^{G_3,M}(x)$ because $f_N^{G_3,M}(x)$ has $f_9^{G_3,M}(x)$ as its factor. By Proposition 5.14, we have

$$\begin{aligned}
f_9^{G,M}(x) = x^{27} &+ 3x^{26} - \frac{128}{9}x^{24} - \frac{214}{9}x^{23} + \frac{62}{9}x^{22} + \frac{5752}{81}x^{21} + \frac{6376}{81}x^{20} \\
&- \frac{3331}{81}x^{19} - \frac{15059}{81}x^{18} - \frac{11686}{81}x^{17} + \frac{8728}{81}x^{16} + \frac{23752}{81}x^{15} + \frac{4316}{27}x^{14} \\
&- \frac{4316}{27}x^{13} - \frac{23752}{81}x^{12} - \frac{8728}{81}x^{11} + \frac{11686}{81}x^{10} + \frac{15059}{81}x^{9} + \frac{3331}{81}x^{8} \\
&- \frac{6376}{81}x^{7} - \frac{5752}{81}x^{6} - \frac{62}{9}x^{5} + \frac{214}{9}x^{4} + \frac{128}{9}x^{3} - 3x - 1
\end{aligned}$$

Thus, $f_9^{G_3,M}(x)$ is monic and in $\mathbf{Q}[x]$ but not in $\mathbf{Z}[x]$. Therefore, by Proposition 5.7, $f_N^{G_3,M}(x)$ has a root which is not a root of unity, and this implies $T_N^{G_3,M} = \infty$.

Then the same method can be used to prove the case of F type. Note that we can obtain the following expansion of $f_9^{G_3,F}(x)$:

$$\begin{aligned}
f_9^{G_3,F}(x) = x^{27} &+ 3x^{26} + 3x^{25} + \frac{25}{9}x^{24} + \frac{26}{9}x^{23} - \frac{2}{9}x^{22} + \frac{46}{81}x^{21} \\
&+ \frac{106}{81}x^{20} - \frac{59}{27}x^{19} + \frac{125}{81}x^{18} - \frac{1}{27}x^{17} - \frac{353}{81}x^{16} - \frac{116}{81}x^{15} - \frac{212}{27}x^{14} \\
&- \frac{212}{27}x^{13} - \frac{116}{81}x^{12} - \frac{353}{81}x^{11} - \frac{1}{27}x^{10} + \frac{125}{81}x^{9} - \frac{59}{27}x^{8} \\
&+ \frac{106}{81}x^{7} + \frac{46}{81}x^{6} - \frac{2}{9}x^{5} + \frac{26}{9}x^{4} + \frac{25}{9}x^{3} + 3x^2 + 3x + 1.
\end{aligned}$$

$\square$

5.3.2 Absolute zeta functions of zeta functions of Grover walks with 3 states

Theorem 5.16. *We have the following explicit expression of absolute zeta functions and their functional equation.*

M type:

$$Z_{\zeta_{C_3}^{G_3,M}}(w,s) = -\sum_{I \subset \{1\}} (-1)^{|I|}\ \zeta_4\left(w, s+9+|I|, (2,2,3,3)\right),$$

$$\zeta_{\zeta_{C_3}^{G,M}}(s) = \prod_{I \subset \{1\}} \Gamma_4\left(s+9+|I|, (2,2,3,3)\right)^{(-1)^{|I|+1}},$$

$$\zeta_{\zeta_{C_3}^{G,M}}(-9-s)^{-1} = \left(\prod_{I \subset \{1\}} S_4\left(s+9+|I|, (2,2,3,3)\right)^{(-1)^{|I|+1}}\right)\zeta_{\zeta_{C_3}^{G,M}}(s).$$

F type:

$$
\begin{aligned}
Z_{\zeta_{C_3}^{G_3,F}}(w,s) &= -\sum_{I\subset\{1\}} (-1)^{|I|}\ \zeta_3\left(w, s+9+|I|, (2,4,4)\right),\\
\zeta_{\zeta_{C_3}^{G_3,F}}(s) &= \prod_{I\subset\{1\}} \Gamma_3\left(s+9+|I|, (2,4,4)\right)^{(-1)^{|I|+1}},\\
\zeta_{\zeta_{C_3}^{G_3,F}}(-9-s) &= \left(\prod_{I\subset\{1\}} S_3\left(s+9+|I|, (2,4,4)\right)^{(-1)^{|I|+1}}\right)\zeta_{\zeta_{C_3}^{G_3,F}}(s).
\end{aligned}
$$

Proof. The proof is almost the same as that of Theorem 5.12. Note that the explicit forms of the zeta functions are as follows:

$$
\begin{aligned}
\zeta_{C_3}^{G_3,M}(u) &= \frac{u-1}{(u^2-1)^2(u^3-1)^2},\\
\zeta_{C_3}^{G_3,F}(u) &= \frac{u-1}{(u^2-1)(u^4-1)^2}.
\end{aligned}
$$

Moreover, both of them are absolute automorphic forms of weight -9. $\square$

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