

論 説

Oscillations in an Extended Uzawa-Lucas Two-Sector Model with Different Exogenous Shocks

Wei-Bin Zhang*

Abstract

This study addresses issues related to economic fluctuations in a model proposed by Zhang (2014) on interactions between gender differences, economic growth and education with endogenous physical and human capital accumulation. Zhang's model is a synthesis of the Solow model (Solow, 1956) and the Uzawa-Lucas two sector growth model (Uzawa, 1965, Lucas, 1988) with Zhang's approach to household behavior. This paper generalizes Zhang's model by treating all the time-independent parameters to be time-dependent. We simulate the model to demonstrate existence of business cycles under different periodic shocks.

Key words: Economic Growth, Economic Cycles, Time Distribution, Human Capital, Exogenous Shocks

JEL Code: O41

1. Introduction

Business cycles are well recorded but not well theoretically explained economic phenomena. Indeed, there are a lot of theoretical studies about mechanisms and phenomena of economic fluctuations (e.g., Zhang, 1991, 2005, 2006; Lorenz, 1993; Flaschel *et al* 1997; Chiarella and Flaschel, 2000; Shone, 2002; Gandolfo, 2005; Puu, 2011). Different studies explain economic business cycles from different perspectives. Lucas (1977) demonstrates how some shocks affect all sectors in an economy. Chatterjee and Ravikumar (1992) construct a neoclassical growth model with seasonal perturbations to taste and technology. Gabaix (2011) tries to show that uncorrelated sectoral shocks are determinants of aggre-

*Ritsumeikan Asia Pacific University
Jumonjibaru, Beppu-Shi, Oita-ken, 874-8577 Japan
E-mail: wbz1@apu.ac.jp

gate fluctuations (see also, Stella, 2015). Although these studies show the existence of business cycles, regular oscillations, irregular fluctuations, and chaos in different economic models, many aspects of economic dynamics with short-run fluctuations and permanent cycles need to be examined. This study makes a contribution to the literature of economic growth and business cycles by re-examining behavior of the Uzawa-Lucas model.

This paper is to introduce exogenous shocks to the growth model proposed by Zhang (2014) on the basis of the Uzawa-Lucas model. Zhang's model deals with gender division of labor and economic growth with endogenous human capital. A main character of economic changes in the past one and half centuries has been the entry of women into the labor force. Bar and Leukhina (2011) observe that married females more than doubled their workforce participation in the last half a century (see also, Blau and Kahn, 2000; Stotsky, 2006; Croson and Gneezy, 2009; Biagetti and Sergio, 2009; and Eckstein and Lifshitz, 2011). Since Becker (1965) published his seminal work in 1965, there are many formal economic theories which explain why the changes in female labor participation take place and examine the factors which are significant determinants of the dynamics (Becker, 1985; Chiappori, 1992; Gomme *et al.* 2001; Campbell and Ludvigson, 2001; Gutierrez, 2003; Tassel, 2004; Fernández, 2007; and Trede and Heimann, 2011). It is commonly observed that education is a significant determinant of woman labor participation. There are many studies on gender differences in education and growth (Bandiera and Natraj, 2013). Inspired by the richness of empirical studies and influenced by different formal economic models, Zhang (2014) develops an integrated analytical framework to study endogenous labor supply and gender division of labor with endogenous human and physical capital accumulation. The model follows the neoclassical growth theory to describe physical capital accumulation. Human capital accumulation is modeled according to the approach by Uzawa (1965) and Lucas (1988). The Uzawa-Lucas model has been extended and generalized in various directions (Jones *et al.* 1993; Stokey and Rebelo, 1995; Mino, 1996, 2001; Alonso-Carrera and Freire-Seren, 2004; and De Hek, 2005). Zhang's model differs from most of the theoretical economic models with gender in that it integrates endogenous human capital, physical capital, and elastic labor supply of man and woman within a comprehensive framework. The main purpose of this study is to generalize Zhang's model by allowing all the time-dependent parameters to be time-dependent. The paper is organized as follows. Section 2 introduces the basic model with wealth accumulation and human capital accumulation. Section 3 simulates the model. Section 4 carries out comparative dynamic analysis with regard to some exogenous shocks. Section 5 concludes the study.

2. The basic model

Following the model by Zhang (2014), we consider that the economy consists of one production sector and one education sector. The economy produces one (durable) good. The production sector basically follows the standard one-sector growth model (Burmeister and Dobell, 1970; Azariadis, 1993; and Barro and Sala-i-Martin, 1995). Saving is undertaken only by households. Assets are owned by households. The production sector uses labor and capital inputs to produce goods. Exchanges take place in perfectly competitive markets. Factor markets work well and factors are fully utilized at every moment. We select the commodity to serve as numeraire, with all the other prices being measured relative to its price. Women and men are different in human capital and preference. The population of each gender is homogeneous. We assume that each family consists of husband and wife. Let subscripts $q=1$ and $q=2$ stand for man and woman respectively. As all the families are identical, the family structure is invariant over time under these assumptions. It should be noted that a similar population structure is accepted by Albanesi and Olivetti (2009). We use $\bar{N}(t)$ to stand for the number of families. We use $T_q(t)$ to stand for the work time of gender q . The flow of labor services used at time t for production $N(t)$ is given by

$$N(t) = [H_1^{\theta_1(t)}(t)T_1(t) + H_2^{\theta_2(t)}(t)T_2(t)]\bar{N}(t),$$

where $H_q(t)$ is the level of human capital of gender q and $\theta_q(t)$ is gender q 's human capital utilization efficiency parameter. We call $H_q^{\theta_q(t)}(t)$ gender q 's level of effective human capital. Wage rates are assumed equal for the same gender in the two sectors. The total stock of physical capital $K(t)$ is fully employed. Let $N_e(t)$ and $K_e(t)$ stand for, respectively, the labor force and capital stocks employed by the education sector, and $N_i(t)$ and $K_i(t)$ for the labor force and capital stocks employed by the production sector. The assumption of full employment of production factors implies

$$K_i(t) + K_e(t) = K(t), N_i(t) + N_e(t) = N(t).$$

We rewrite the above relations

$$n_i(t)k_i(t) + n_e(t)k_e(t) = k(t), n_i(t) + n_e(t) = 1, \tag{1}$$

where

$$k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, n_j(t) \equiv \frac{N_j(t)}{N(t)}, k(t) \equiv \frac{K(t)}{N(t)}, j=i, e.$$

The production sector

We use $F_i(t)$ to present the output level of the production sector at time t . The production function is taken on the following form

$$F_i(t) = A_i(t) K_i^{\alpha_i(t)}(t) N_i^{\beta_i(t)}(t), A_i(t), \alpha_i(t), \beta_i(t) > 0, \alpha_i(t) + \beta_i(t) = 1, \quad (2)$$

where $A_i(t)$ is the total productivity of the production sector, and $\alpha_i(t)$ and $\beta_i(t)$ are respectively the output elasticities of capital and qualified labor input. The rate of interest, $r(t)$, and wage rate per unit of work time, $w_q(t)$, are determined by markets. The marginal conditions are

$$r(t) + \delta_k(t) = \frac{\alpha_i(t) F_i(t)}{K_i(t)}, w_q(t) = \frac{\beta_i(t) H_q^{\theta_q(t)}(t) F_i(t)}{N_i(t)}, q=1, 2, \quad (3)$$

where $\delta_k(t)$ is the depreciation rate of physical capital.

The education sector

The education sector is perfect competitive and all the students pay the same fee for per unit of study time. Let $p(t)$ stand for the education fee per unit of time. The education sector pays teachers and capital with the market rates. The production function of the education sector is specified as follows

$$F_e(t) = A_e(t) K_e^{\alpha_e(t)}(t) N_e^{\beta_e(t)}(t), \alpha_e(t), \beta_e(t) > 0, \alpha_e(t) + \beta_e(t) = 1, \quad (4)$$

where $A_e(t)$, $\alpha_e(t)$ and $\beta_e(t)$ are positive parameters. The marginal conditions for the education sector are

$$r(t) + \delta_k(t) = \frac{\alpha_e(t) p(t) F_e(t)}{K_e(t)}, w_q(t) = \frac{\beta_e(t) p(t) H_q^{\theta_q(t)}(t) F_e(t)}{N_e(t)}, q=1, 2. \quad (5)$$

Behavior of consumers

This study uses the approach to consumer behaviour proposed by Zhang (1993). The households make choice of consumption levels of services and commodities, education time and leisure time as well as on amount of saving. We use $\bar{k}(t)$ to stand for wealth per household, i.e., $\bar{k}(t) = K(t)/\bar{N}$. The per capita current income $y(t)$ from the interest and wage payments is

$$y(t) = r(t) \bar{k}(t) + w_1(t) T_1(t) + w_2(t) T_2(t).$$

The per capita disposable income of the household is defined as the sum of the current income and the wealth available for purchasing consumption goods and saving: $\hat{y}(t) = y(t) + \bar{k}(t)$. The household distributes the total available budget between saving

$s(t)$, consuming good $c(t)$, and receiving education $\tilde{T}_1(t)$, and $\tilde{T}_2(t)$. The budget constraint is given by

$$c(t) + s(t) + p(t)\tilde{T}_1(t) + p(t)\tilde{T}_2(t) = \hat{y}(t). \quad (6)$$

Let $\bar{T}_q(t)$ stand for the leisure time of gender q at time t . The time constraint for each person is

$$T_q(t) + \tilde{T}_q(t) + \bar{T}_q(t) = T_0,$$

where T_0 is the total available time for work, education and leisure. Insert the time constraints in (6)

$$c(t) + s(t) + \bar{p}_1(t)\tilde{T}_1(t) + \bar{p}_2(t)\tilde{T}_2(t) + w_1(t)\bar{T}_1(t) + w_2(t)\bar{T}_2(t) = \bar{y}(t), \quad (7)$$

where we use $\hat{y}(t) = y(t) + k(t)$, and

$$\bar{p}_q(t) \equiv p(t) + w_q(t), \quad \bar{y}(t) \equiv (r(t) + 1)k(t) + w_1(t)T_0 + w_2(t)T_0.$$

The utility level $U(t)$ is taken on the following form

$$U(t) = u(t) \bar{T}_1^{\sigma_{01}(t)}(t) \bar{T}_2^{\sigma_{02}(t)}(t) \tilde{T}_1^{\eta_{01}(t)}(t) \tilde{T}_2^{\eta_{02}(t)}(t) c^{\xi_0(t)}(t) s^{\lambda_0(t)}(t), \quad (8)$$

$$\sigma_{0q}(t), \eta_{0q}(t), \xi_0(t), \lambda_0(t) > 0,$$

where u is a time-dependent variable, $\sigma_{0q}(t)$ and $\eta_{0q}(t)$ are called respectively gender q 's propensities to use leisure time and to receive education, and $\xi_0(t)$ and $\lambda_0(t)$ respectively the family's propensities to consume good and to hold wealth. Maximizing (8) subject to (7) yields

$$w_q(t)\bar{T}_q(t) = \sigma_q(t)\bar{y}(t), \quad \bar{p}_q(t)\tilde{T}_q(t) = \eta_q(t)\bar{y}(t), \quad c(t) = \xi(t)\bar{y}(t), \quad s(t) = \lambda(t)\bar{y}(t), \quad (9)$$

where

$$\rho(t) \equiv \frac{1}{\sigma_{01}(t) + \sigma_{02}(t) + \eta_{01}(t) + \eta_{02}(t) + \xi_0(t) + \lambda_0(t)},$$

$$\sigma_q(t) \equiv \rho(t)\sigma_{0q}(t), \quad \eta_q(t) \equiv \rho(t)\eta_{0q}(t), \quad \xi(t) \equiv \rho(t)\xi_0(t), \quad \lambda(t) \equiv \rho(t)\lambda_0(t).$$

Accumulation of human capital

Following Zhang (2014), we describe human capital accumulation

$$\dot{H}_q(t) = \frac{\nu_{qe}(t)(F_e(t)/2\bar{N}(t))^{a_{qe}(t)}(H_q^{\theta_q}(t)\tilde{T}_q(t))^{b_{qe}(t)}}{H_q^{\pi_{qe}(t)}(t)} - \delta_{qh}(t)H_q(t), \quad q=1, 2, \quad (10)$$

where $\delta_{qh}(t)$ (>0) is the depreciation rate of human capital, $\nu_{qe}(t)$, $a_{qe}(t)$, and $b_{qe}(t)$ are non-negative parameters. Human capital tends to increase with an increase in the level of education service, $F_{qe}(t)/2\bar{N}(t)$, and in the (qualified) study time, $H_q^{g_q(t)}(t)T_{qe}(t)$. The term $H^{\pi_{qe}(t)}(t)$ implies that as the level of human capital of the population increases, it may be more difficult (in the case of $\pi_{qe}(t)$ being large) or easier (in the case of $\pi_{qe}(t)$ being small) to accumulate more human capital through formal education. We will simulate the model when returns to scale are not strong.

Wealth accumulation

We now find dynamics of capital accumulation. According to the definition of $s(t)$, the change in the household's wealth is given by

$$\dot{\bar{k}}(t) = s(t)\bar{k}(t) - \frac{\dot{\bar{N}}(t)\bar{k}(t)}{\bar{N}(t)}. \quad (11)$$

Balance of demand and supply

For the education sector, the demand for and supply of education balance at any point of time

$$\tilde{T}_1(t)\bar{N}(t) + \tilde{T}_2(t)\bar{N}(t) = F_e(t). \quad (12)$$

As output of the production sector is equal to the sum of the level of consumption, the depreciation of capital stock and the net savings, we have

$$C(t) + S(t) - K(t) + \delta_k(t)K(t) = F_i(t), \quad (13)$$

where $C(t)$ is the total consumption, $S(t) - K(t) + \delta_k(t)K(t)$ is the sum of the net saving and depreciation and

$$C(t) = c(t)\bar{N}(t), S(t) = s(t)\bar{N}(t). \quad (14)$$

We have thus built the dynamic model. We now examine dynamics of the model.

3. The dynamics and its properties

This section examines dynamics of the model. First, we show that the dynamics can be expressed by the three differential equations system with $k_i(t)$, $H_1(t)$, $H_2(t)$, and t as the variables.

Lemma

The dynamics of the economic system is given by the three-dimensional differential equations

$$\begin{aligned}\dot{H}_q &= \Omega_q(k_i, H_1, H_2, t), \quad q=1, 2, \\ \dot{k}_i &= \Omega(k_i, H, H_2, t),\end{aligned}\tag{15}$$

where $\Omega_q(k_i, H_1, H_2, t)$, and $\Omega(k_i, H_1, H_2, t)$ are functions of $k_i(t)$, $H_1(t)$, $H_2(t)$, and t defined in the Appendix. Moreover, all the other variables are determined as functions of $k_i(t)$, $H_1(t)$, $H_2(t)$, and t at any point of time by the following procedure: $k(t)$ by (A17) $\rightarrow n_i(t)$ and $n_h(t)$ by (A3) $\rightarrow r(t)$ and $w_q(t)$ by (A4) $\rightarrow k_e(t)$ by (A1) $\rightarrow p(t)$ by (A2) $\rightarrow T_1(t)$ and $T_2(t)$ by (A3) $\rightarrow N(t)$ by its definition $\rightarrow \bar{k}(t) = k(t)N(t)/\bar{N} \rightarrow K(t) = k(t)N(t) \rightarrow N_j(t) = n_j(t)N(t)$, $j=i, e \rightarrow K_j(t) = k_j(t)N_j(t) \rightarrow F_j(K_j(t), N_j(t)) \rightarrow \bar{y}(t)$ by (A6) $\rightarrow \bar{T}_q(t)$, $\tilde{T}_q(t)$, $c(t)$, $s(t)$ by (9).

The system (15) contains the following variables: $k_i(t)$, $H_1(t)$, $H_2(t)$, and t . We omit analyzing the model as the expressions are too complicated. We simulate the model to illustrate behavior of the system. In the remainder of this study, we specify the depreciation rates by $\delta_k=0.05$, $\delta_{jh}=0.04$, and let $T_0=1$. Following Zhang (2014), we specify the other parameters as follows

$$\begin{aligned}N_0 &= 20, A_i = 1.2, A_e = 0.9, \theta_1 = 0.6, \theta_2 = 0.55, \lambda_0 = 0.65, \xi_0 = 0.08, \\ \eta_{01} &= 0.012, \eta_{02} = 0.01, \sigma_{01} = 0.18, \sigma_{02} = 0.22, \alpha_i = 0.35, \alpha_e = 0.3, \alpha_{qe} = 0.3, \\ b_{qe} &= 0.5, \pi_{qe} = 0.2, v_{1e} = 0.8, v_{2e} = 0.75.\end{aligned}$$

We specify the initial conditions

$$k_i(0) = 16, H_1(0) = 1.2, H_2(0) = 1.$$

The simulation result is plotted in Figure 1. In Figure 1 the total national output is given by

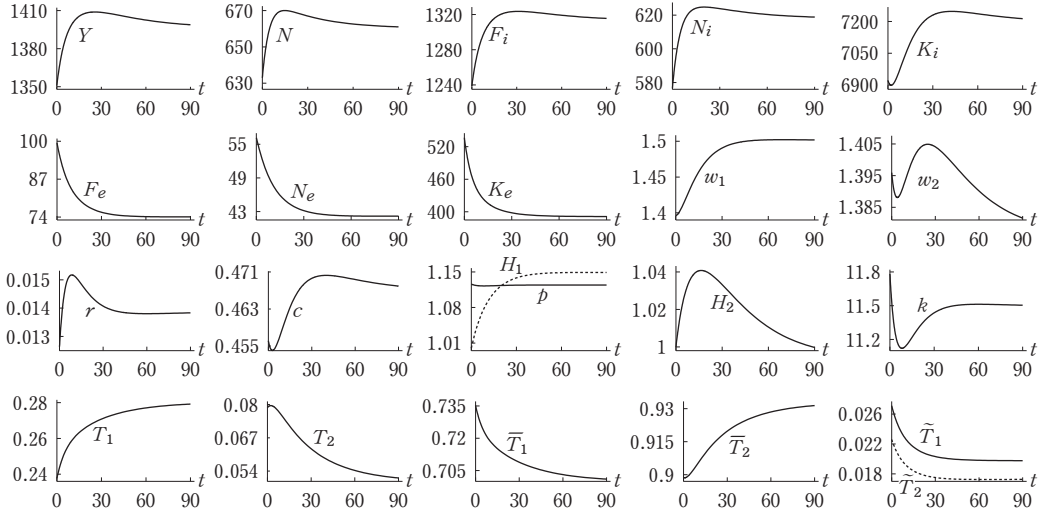
$$Y(t) = F_i(t) + p(t)F_e(t).$$

We calculate the equilibrium values of the variables as follows

$$\begin{aligned}N &= 6.70, H_1 = 1.14, H_2 = 0.99, N_i = 6.17, N_e = 0.53, K_i = 109.61, \\ K_e &= 7.49, F_i = 20.27, F_e = 1.06, k = 17.48, k_i = 17.77, k_e = 14.14, p = 1.53, \\ w_1 &= 2.31, w_2 = 2.12, \bar{T}_1 = 0.70, \bar{T}_2 = 0.93, \tilde{T}_1 = 0.03, \tilde{T}_2 = 0.02, T_1 = 0.27, \\ T_2 &= 0.04, \bar{k} = 5.86, c = 0.72.\end{aligned}$$

It is straightforward to calculate the three eigenvalues

Figure 1. The motion of the economic system



$-0.201, -0.05, -0.03.$

As the three eigenvalues are negative, the unique equilibrium is locally stable.

4. Comparative dynamics analyses

The previous section summarizes the results in Zhang (2014). We may interpret those parameter values as the long-term trends. We are interested in what happen to the economic system when some parameters experience time-dependent changes. As section 2 generalized Zhang’s model by treating all the parameters time-dependent, we can carry out comparative dynamic analysis by following the computational procedure given by the lemma.

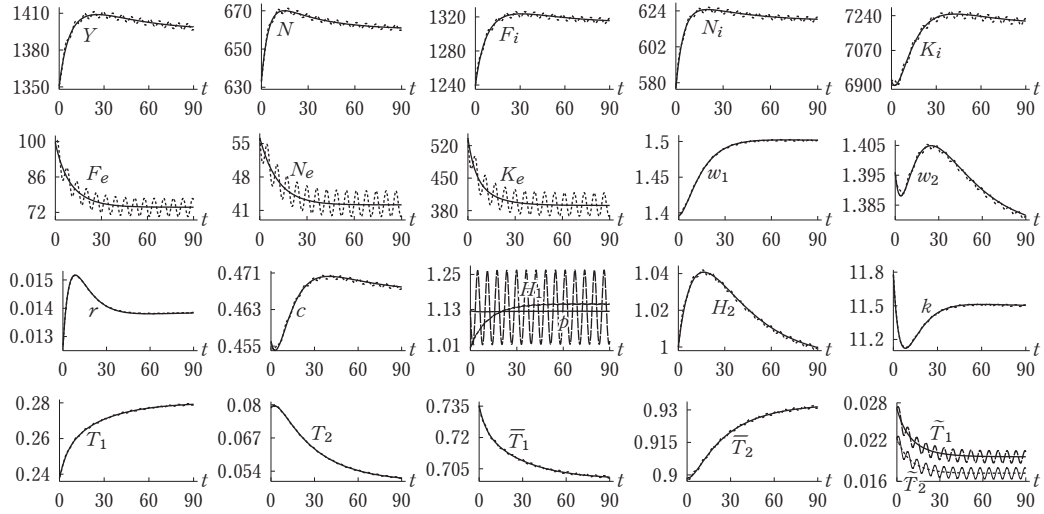
Oscillations in the total factor productivity of the education sector

First, we examine the case that the total factor productivity of the education sector experiences the following exogenous shocks

$$A_e(t)=0.9+0.1\sin(t).$$

The simulation results are plotted in Figure 2. Figure 2 shows how the system under the exogenous fluctuations moves near its long-term trends in Figure 1. The education sector’s total output and two inputs experience fluctuations with large amplitudes. The national output level, the total labor input and the production sector’s total output and two inputs experience fluctuations with small amplitudes. The education fee and each gender’s

Figure 2. Oscillations in the total factor productivity of the education Sector



individual education time also fluctuates with large amplitudes. The rate of interest and wage rates are slightly changed. The household’s consumption level and work times are slightly affected.

Oscillations in the woman’s propensity to receive education

The neoclassical approach holds that gender inequalities resulting from disparities in human capital will wither away in association with economic development (e.g., Beneria and Feldman, 1992, Forsythe, *et al.* 2000). Stotsky (2006: 18) argues, “the neoclassical approach examines the simultaneous interaction of economic development and the reduction of gender inequalities. It sees the process of economic development leading to the reduction of these inequalities and also inequalities hindering economic development.” As gender relations in the family change over time, it is significant to see how fluctuations in gender relations affect development. We now allow the woman’s propensity to receive education to fluctuate in the following way

$$\eta_{02}(t)=0.01+0.002\sin(t).$$

The simulation results are demonstrated in Figure 3. As the woman’s propensity to receive education oscillates, the national output, the total labor supply, the two sectors’ output levels, and the two sectors’ factor inputs are oscillatory. The woman’s wage rate and study time fluctuate, while the man’s wage rate and study time fluctuate with small amplitudes.

Oscillations in man’s human capital utilization efficiency

Another important issue is how the accumulated human capital is applied. We now

Figure 3. Oscillations in the woman's propensity to receive education

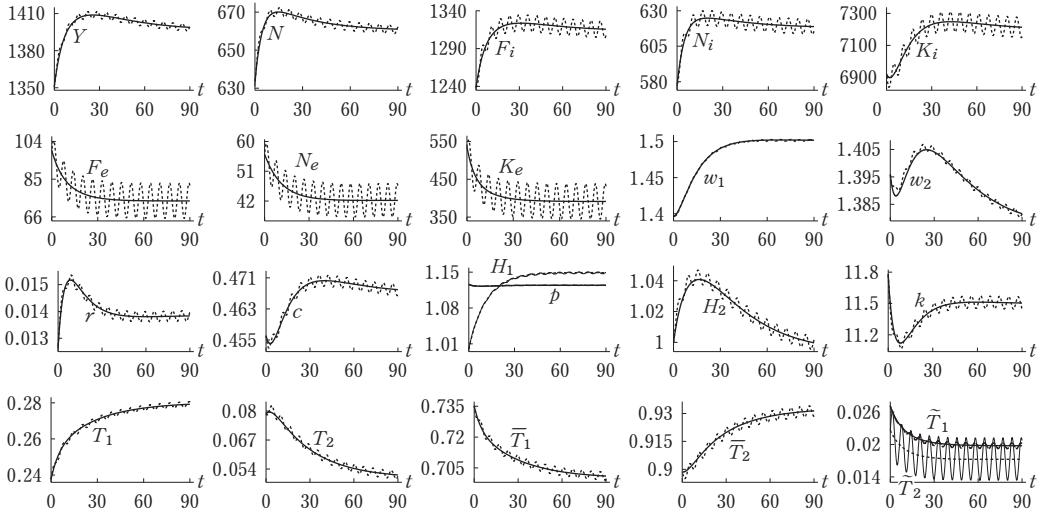
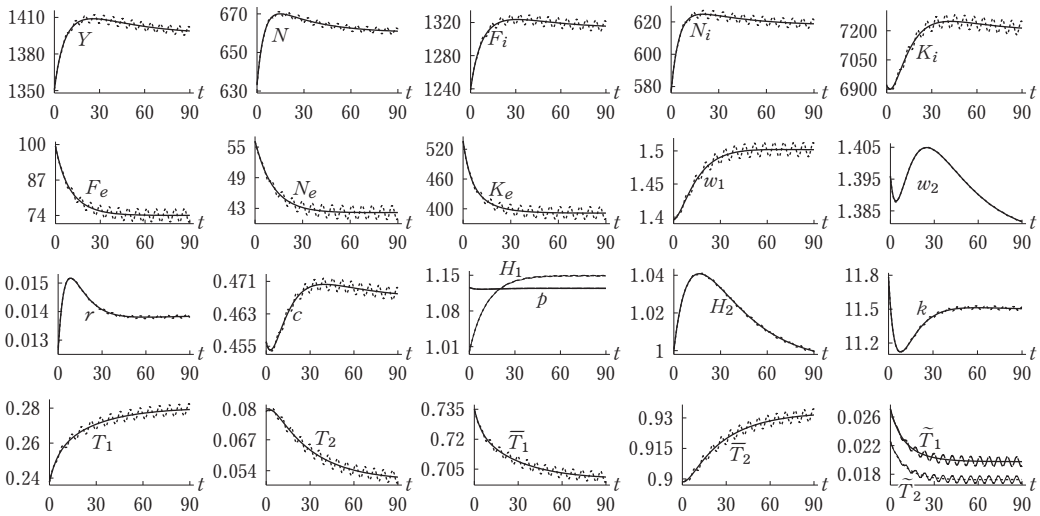


Figure 4. Oscillations in man's human capital utilization efficiency

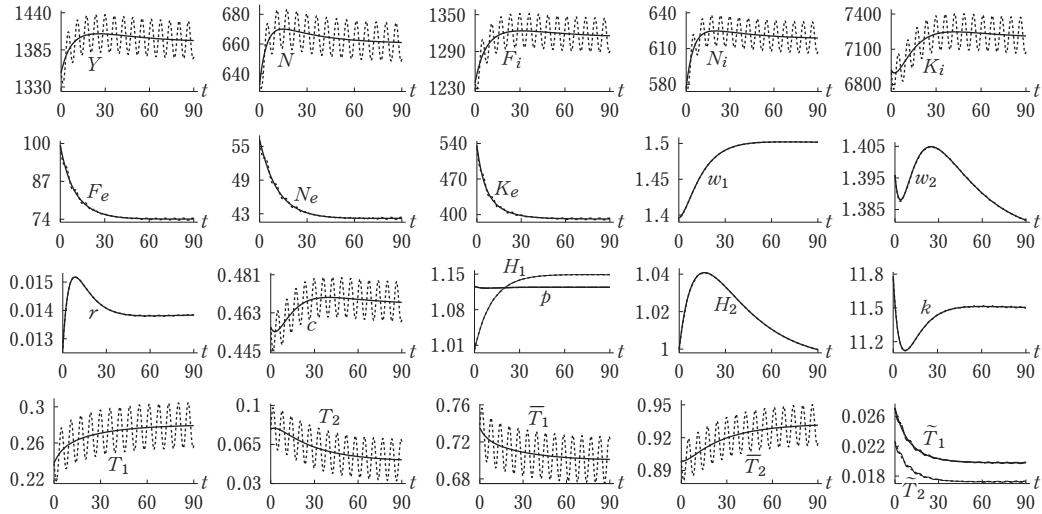


study the impact of the following oscillations in man's human capital utilization efficiency

$$\theta_1(t) = 0.6 + 0.05\sin(t).$$

As the man's efficiency is fluctuated, his wage is fluctuated. The two genders' human capital levels are slightly affected and woman's wage is changed slightly. Both man and woman's time distributions are oscillatory. The national output, the total labor supply, the two sectors' output levels, and the two sectors' factor inputs are oscillatory. We see that oscillations in man's human capital utilization efficiency may be a determinant of business cycles.

Figure 5. Oscillations in man's propensity to stay at home



Oscillations in man's propensity to stay at home

There is an immense body of empirical and theoretical literature on economic growth with time distribution between home and non-home economic and leisure activities. In an empirical study on patterns of women's work and determinants of the gender division of labor in rural Bangladesh, Bose *et al.* (2009) find that both economic and socio-cultural factors are important in explaining the persistent gender division of labor. In a study some Asian economies, Banerjee (1999) shows that the state interventions and women's own changing perceptions are important in determining gender relations in the household. Although our study does not explicitly model endogenous changes in preferences, we can examine the effects of changes in preferences on the economic system. We now examine the following oscillations in man's propensity to stay at home

$$\sigma_{01}(t) = 0.18 + 0.01\sin(t).$$

The results are plotted in Figure 5. Although the oscillations cause fluctuations in both man's and woman's work and leisure hours, man's and woman's study hours are slightly affected. The wage rates, human capital and education sector's output and two inputs are slightly affected. The national output, the total labor supply, the production sectors' output level and two inputs are oscillatory.

Oscillations in the propensity to save

We specify perturbations in the propensity to save in the following way

$$\lambda_0(t) = 0.65 + 0.02\sin(t).$$

Figure 6. Oscillations in the propensity to save

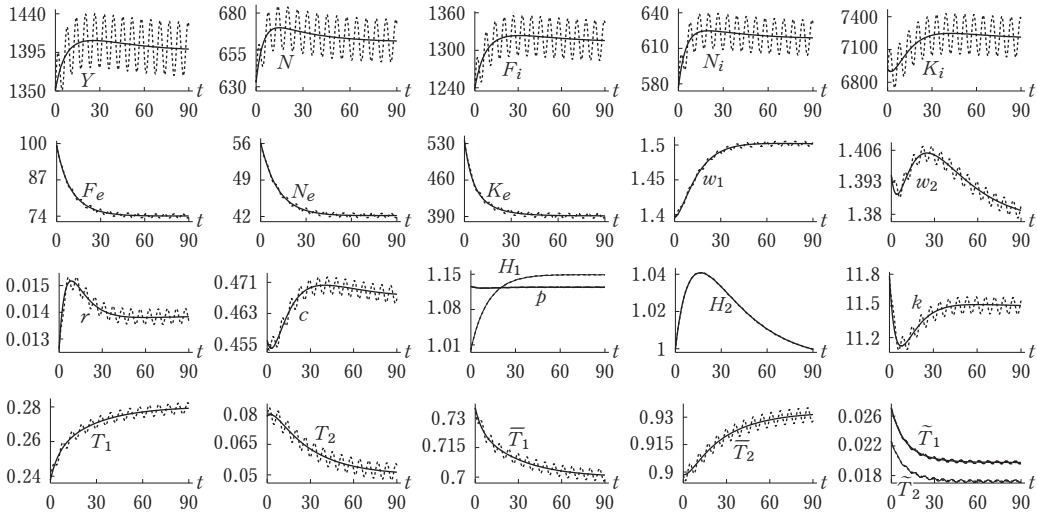
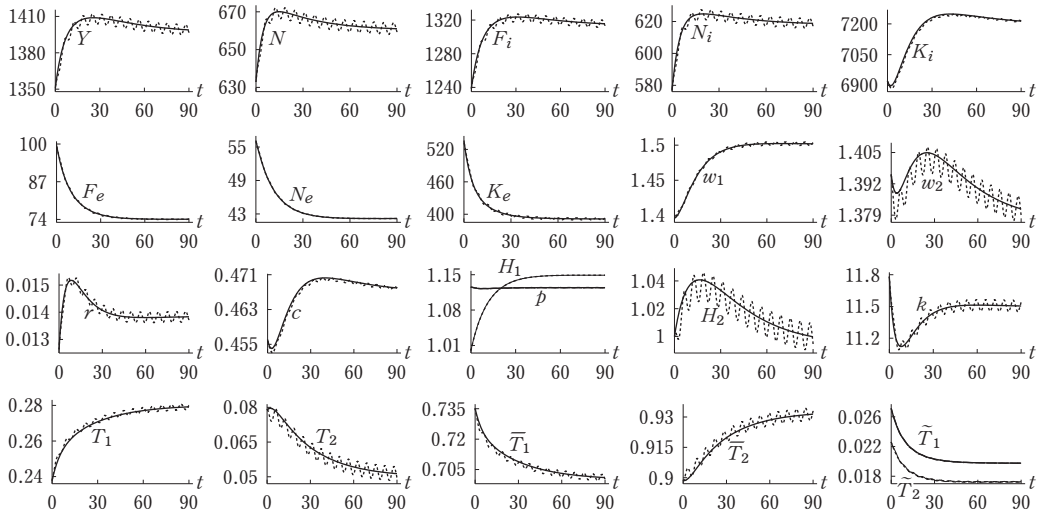


Figure 7. Oscillations in woman's depreciation rate of human capital



We plot the simulation results in Figure 6. As the propensity to save oscillates around its trend value, the variables also show oscillatory behavior. The national output and national wealth fluctuate slightly. The production sector's output level and two inputs are oscillatory. The education sector's output level and two inputs are oscillatory. The rate of interest also shows large fluctuations due to the perturbations in the propensity to save. As the system is stable, we see that small exogenous perturbations don't lead the system to be far from its trend.

Oscillations in woman's depreciation rate of human capital

We specify perturbations in woman's depreciation rate of human capital

$$\delta_{2h}(t) = 0.03 + 0.01\sin(t).$$

We plot the simulation results in Figure 7. As the depreciation rate of human capital fluctuates, the national output and national wealth fluctuate slightly. The production sector's output level and two inputs are oscillatory. The education sector's output level and two inputs are slightly affected. The rate of interest is oscillatory. The man's human capital and wage rate are slightly affected and woman's human capital and wage rate fluctuate.

5. Concluding remarks

This study addressed issues related to economic fluctuations in a model proposed by Zhang (2014) on interactions between gender differences, economic growth and education with endogenous physical and human capital accumulation. Zhang's model is a synthesis of the Solow model (Solow, 1956) and the Uzawa-Lucas two sector growth model (Uzawa, 1965, Lucas, 1988) with Zhang's approach to household behavior. This paper generalized Zhang's model by treating all the time-independent parameters to be time-dependent. We emphasized the impact of the gender-differentiated preferences and human capital utilization efficiencies upon the gender-differentiated time distribution and human capital and wage rates. We took account of learning by education in modeling human capital accumulation. We simulated the model to demonstrate existence of business cycles under different periodic shocks. The model can be extended in different directions. We may follow Funke and Strulik (2000) to integrate the two separate lines of research on growth with knowledge — the Uzawa model with education and the endogenous growth models (see also, Lacobetta, 2010).

Appendix: Proving the Lemma

We now check the lemma. In the appendix we omit time in expressions wherever there is no confusion. Insert (3) in (6)

$$\frac{K_e}{N_e} = \alpha \frac{K_i}{N_i}, \text{ i.e., } k_e = \alpha k_i, \tag{A1}$$

where $\alpha \equiv \alpha_e \beta_i / \alpha_i \beta_e$ ($\neq 1$ assumed). From (3) and (6), we obtain

$$\frac{\alpha_i F_i}{K_i} = \frac{\alpha_e p F_e}{K_e}.$$

Substituting the two production functions and (A1) into the above equation yields

$$p = A_p k_i^\beta, \quad (\text{A2})$$

where

$$A_p \equiv \frac{\alpha_i A_i \alpha^{\beta_e}}{\alpha_e A_e}, \quad \beta \equiv \beta_e - \beta_i.$$

Solve (A1) and (1)

$$n_i = \frac{\alpha k_i - k}{\alpha k_i}, \quad n_e = \frac{k - k_i}{\alpha k_i}, \quad (\text{A3})$$

where $\bar{\alpha} \equiv \alpha - 1$. Equations (3) imply

$$r = \frac{\alpha_i A_i}{k_i^{\beta_i}} - \delta_k, \quad w_q = \beta_i A_i H_q^{\beta_q} k_i^{\alpha_i}, \quad q = 1, 2. \quad (\text{A4})$$

Dividing (13) by \bar{N} yields

$$c + s - \delta \bar{k} = \frac{A_i n_i k_i^{\alpha_i} N}{\bar{N}},$$

where $\delta \equiv 1 - \delta_k$ and we use $F_i = A_i K_i^{\alpha_i} N_i^{\beta_i}$. Substituting $c = \bar{\xi} \bar{y}$ and $s = \lambda \bar{y}$ into the above equation yields

$$\bar{y} = \left(\frac{\alpha A_i k_i^{\alpha_i}}{\bar{\alpha}} - \frac{A_i k}{\alpha k_i^{\beta_i}} + \delta k \right) \frac{N}{(\bar{\xi} + \lambda) \bar{N}}, \quad (\text{A5})$$

where we use (A3) and $\bar{k} = kN/\bar{N}$. Insert (2) and $\bar{k} = kN/\bar{N}$ into the definition of \bar{y}

$$\bar{y} = (r + 1) \frac{kN}{\bar{N}} + w_1 T_0 + w_2 T_0. \quad (\text{A6})$$

Solve (A5) and (A6)

$$\frac{(A_\xi k_i^{\alpha_i} + Rk)N}{\bar{N}} = w_1 T_0 + w_2 T_0, \quad (\text{A7})$$

where

$$R(k_i) \equiv \frac{\delta}{(\bar{\xi} + \lambda)} - r - 1 - \frac{A_i}{(\bar{\xi} + \lambda) \alpha k_i^{\beta_i}}, \quad A_\xi \equiv \frac{\alpha A_i}{\alpha (\bar{\xi} + \lambda)}.$$

From (8), we have

$$\bar{T}_1 + \bar{T}_2 = \left(\frac{\alpha A_i k_i^{\alpha_i}}{\alpha} - \frac{A_i k}{\alpha k_i^{\beta_i}} + \delta k \right) \frac{\bar{W} N}{N}, \quad (A8)$$

where we use (A5) and

$$\bar{W}(k_i, H_1, H_2) \equiv \left(\frac{\sigma_1}{w_1} + \frac{\sigma_2}{w_2} \right) \frac{1}{(\xi + \lambda)}.$$

From (12) and (5), we have

$$\tilde{T}_1 + \tilde{T}_2 = \frac{A_e k_e^{\alpha_e} N_e}{N}. \quad (A9)$$

From (A9), $k_e = \alpha k_i$ and (A3), we have

$$\tilde{T}_1 + \tilde{T}_2 = \frac{\tilde{\alpha} k_i^{\alpha_e} N}{N} \left(\frac{k - k_i}{k_i} \right), \quad (A10)$$

where $\tilde{\alpha} \equiv A_e \alpha^{\alpha_e} / \alpha$. From $T_q + \tilde{T}_q + \bar{T}_q = T_0$, we have

$$T_1 + T_2 + \tilde{T}_1 + \tilde{T}_2 + \bar{T}_1 + \bar{T}_2 = 2T_0.$$

Insert (A8) and (A10) in the above equation

$$(T_1 + T_2) \bar{N} + \Delta N = 2T_0 \bar{N}, \quad (A11)$$

where

$$\Delta(k, k_i, H_1, H_2) \equiv \tilde{\alpha} k_i^{\alpha_e} \left(\frac{k - k_i}{k_i} \right) + \left(\frac{\alpha A_i k_i^{\alpha_i}}{\alpha} - \frac{A_i k}{\alpha k_i^{\beta_i}} + \delta k \right) \bar{W}.$$

Insert (1) in (A11)

$$(1 + \Delta H_1^{\theta_1}) T_1 + (1 + \Delta H_2^{\theta_2}) T_2 = 2T_0. \quad (A12)$$

Insert (1) in (A7)

$$H_1^{\theta_1} T_1 + H_2^{\theta_2} T_2 = \tilde{\Delta} \equiv \frac{w_1 T_0 + w_2 T_0}{A_s k_i^{\alpha_i} + Rk}. \quad (A13)$$

We solve (A12) and (A13)

$$T_1 = (2T_0 H_2^{\theta_2} - (1 + \Delta H_2^{\theta_2}) \tilde{\Delta}) \bar{\Delta}, \quad T_2 = ((1 + \Delta H_1^{\theta_1}) \tilde{\Delta} - 2T_0 H_1^{\theta_1}) \bar{\Delta}, \quad (A14)$$

where

$$\bar{\Delta} \equiv \frac{1}{H_2^{\beta_2} - H_1^{\beta_1}}.$$

Insert $\tilde{T}_q = \eta_q \bar{y} / \bar{p}_q$ in (A9)

$$\left(\frac{\eta_1}{\bar{p}_1} + \frac{\eta_2}{\bar{p}_2} \right) \bar{y} = \frac{A_e k_e^{\alpha_e} N_e}{N}. \quad (\text{A15})$$

Substituting (A5) into (A15) yields

$$\frac{\alpha A_i k_i^{\alpha_i}}{\bar{\alpha}} - \frac{A_i k}{\alpha k_i^{\beta_i}} + \delta k = \bar{p}_\eta n_e, \quad (\text{A16})$$

where

$$\bar{p}_\eta(k_i, H_1, H_2) \equiv (\xi + \lambda) A_e k_e^{\alpha_e} \left(\frac{\eta_1}{\bar{p}_1} + \frac{\eta_2}{\bar{p}_2} \right)^{-1}.$$

Inserting n_e from (A3) in (A16), we solve

$$k = \Lambda(k_i, H_1, H_2) \equiv (\alpha A_i k_i^{\alpha_i} + \bar{p}_\eta) \left(\frac{\bar{p}_\eta}{k_i} + \frac{A_i}{k_i^{\beta_i}} - \bar{\alpha} \delta \right)^{-1}. \quad (\text{A17})$$

By (A17), we express k as functions of k_i, H_1 and H_2 at any point of time. By (A14) we express T_1 and T_2 as functions of k_i, H_1 and H_2 . By (1) we express N as a function of k_i, H_1 and H_2 . By (A6), $\bar{k} = kN/\bar{N}$, and $K = kN$, we express K, \bar{k} , and \bar{y} as functions of k_i, H_1 and H_2 . We write \bar{k} as a function of $k_i(t), H_1(t), H_2(t)$, and t

$$\bar{k} = \bar{\Lambda}(k_i, H_1, H_2, t).$$

By $k_e = \alpha k_i$ and (A3), we solve k_e, n_i and n_e as functions k_i, H_1 and H_2 . We have

$$N_j = n_j N, K_j = k_j N_j, j = i, e.$$

We solve F_i and F_e as functions of k_i, H_1 and H_2 . From (9), we solve c, s , and \bar{T}_q as functions of k_i, H_1 , and H_2 . From (A9) and (9), we solve

$$\tilde{T}_1 = \frac{\eta \bar{p} \tilde{T}}{1 + \eta \bar{p}}, \tilde{T}_2 = \frac{\tilde{T}}{1 + \eta \bar{p}}, \quad (\text{A18})$$

where

$$\tilde{T}(k_i, H_1, H_2) \equiv \frac{\bar{\alpha} k_i^{\alpha_i} N}{N} \left(\frac{k - k_i}{k_i} \right), \eta \equiv \frac{\eta_1}{\eta_2}, \bar{p} \equiv \frac{\bar{p}_2}{\bar{p}_1}.$$

By (A18), we \tilde{T}_q solve \tilde{T}_q as functions of $k_i(t), H_1(t), H_2(t)$, and t . Hence, from (4) it is

straightforward to show that the motion of human capital can be expressed as functions of $k_i(t)$, $H_1(t)$, $H_2(t)$, and t at any point of time

$$\dot{H}_q = \Omega_q(k_i, H_1, H_2, t), \quad q=1, 2. \quad (\text{A19})$$

First, from (11) it is straightforward to express the change of wealth as a function of $k_i(t)$, $H_1(t)$, $H_2(t)$, and t as follows

$$\dot{\bar{k}} = \Omega_0(k_i, H_1, H_2, t) \equiv s - \bar{k}. \quad (\text{A20})$$

Taking derivatives of $\bar{k} = \bar{\Lambda}$ with respect to time, we have

$$\dot{\bar{k}} = \frac{\partial \bar{\Lambda}}{\partial t} + \frac{\partial \bar{\Lambda}}{\partial k_i} \dot{k}_i + \Omega_1 \frac{\partial \bar{\Lambda}}{\partial H_1} + \Omega_2 \frac{\partial \bar{\Lambda}}{\partial H_2}, \quad (\text{A21})$$

where we use (A14). Substituting (A21) into (A20) yields

$$\dot{k}_i = \Omega(k_i, H_1, H_2, t) \equiv \left(\Omega_0 - \frac{\partial \bar{\Lambda}}{\partial t} - \Omega_1 \frac{\partial \bar{\Lambda}}{\partial H_1} - \Omega_2 \frac{\partial \bar{\Lambda}}{\partial H_2} \right) \left(\frac{\partial \bar{\Lambda}}{\partial k_i} \right)^{-1}. \quad (\text{A22})$$

The three differential equations, (A22) and (A19), contain three variables $k_i(t)$, $H_1(t)$, $H_2(t)$, and t . We thus proved the lemma.

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