Doctoral Thesis

Nekrasov's Formula for Gauged Argyres-Douglas Theories and S-duality

March 2023

Doctoral Program in Advanced Mathematics and Physics

Graduate School of Science and Engineering

Ritsumeikan University

KIMURA Takuya

Doctoral Thesis Reviewed
by Ritsumeikan University

Nekrasov's Formula for Gauged Argyres-Douglas Theories and S-duality
（ゲージ化されたアルジレス-ダグラス理論における ネクラソフ公式とS双対性）

March 2023

2023年3月

Doctoral Program in Advanced Mathematics and Physics

Graduate School of Science and Engineering

Ritsumeikan University

立命館大学大学院理工学研究科

基礎理工学専攻博士課程後期課程

KIMURA Takuya

木村　拓也

Supervisor：Professor SUGAWARA Yuji

研究指導教員：菅原　祐二　教授

Abstract

In this thesis, we study the S-duality for the four-dimensional $\mathcal{N} = 2$, $SU(2)$ superconformal field theories coupled to the (A_1, D_N) type Argyres-Douglas theories. These superconformal theories cannot be engineered by compactifying the 6d $\mathcal{N} = (2, 0)$ superconformal field theory of type A_1 on Riemann surface. We propose that the Nekrasov formula for the Argyres-Douglas theories by extending the generalized AGT correspondence to the case of $U(2)$ gauge group, which is induced by defining the irregular state of the direct sum of Virasoro algebra and Heisenberg algebra. This formula is regarded as the contribution from the (A_1, D_N) theory to the instanton part of the Nekrasov partition function. As we apply this Nekrasov formula to these four-dimensional superconformal field theories, we evaluate the Nekrasov partition function of these theories and find that the prepotential is related to that of the $SU(2)$ superconformal QCD by the non-trivial replacement of the UV gauge coupling. From the relation of the prepotentials, we read off the action of the S-duality group on the UV gauge coupling of these theories via that of the $SU(2)$ superconformal QCD.

Contents

1 Introduction

Supersymmetry imposes strong constraints on the quantum field theories. The four-dimensional maximally supersymmetric gauge theories are $\mathcal{N} = 4$ and are known as the superconformal field theories. The $\mathcal{N} = 4$ supersymmetric Yang-Mills theories have quantum corrections; the gauge couplings in UV and IR region are equivalent, $\tau = \tau_{\mathrm{IR}}$. Seiberg and Witten have uncovered that the prepotential of the low-energy effective theory for the $\mathcal{N} = 2$ theories is explicitly determined by the Seiberg-Witten (SW) curve and the SW 1-form [1,2]. This means that one can understand the behavior of the quantum correction of the low-energy effective theory for the $\mathcal{N} = 2$ theories by only geometrical properties.

The S-duality of the $\mathcal{N} = 4$ $SU(N)$ SYMs, called the Montonen-Olive duality, implies that the theory with the weak coupling τ_{IR} is equivalent to that with the strong coupling $-1/\tau_{\mathrm{IR}}$, where $\tau_{\mathrm{IR}} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, and then we exchange the minimal magnetic monopole with the W-boson of the two theories [3–6]. These SYMs also have the symmetry of the theta angle, $\tau_{\mathrm{IR}} \to \tau_{\mathrm{IR}} + 1$, so-called the T-transformation. On the other hand, for the $\mathcal{N} = 2$, $SU(2)$ with four flavors, the S-duality called the Seiberg-Witten duality is the exchanging of the minimal magnetic monopole with the quark whose electric charge is $1/2$ of that of the W-boson, where the convention of the complex gauge coupling is $\tau_{\mathrm{IR}} = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$. The T-transformation $\tau_{\mathrm{IR}} \to \tau_{\mathrm{IR}} + 1$ means that the shift of the theta angle $\theta_{\mathrm{IR}} \to \theta_{\mathrm{IR}} + \pi$ [1]. The fundamental domain in the space of the gauge couplings show in Figure 1.

Although it is difficult to evaluate the path integral, including the non-perturbative contribution for general quantum field theories because of an infinite dimensional integral, the $\mathcal{N} = 2$ theories with Lagrangian description have been evaluated by the supersymmetric localization technique [7]. Then this path integral reduces to a sum over the fixed points, described by a set of Young diagrams, of the torus action on the instanton moduli space, which has been discovered by [8, 9], the so-called "Nekrasov partition function". The Nekrasov partition function is the partition function on the spacetime $\mathbb{R}^4$ deformed by Ω-background, whose parameters are denoted (ϵ_1, ϵ_2) and related to the rotations on spacetime. The Nekrasov partition function rederives the prepotential in the classical limit as $\epsilon_i \to 0$ [9–11].

$$\mathcal{F} \equiv \lim_{\epsilon_i \to 0} \left(-\epsilon_1\epsilon_2 \log \mathcal{Z}_{\mathrm{Nek}}\right) \ . \tag{1.1}$$

The instanton part of the Nekrasov partition function is known to be wirtten as a simple product of the contributions from the "matter sectors" and the "gauge sectors". For instance, we consider the $U(2)$ gauge theory with four flavors, and then the Nekrasov partition

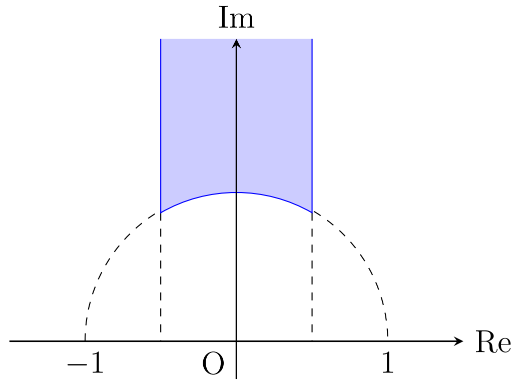

Figure 1: The fundamental domain in $\tau_{\rm IR}$

function of this theory is written as

$$\mathcal{Z}^{N_f=4}_{U(2)} = \mathcal{Z}_{\rm pert} \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} \mathcal{Z}^{\rm vec}_{Y_1,Y_2}(a) \prod_{i=1}^{4} \mathcal{Z}^{\rm fund}_{Y_1,Y_2}(a, m_i) \ , \tag{1.2}$$

where q is the exponential of the gauge coupling, and the parameters a and m_i are the VEV of the scalar in the $SU(2)$ vector multiplet and the mass parameters in the fundamental hypermultiplets. The factors $\mathcal{Z}^{\rm vec}_{Y_1,Y_2}$ and $\mathcal{Z}^{\rm fund}_{Y_1,Y_2}$ express the contributions from the gauge sector and the fundamental hypermultiplet in the matter sector.

Compactifying the 6d superconformal field theories (SCFTs) on a two-dimensional surface, a class of the four-dimensional $\mathcal{N} = 2$ theories can be obtained [12–17]. In particular, so-called the class $\mathcal{S}$ theories are obtained by compactifying the 6d $\mathcal{N} = (2, 0)$ SCFTs of the type of the simply-laced Lie algebra $\mathfrak{g}$ on a punctured Riemann surface [12–14, 18, 19]. The main advantage of the class $\mathcal{S}$ construction for type $\mathfrak{g} = A_1$ is to be available a 2d/4d correspondence, called the AGT correspondence [20, 21].

For instance, there is an exciting series of strongly-coupled SCFTs in the class $\mathcal{S}$ theories, called Argyres-Douglas (AD) theories. These theories have the relevant operators and their corresponding relevant couplings, whose scaling dimensions are fractional [14, 22–26]. Since the AD theories have no Lagrangian description, the Nekrasov partition function of the AD theories cannot be directly evaluated by the supersymmetric localization technique, while this can be indirectly evaluated via the (generalized) AGT correspondence [21,26,27]. These developments were studied by [28–33].

There is an interesting series of the four-dimensional $\mathcal{N} = 2$ SCFTs including the AD theories in their matter sector, which is described by the quiver diagram in Figure 2. We call such SCFTs "*conformally gauged AD theories*". The supersymmetric localization

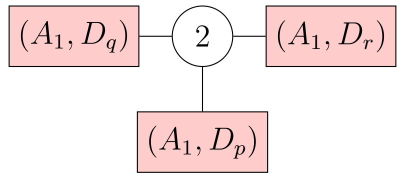

Figure 2: The quiver diagram for the four-dimensional superconformal $SU(2)$ gauge theories coupled to three AD theories with $(p, q, r) = (2, 4, 4)$, $(2, 3, 6)$, or $(3, 3, 3)$. The middle circle stands for $SU(2)$ gauge group and each boxes stand for the AD theories. $SU(2)$ sub-group of the flavor symmetry of each of the AD theories is diagonally gauged by the $SU(2)$ vector multiplet.

technique cannot directly evaluate the Nekrasov partition function of these theories due to the AD theories in the matter sectors. Moreover, we cannot also evaluate this partition function via the AGT correspondence since these theories have no engineering by compactifying the 6d $\mathcal{N} = (2, 0)$ SCFT of type A_1 on a punctured Riemann surface. Since these theories have a vanishing one-loop beta function of the gauge coupling, we expect that these theories will provide a new S-duality. [34–41] In particular, for the conformally gauged AD theory for $(p, q, r) = (2, 4, 4)$ in Figure 2, the S-duality represents the minimal generalization of the Seiberg-Witten duality, which is studied by [34].

This thesis proposes a way to compute the Nekrasov partition function of the conformally gauged AD theories. Using the proposal, we study the S-duality for these theories based on our works [42, 43]. Our strategy is to introduce the Nekrasov formula for the AD sectors, say $\mathcal{Z}^{(A_1,D_{2n})}_{Y_1,Y_2}$, via the generalized AGT correspondence. To that end, we consider the "*non*-conformally gauged AD theory" described in Figure 3, where the subgroup $SU(2)$ of the flavor symmetry of the AD theories is gauged by the $SU(2)$ vector multiplet. This theory is constructed by compactifying the 6d $\mathcal{N} = (2, 0)$ SCFT of type A_1 on the Riemann surface with two irregular punctures, and therefore we can apply this theory to the AGT correspondence. Since the AGT correspondence is for $SU(2)$ gauge group while the Nekrasov partition function is obtained from $U(2)$ gauge theory, we need a $U(2)$ version of the generalized AGT correspondence so that the resulting partition function follows the same form of (1.2) decomposed as a sum over the two Young diagrams (Y_1, Y_2). According to the $U(2)$ version of the original AGT correspondence, the extension to the $U(2)$ is obtained by defining the primary vertex operator of the direct sum of Virasoro and Heisenberg algebras $(Vir \oplus H)$ instead of only the Virasoro algebra on the two-dimensional CFT [44, 45].

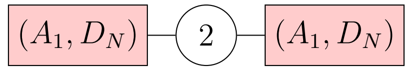

Figure 3: Asymptotically free theory with $SU(2)$ gauge group coupled to the two (A_1, D_N) theories, where N is a positive integer.

This thesis is organized as follows; in section 2, we will review the four-dimensional $\mathcal{N} = 2$ theories and the AGT correspondence. In the review part of the AGT correspondence, we first discuss the original AGT correspondence based on [20] and extend it to the $U(2)$ version. Furthermore, we give a brief review of the generalization of the AGT correspondence.

In section 3, we provide the $U(2)$-version of the generalized AGT correspondence. In particular, for the case of the (A_1, D_N) theories for even N, we discuss that an irregular state of $Vir \oplus H$ is characterized by an extension of the colliding limit construction in [27]. At the same time, we provide an irregular state for odd N by considering the classical limit $\epsilon_i \to 0$ and turning off the relevant couplings and the VEV of the relevant operators. We then introduce the Nekrasov formula for the (A_1, D_N) theories.

In sections 4 and 5, we apply our formula to the conformally gauged AD theories for $(p, q, r) = (2, 4, 4)$ and $(2, 3, 6)$ in Figure 2, which are called the (A_3, A_3) theory and the (A_2, A_5) theory [46]. These include two AD theories and a fundamental hypermultiplet in their matter sector. We show that when we turn off all of the dimensionful parameters except for the VEV the scalar in the $SU(2)$ vector multiplet a and the Ω-background parameters (ϵ_1, ϵ_2), the prepotentials of these theories are related to that of the $SU(2)$ SQCD with four flavors $\mathcal{F}_{N_f=4}$ as follows:

$$2\mathcal{F}_{(A_3,A_3)}(q; a) = \mathcal{F}_{N_f=4}(q^2; a) \ , \tag{1.3}$$

$$3\mathcal{F}_{(A_2,A_5)}(q; a) = \mathcal{F}_{N_f=4}(q^3; a) \ . \tag{1.4}$$

where $\mathcal{F}_{(A_3,A_3)}$ and $\mathcal{F}_{(A_2,A_5)}$ are the prepotentials of the (A_3, A_3) theory and the (A_2, A_5) theory, and q is the exponential of the gauge coupling. Moreover, we discuss the S-duality of these theories via that of the $SU(2)$ SQCD with four flavors, and we check the consistency of the above from the SW curve. In section 6, we conclude by presenting several future directions.

2 Review of AGT correspondence

This section will review the AGT correspondence discovered by Alday, Gaiotto, and Tachikawa [20, 21]. This is a strong correspondence between the four-dimensional $\mathcal{N} = 2$ theory and the two-dimensional Liouville CFT. In particular, this implies that the instanton part of the Nekrasov partition function is equivalent to the conformal block, and this consistency has been checked by [45, 47–52]. The AGT correspondence was generalized for the AD theories [26, 27].

In subsection 2.1, we briefly review the four-dimensional $\mathcal{N} = 2$ theories. In subsection 2.2, we review the original AGT correspondence based on [20] and give the $U(2)$-version of this correspondence. In subsection 2.3, we explain to be generalized for the AD theories.

2.1 Four dimensional $\mathcal{N} = 2$ theories

The low energy effective theory of $\mathcal{N} = 2$ theories is characterized by the moduli space of vacua called the Coulomb branch. This is parametrized the VEV of the scalr field in $\mathcal{N} = 2$ vector multiplet. For instance, we consider $\mathcal{N} = 2$, $SU(2)$ gauge theory. Then the moduli parameter is $u = \mathrm{tr}\langle\phi^2\rangle$, where ϕ is the scalar field. As is well known, the spontaneous symmetry breaking reduces this theory to the $U(1)$ gauge theory when $u \neq 0$, while the gauge group is unbreakable when $u = 0$. However, the Coulomb branch has a non-trivial structure by the quantum corrections. The low energy effective action of this theory is determined by the prepotential $\mathcal{F}$ as

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \mathrm{Im} \int d^4\theta \left[\Phi^\dagger \frac{\partial \mathcal{F}}{\partial \Phi} + \frac{\partial^2 \mathcal{F}}{\partial \Phi^2} W_\alpha W^\alpha \right] \ , \tag{2.1}$$

where Φ is the $\mathcal{N} = 1$ chiral multiplet, including the scalar field, and W_α is the $\mathcal{N} = 1$ vector multiplet, including the vector field, in both the adjoint of the gauge group. Since the $\mathcal{N} = 2$ theories are constrained by the supersymmetry, the prepotential is a holomorphic function of Φ and this has only quantum corrections:

$$\mathcal{F} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{\text{one-loop}} + \mathcal{F}_{\text{instanton}} \ . \tag{2.2}$$

The relation between this and the gauge coupling τ_{IR} of the resulting theory is given as $\tau_{\text{IR}} = \partial^2\mathcal{F}/\partial\Phi^2$. The problem of the global structure of the Coulomb branch reduces to the prepotential, which is explicitly determined by the SW curve and the SW 1-form [1]. The SW curve stands for the algebraic curve, which is identical to the fibration over the Coulomb branch, while the SW 1-form relates to the masses of the BPS particles. For

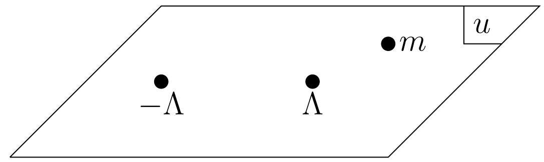

Figure 4: The Coulomb branch for the $SU(2)$ theory with one flavor. The black dots stand for the singular points. The massless particles are the dyon, the monopole, and the quark at $u = -\Lambda, \Lambda$, and m, respectively.

$SU(2)$ SQCD with one flavor, the SW curve stands for the torus, which degenerate at the three singular points on the Coulomb branch. The low energy effective theories at the singular points reduce to the $U(1)$ gauge theory with a massless particle, which is monopole, dyon, or quark, see Figure 4. Let us consider the colliding of these singular points. Then we obtain the Argyres-Douglas theory at this singular point, called the Argyres-Douglas point [22–24]. For instance, we start with the $SU(2)$ gauge theory with one flavor. the AD theory arises in the low energy region when the mass parameter m is related to the dynamical scale Λ as $m \sim \Lambda$. This AD theory is the $U(1)$ gauge theory with a massless monopole and a massless quark. This theory has no Lagrangian description and is believed the strongly coupled superconformal field theory. We review the generalize AD theories in the following.

String theory, M-theory, and F-theory can obtain the four-dimensional $\mathcal{N} = 2$ theories [53–60]. In this subsection, we discuss a series of the $\mathcal{N} = 2$ theories. In the last of this subsection, we denote the Nekrasov partition function of the four-dimensional $\mathcal{N} = 2$ theories with Lagrangian description. In particular, we focus on the $SU(2)$ SQCD with four flavors and briefly review the S-duality for this theory.

2.1.1 $\mathcal{N} = 2$ theories of class $\mathcal{S}$

Compactifying 6d $\mathcal{N} = (2, 0)$ SCFTs of type A_1 on a punctured Riemann surface $\mathcal{C}$ obtains a class of the four-dimensional $\mathcal{N} = 2$ theories [12–14, 18, 19]. We here denote by $\mathcal{T}_{\mathcal{C}}$ the four-dimensional $\mathcal{N} = 2$ SCFT. In particular, It is known that the punctures on $\mathcal{C}$ can be classified into two types: "regular puncture" and "irregular puncture" [13, 14, 18]. These punctures are characterized by the behavior of the Hitchin field $\Phi(z)$ in the Hitchin system corresponding to the 4d theory $\mathcal{T}_{\mathcal{C}}$. $\Phi(z)$ has a simple pole at the regular puncture in the neighborhood of $z = 0$, while it has a heigher pole at the irregular puncture. In the following sub-section, we will see that for more details of the irregular punctures. The

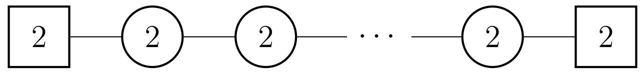

Figure 5: The quiver diagram of $\mathcal{T}_{\mathcal{C}}$ compactifying by sphere $\mathcal{C}$ with $(n+2)$ regular punctures. There are $(n-1)$ circles, each of which stands for an $SU(2)$ gauge group.

spectral curve of the Hitchin system identifies the SW curve of the 4d theory $\mathcal{T}_{\mathcal{C}}$ as [14, 61]

$$\det(x - \Phi(z)) = 0 \ , \tag{2.3}$$

where the SW 1-form is $\lambda \equiv x dz$.

For instance, let us consider $\mathcal{C}$ as the sphere with $(n+2)$ regular punctures. Then one obtains the linear-quiver gauge theory [13, 62–64], which has $SU(2)^{n-1}$ gauge group, see Figure 5. The beta function of all of the gauge couplings is 0. Hence we see that this theory is superconformal.

2.1.2 General Argyres-Douglas theories

The AD theories are a series of strongly coupled four-dimensional SCFTs with Coulomb branch operators and relevant couplings whose scaling dimensions are fractional. Hence these theories have no Lagrangian description. The AD theories cannot apply to the supersymmetric localization technique, which differs from the Lagrangian theories. The original AD theories were discovered as an IR SCFT at a special point on the Coulomb branch of the $SU(3)$ SYM [22] and the $SU(2)$ SQCD with various flavors: $N_f = 1, 2, 3$ [23].[1] It is further generalized to $\mathcal{N} = 2$ theory with a general gauge group and fundamental hypermultiplet [24]. The AD theories are generally constructed by compactifying 6d $\mathcal{N} = (2, 0)$ SCFTs of type A_1 on a punctured sphere $\mathcal{C}$ [13, 14, 25, 26, 65–67] and by type IIB string theory on Calabi-Yau singularities [46, 68].

As shown in [25], the four-dimensional theory $\mathcal{T}_{\mathcal{C}}$ is conformal theory only if the Riemann surface $\mathcal{C}$, whose genus is zero, has an irregular puncture and one or zero regular puncture. In the neighborhood of the origin, The irregular punctures behave as $\Phi(z) \sim 1/z^{\frac{N}{2}+1}$ for integer N, it is called rank $N/2$ irregular puncture. These AD theories are called the (A_1, D_N) and (A_1, A_{N-3}) theory, respectively. Here A_n and D_n stand for the type of isolated singularity of the SW curve [46, 68]. The UV theories for these AD theories are the quiver diagrams described in Figure 6.[2] In particular, for the (A_1, D_{even}) theories, the UV theory

[1]The AD theory from $SU(2)$ SQCD with $N_f = 1$ coincides with that from $SU(3)$ SYM.

[2]The AD theories obtained from $SU(2)$ SQCD with $N_f = 1, 2, 3$ flavors are called the (A_1, A_2), (A_1, A_3)

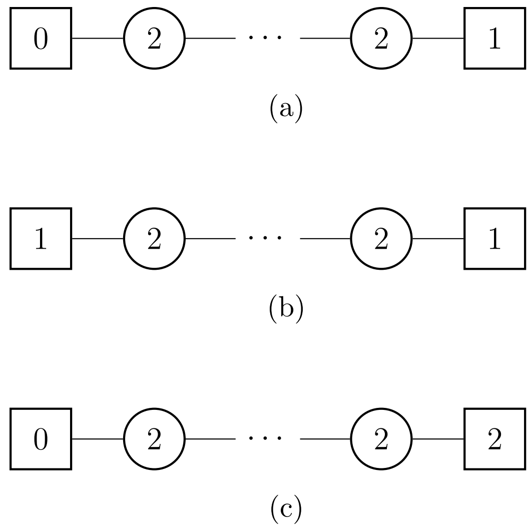

Figure 6: The quiver diagrams (a), (b) and (c) are of the UV theories for The (A_1, A_{2n-2}) theory, The (A_1, A_{2n-1}) theory and The (A_1, D_{2n-1}) theory, respectively for integer n. There are $(n-1)$ circles for each of the diagrams. The boxes stand for fundamental hypermultiplets of $SU(2)$.

is the superconformal $SU(2)$ linear quiver diagram described in Figure 5 for $n = N/2$ [25]. Note that the (A_1, D_2) theory is identified with a free fundamental hypermultiplet in the doublet of $U(2)$. This thesis focuses on the (A_1, D_N) theories. The (A_1, D_N) theory has flavor symmetry $SU(2) \times U(1)$. For $N = 4$, this flavor symmetry is enhanced to $SU(3)$.

2.1.3 Nekrasov partition function

The Nekrasov partition function is known as the partition function on $\mathbb{R}^4$ deformed by the Ω-background, which was first studied by [8, 9]. Ω-background parameters (ϵ_1, ϵ_2) are associated with the rotation $SO(4)$, and therefore the Nekrasov partition function is a function of (ϵ_1, ϵ_2). This consists of the perturbative part $\mathcal{Z}_{\text{pert}}$ and the instanton part $\mathcal{Z}_{\text{inst}}$ as

$$\mathcal{Z}_{\text{Nek}}(\epsilon_1, \epsilon_2; a; \cdots) = \mathcal{Z}_{\text{pert}} \mathcal{Z}_{\text{inst}} \ , \tag{2.4}$$

where a stands for the VEV of the scalar field in the vector multiplet, and "$\cdots$" stands for the other parameters in this theory. the Nekrasov partition function of four-dimensional

and (A_1, D_4) theories, respectively. In particular, since A_3 and D_3 have the same Dynkin diagram, we see that the (A_1, D_3) theory is equivalent to the (A_1, A_3) theory.

$\mathcal{N}=2$ $U(N)$ theory with Lagrangian description can be evaluated by using the supersymmetric localization technique. Then the instanton part is given by a sum over the fixed points of the torus action on the $U(N)$ instanton moduli space.[3] These fixed points are labeled by a set of N Young diagrams $\{Y_k\}$, i.e., we combinatorially evaluate the partition function.

Below, let us consider that the gauge group is $U(2)$. For the $U(2)$ gauge theory with N_f flavors, the fixed points of the torus action on the $U(2)$ instanton moduli space are described by two Young diagrams (Y_1, Y_2). The instanton part of this theory is given by a simple product expression as [69–71]

$$\mathcal{Z}_{\text{inst}} = \sum_{Y_1,Y_2} \Lambda^{b_0(|Y_1|+|Y_2|)} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \prod_{i=1}^{N_f} \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, m_i) \ , \tag{2.5}$$

where $b_0 \equiv 4-N_f$ is the coefficient of the beta function, Λ and a are respectively dynamical scale and the VEV of the scalar field in the $SU(2)$ vector multiplet, and m_i are the mass parameters of the fundamental hypermultiplets. When $N_f = 4$, Λ is replaced by the exponential of the marginal gauge coupling q. $\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}$ stands for the contribution from the fundamental hypermultiplet and is written as[4]

$$\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a,m) \equiv \prod_{i=1}^{2} \prod_{s\in Y_i} (\phi(a_i, s) - m + \epsilon_1 + \epsilon_2) \ , \tag{2.7}$$

where $a_1 = -a_2 = a$ and

$$\phi(a, s) \equiv a + \epsilon_1(i-1) + \epsilon_2(j-1) \ . \tag{2.8}$$

$\mathcal{Z}^{\text{vec}}_{Y_1,Y_2}$ stands for the contribution from the vector multiplet and is written as

$$\mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \equiv \prod_{i,j=1}^{2} \prod_{s\in Y_i} \frac{1}{-E_{Y_i,Y_j}(a_i - a_j, s) + \epsilon_1 + \epsilon_2} \prod_{t\in Y_j} \frac{1}{E_{Y_j,Y_i}(a_j - a_i, t)} \ , \tag{2.9}$$

where the factor E_{Y_1,Y_2} is defined as

$$E_{Y_1,Y_2}(a, s) \equiv a - \epsilon_1 L_{Y_2}(s) + \epsilon_2(A_{Y_1}(s) + 1) \ . \tag{2.10}$$

[3] The torus action is related to the maximal torus of $U(N)$ and $SO(4)$.

[4] The contribution from the anti-fundamental hypermultiplet related to $\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a,m)$:

$$\mathcal{Z}^{\text{anti-fund}}_{Y_1,Y_2}(a,m) = \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, \epsilon_1 + \epsilon_2 - m) \tag{2.6}$$

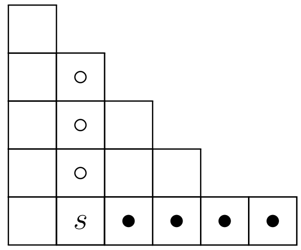

Figure 7: For $s = (2, 1)$, the leg-length is the number of black dots and the arm-length is the number of white dots.

Let us denote Young diagram by $Y = \{\lambda_1 \geq \lambda_2 \geq \cdots\}$, where λ_i is the height of i-th column, and denote its transpose by $Y^T = \{\lambda_1^{'} \geq \lambda_2^{'} \geq \cdots\}$, where λ_i is the length of i-th width. For a box $s = (i, j)$, the leg-length L_Y and the arm-length A_Y are defined as (see Figure 7)

$$L_Y(s) = \lambda_j^{'} - i \ , \qquad A_Y(s) = \lambda_i - j \ . \tag{2.11}$$

The contribution from the bi-fundamental hypermultiplet of $U(2) \times U(2)$ is defined as

$$\begin{aligned} \mathcal{Z}^{\text{bifund}}_{Y_1,Y_2;W_1,W_2}(a, b, \alpha) \equiv \prod_{i,j=1}^{2} \prod_{s \in Y_i} &\left(E(a_i - b_j, Y_i, W_j, s) - \alpha\right) \\ \times \prod_{t \in W_j} &\left(\epsilon_1 + \epsilon_2 - E(b_j - a_i, W_j, Y_i, t) - \alpha\right) \ , \end{aligned} \tag{2.12}$$

where a and b stand for the VEVs of the scalar field in each vector multiplets, and α is a mass parameter of the bi-fundamental hypermultiplet. Note that this is related to the contribution from the vector multiplet as

$$\mathcal{Z}^{\text{bifund}}_{Y_1,Y_2;Y_1,Y_2}(a, a, 0) = 1/\mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \ . \tag{2.13}$$

2.1.4 S-duality for the $SU(2)$ SQCD with four flavors

Here, we review S-duality for the $SU(2)$ SQCD with four flavors. This theory is known to be self-dual. The Nekrasov partition function of this theory is given as

$$\mathcal{Z}^{N_f=4}_{U(2)} = \mathcal{Z}_{\text{pert}} \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \prod_{i=1}^{4} \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, m_i) \ , \tag{2.14}$$

where we replaced $SU(2)$ gauge group with $U(2)$. The exponential of the gauge coupling is defined as $q = e^{i\pi\tau}$.[5] When the mass parameters are turned off, from (1.1), the prepotential is obtained as the sum of the perturbative part $(\log q - \log 16)a^2$ and instanton part:

$$\mathcal{F}_{\text{inst}}^{N_f=4}(q;a) = \left(\frac{1}{2}q + \frac{13}{64}q^2 + \frac{23}{192}q^3 + \frac{2701}{32768}q^4 + \cdots\right)a^2 \ . \tag{2.15}$$

On the other hand, in terms of q_{IR}, the prepotential is defined as

$$\mathcal{F}_{N_f=4} = (\log q_{\text{IR}})a^2 \ , \tag{2.16}$$

where q_{IR} is the exponential of IR gauge coupling and is defined as $q_{\text{IR}} = e^{i\frac{\theta_{\text{IR}}}{\pi} - \frac{8\pi^2}{g_{\text{IR}}^2}}$. This implies that the relation between q and q_{IR} is [72]

$$q = \frac{\theta_2(q_{\text{IR}})^4}{\theta_3(q_{\text{IR}})^4} \ , \tag{2.17}$$

where $\theta_2(q) = \sum_{n\in\mathbb{Z}} q^{(n-\frac{1}{2})^2}$ and $\theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$. This relation means that

$$\tau_{\text{IR}} \equiv \frac{1}{\pi i}\log q_{\text{IR}} = \frac{\theta_{\text{IR}}}{\pi} + \frac{8\pi i}{g_{\text{IR}}^2} \tag{2.18}$$

is the modulus of the elliptic curve, which corresponds to the double cover of a sphere with four punctures whose cross ratio is q. This elliptic curve is known to be invariant under $PSL(2,\mathbb{Z})$. The action of $PSL(2,\mathbb{Z})$ on the IR gauge coupling generated by

$$T : \tau_{\text{IR}} \to \tau_{\text{IR}} + 1 \ , \qquad S : \tau_{\text{IR}} \to -\frac{1}{\tau_{\text{IR}}} \ . \tag{2.19}$$

This implies that the theory is invariant under T and S transformations on the UV gauge coupling as follows:

$$T : q \to \frac{q}{q-1} \ , \qquad S : q \to 1 - q \ . \tag{2.20}$$

Let us now turn on all the mass parameters. then the instanton part of the prepotential is modified as

$$\begin{aligned}
&\mathcal{F}_{\text{inst}}^{N_f=4}(q;a,m_i) \\
&\quad = \frac{1}{2}(a^2 + m_1m_2m_3m_4a^{-2})q \\
&\qquad + \frac{1}{64}\Big(13a^2 + (16m_1m_2m_3m_4 + m_3^2m_4^2 + m_2^2(m_3^2+m_4^2) + m_1^2(m_2^2+m_3^2+m_4^2))a^{-2} \\
&\qquad - 3(m_2^2m_3^2m_4^2 + m_1^2(m_3^2m_4^2 + m_2^2(m_3^2+m_4^2)))a^{-4} + 5m_1^2m_2^2m_3^2m_4^2a^{-6}\Big)q^2 + \cdots \ .
\end{aligned} \tag{2.21}$$

[5] We use the convention $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$

In this case, the theory is invariant under the transformations (2.20) combined with the exchanging the mass parameters by the action of $PSL(2,\mathbb{Z})$.

2.2 AGT correspondence

In this subsection, we review the original AGT correspondence [20] and discuss to extension to $U(2)$-version [44].

2.2.1 Original AGT correspondence

The original AGT correspondence is a relation between the four-dimensional $\mathcal{N}=2$ theory $\mathcal{T}_{\mathcal{C}}$ and two-dimensional Liouville CFT [20]. Let us consider the case of $\mathcal{T}_{\mathcal{C}}$ for $\mathcal{C}$ being a sphere with $(n+2)$ regular punctures. Recall that this theory is described in the quiver diagram Figure 5. This theory is a linear-quiver gauge theory that has $\mathrm{SU}(2)^{n-1}$ gauge group. The original AGT correspondence implies that the instanton part of the Nekrasov partition function of this theory is identified by the $(n+2)$-point conformal block of the Virasoro primary vertex operator $V_\alpha \equiv e^{2\alpha\phi(z)}$ with the conformal weight $\Delta_\alpha \equiv \alpha(Q-\alpha)$ of the Liouville CFT as

$$\mathcal{Z}_{inst}(\vec{a};m_0,\cdots,m_{n+1}) = \mathcal{F}_{\alpha_0}{}^{\alpha_1}{}_{\beta_1}{}^{\alpha_2}{}_{\beta_2}{}^{\alpha_3}\cdots{}_{\beta_{n-1}}{}^{\alpha_n}{}_{\alpha_{n+1}} \ , \tag{2.22}$$

where $\vec{a}=(a_1,\cdots,a_{n-1})$ and m_i are respectively the VEV of Coulomb branch operators and the mass parameters in the four-dimensional theory; α_i and β_i are the external and intermediate momenta in the Liouville theory, respectively. The relations between The 4d and 2d parameters are of

$$\frac{a_i}{\sqrt{\epsilon_1\epsilon_2}} = \beta_i - \frac{Q}{2} \ , \qquad \frac{m_i}{\sqrt{\epsilon_1\epsilon_2}} = \alpha_i - \frac{Q}{2} \ , \tag{2.23}$$

where ϵ_i is the Ω-background parameters that correspond to the Liouville charge by $Q = (\epsilon_1+\epsilon_2)/\sqrt{\epsilon_1\epsilon_2}$. In this paper, we rescale the Ω-background parameters by $\sqrt{\epsilon_1\epsilon_2}=1$. The SW curve of $\mathcal{T}_{\mathcal{C}}$ (2.3) is also identified as the insertion of the energy-momentum tensor into the $n+2$-points function in the classical limit $\epsilon_i \to 0$ as

$$x^2 = -\frac{\langle\alpha_0|T(z)V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)|\alpha_{n+1}\rangle}{\langle\alpha_0|V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)|\alpha_{n+1}\rangle} \ , \tag{2.24}$$

where the Virasoro primary state is defined by the state-operator map as $|\alpha\rangle \equiv \lim_{z\to 0} V_\alpha(z)|0\rangle$.

The AGT correspondence can be extended to $SU(N)$ gauge theory [73]. Then the algebra on two-dimensional theory is of W_N algebra.[6]

2.2.2 $U(2)$-version of original AGT correspondence

In [20], the Nekrasov partition function of a $U(2)$ gauge theory $\mathcal{Z}_{U(2)}$ is generally decomposed into that of the $SU(2)$ gauge theory $\mathcal{Z}_{SU(2)}$ and $U(1)$ factor $\mathcal{Z}_{U(1)}$, by

$$\mathcal{Z}_{U(2)} = \mathcal{Z}_{SU(2)}\mathcal{Z}_{U(1)} \ , \tag{2.25}$$

where the instanton part of $\mathcal{Z}_{SU(2)}$ is identical to a conformal block of the two-dimensional Liouville CFT as reviewed in subsection 2.2, and $\mathcal{Z}_{U(1)}$ is interpreted to the contribution of the $U(1)$ part of the gauge theory. On the 2d side, the $U(1)$ factor corresponds to the n-points function of chiral vertex operators for an extra Heisenberg algebra, which was studied in [45].

Suppose the linear quiver gauge theory described by the quiver diagram Figure 5, $\mathcal{Z}_{SU(2)}$ is identical to the $(n+2)$-points function of the Liouville CFT, while the $U(1)$ factor is identified as [44]

$$\mathcal{Z}_{U(1)} = \langle V^H_{\alpha_0}(z_0)\cdots V^H_{\alpha_{n+1}}(z_{n+1})\rangle \ , \tag{2.26}$$

where $V^H_\alpha(z) \equiv \exp\left(2(\alpha-Q)i\sum_{k<0}\frac{a_k}{k}z^{-k}\right)\exp\left(2\alpha i\sum_{k>0}\frac{a_k}{k}z^{-k}\right)$ is of the chiral vertex operator for Heisenberg algebra, and each of the coordinates z_k of these operators coincides with those of the Virasoro primary vertex operators in (2.22). Thus, we see that the Nekrasov partition function $Z_{U(2)}$ is identified with the $(n+2)$-points function of chiral vertex operators of the form

$$\widehat{V}_\alpha(z) \equiv V_\alpha(z)\otimes V^H_\alpha(z) \ . \tag{2.27}$$

Therefore we note that the original AGT correspondence is extended $U(2)$-version in case of the direct sum of Virasoro and Heisenberg algebra which we denote $Vir\oplus H$. Note that Virasoro generators L_k and Heisenberg generators a_k are commutative since we defined these algebras as the direct sum of the two algebras. Thus, our convention for the algebra $Vir\oplus H$ is such that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \ , \tag{2.28}$$

$$[a_n, a_m] = \frac{n}{2}\delta_{n+m,0} \ , \qquad [L_n, a_m] = 0 \ , \tag{2.29}$$

[6] In particular, for $N=3$, it has been checked by [74, 75].

and the action of $Vir \oplus H$ on $\widehat{V}_\alpha(z)$ is characterized by[7]

$$[L_n, \widehat{V}_\alpha(z)] = (z^{n+1} + (n+1)\alpha(Q-\alpha)z^n)\widehat{V}_\alpha(z) \ , \tag{2.31}$$

$$[a_n, \widehat{V}_\alpha(z)] = \begin{cases} -i\alpha z^n \widehat{V}_\alpha(z) & (n<0) \\ i(Q-\alpha)z^n\widehat{V}_\alpha(z) & (n>0) \end{cases} \ . \tag{2.32}$$

2.3 Generalized AGT correspondence

In this subsection, we will discuss the generalized AGT correspondence. The same argument as reviewed in subsection 2.2, the generalized AGT correspondence relates the instanton part of the Nekrasov partition function of the AD theories with an "irregular" conformal block of the Liouville theory. Moreover, we review the construction of the irregular state corresponding to the (A_1, D_N) theory for even N.

2.3.1 Generalization of AGT correspondence

According to the generalized AGT correspondence [21,26,27], the regular puncture and the irregular puncture correspond to a primary state $|a\rangle$ and irregular state $|I^{(N/2)}\rangle$ of Virasoro algebra, respectively. The irregular state is a simultaneous eigenstate of L_k for $k \geq \lfloor N/2 \rfloor$ as follows:[8]

$$L_k|I^{(N/2)}\rangle = \begin{cases} 0 & \text{for} \quad N < k \\ \lambda_k|I^{(N/2)}\rangle & \text{for} \quad \left\lfloor \frac{N}{2} \right\rfloor \leq k \leq N \end{cases} \ . \tag{2.33}$$

The generalized AGT correspondence implies that the Nekrasov partition function of the (A_1, D_N) theory is identified with the inner product of the regular state $|a\rangle$ and the irregular state $|I^{(N/2)}\rangle$ as

$$\mathcal{Z}_{(A_1,D_N)} = \langle a|I^{(N/2)}\rangle \ , \tag{2.34}$$

and SW curve of the theory is also identified as the classical limit of the insertion of the energy-momentum tensor into the inner product

$$x^2 = -\frac{\langle a|T(z)|I^{(N/2)}\rangle}{\langle a|I^{(N/2)}\rangle} \ . \tag{2.35}$$

[7] From the state-operator map, one obtains the highest weight state $|a\rangle$ of $Vir \oplus H$ as

$$|a\rangle = |a_V\rangle \otimes |a_H\rangle \ , \tag{2.30}$$

where $|a_V\rangle$ is the highest weight state of Virasoro algebra, and $|a_H\rangle$ is that of Heisenberg algebra.

[8] $\lfloor s \rfloor$ is the integer part of s.

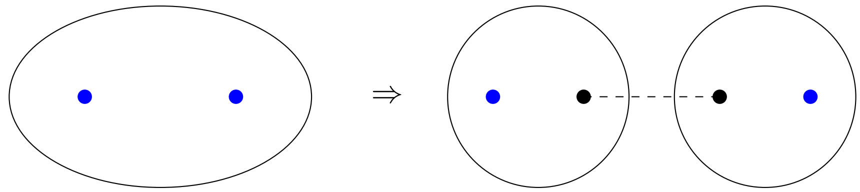

Figure 8: LHS: The Riemann surface of the theory described in Figure 3 being sphere with two irregular punctures. RHS: The surface stands for a weak coupling limit of the LHS theory.

Since the Virasoro generators define the energy-momentum tensor as $T(z) \equiv \sum y^{-k-2} L_k$, we see that (2.35) is evaluated from (2.33) as

$$x^2 = -\frac{\lambda_N}{z^{N+2}} - \frac{\lambda_{N-1}}{z^{N+1}} - \cdots - \frac{\Delta_a}{z^2} \ . \tag{2.36}$$

Let us now consider that $\mathcal{C}$ has two irregular punctures of rank $N/2$. In this case, the four-dimensional theory $\mathcal{T}_\mathcal{C}$ is asymptotically free, namely non-conformal field theory, and then the quiver diagram for this theory is described in Figure 3; we now call this "non-conformally gauged AD theory". Here, the flavor symmetry $SU(2) \subset SU(2) \times U(1)$ of the (A_1, D_N) theories are gauged by the $SU(2)$ vector multiplet.[9] This theory is engineered by compactifying on a surface $\mathcal{C}$ with two irregular punctures, which is represented by connecting the regular punctures of each of the punctured Riemann surfaces corresponding to the AD sectors, see Figure 8. The generalized AGT correspondence implies that the Nekrasov partition function of this theory is written as

$$\mathcal{Z}_{SU(2)}^{2\times(A_1,D_{2n})} = \langle I^{(n)} | I^{(n)} \rangle \ , \tag{2.37}$$

where $\langle I^{(N/2)}|$ and $|I^{(N/2)}\rangle$ correspond to each of the AD sectors.

2.3.2 *Colliding-limit construction*

Here, we show the construction of $|I^{(N/2)}\rangle$ for even N, based on [27]. The irregular state $|I^{(N/2)}\rangle$ for even N can be constructed by colliding $(\frac{N}{2}+1)$ Virasoro primary operators, see Figure 9. To see that, we define the state

$$|\phi_{\frac{N}{2}}(z_1, \ldots, z_{\frac{N}{2}})\rangle \equiv \ :\left(\prod_{i=1}^{N/2} V^L_{\alpha_i}(z_i)\right) V^L_{\alpha_0}(0) \colon |0\rangle \ , \tag{2.38}$$

[9]When N is equal to 4, the flavor symmetry is enhanced to $SU(3)$.

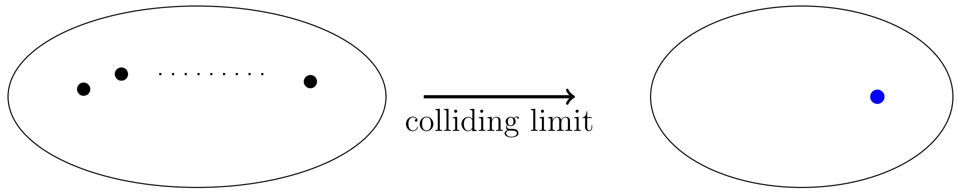

Figure 9: Left: There are $(\frac{N}{2}+1)$ black dots, each of wihch stand for Virasoro primary vertex operators. Right: Blue dot stand for a rank n irregular state constructed by *colliding limit*.

where $: AB :$ is the normal-orderd product of A and B. The action of the encoding singular behavior of the stress tensor $T_>(y) \equiv \sum_{k\geq -1} y^{-k-2} L_k$ on this state is given by

$$T_>(y)|\phi_{\frac{N}{2}}(z_1,\ldots,z_{\frac{N}{2}})\rangle = \sum_{i=0}^{N/2}\left(\frac{\alpha_i(Q-\alpha_i)}{y-z_i} + \frac{1}{y-z_i}\frac{\partial}{\partial z_i}\right)|\phi_{\frac{N}{2}}(z_1,\ldots,z_{\frac{N}{2}})\rangle \ , \tag{2.39}$$

where z_o is zero. Note that we consider the limit $z_i \to 0$ and $\alpha_i \to \infty$ with $c_k \equiv \sum_{i=0}^{N/2} \alpha_i z_i^k$ kept finite for $k = 0, \cdots, \frac{N}{2}$, and then the action (2.39) is reduced to

$$T_>(y)|I^{(N/2)}\rangle = \left(\sum_{k=N/2}^{N} \frac{\lambda_k}{y^{k+2}} + \sum_{k=0}^{N/2-1} \frac{\lambda_k + \sum_{\ell=1}^{N/2-k} \ell c_{\ell+k}\frac{\partial}{\partial c_\ell}}{y^{k+2}} + \frac{L_{-1}}{y}\right)|I^{(N/2)}\rangle \ , \tag{2.40}$$

where the irregular state and λ_k are defined as

$$|I^{(N/2)}\rangle \equiv \lim_{\text{colliding limit}} |\phi_{\frac{N}{2}}(z_1,\cdots,z_{\frac{N}{2}})\rangle \ , \tag{2.41}$$

and

$$\lambda_k = \begin{cases} -\sum_{\ell=k-N/2}^{N/2} c_\ell c_{k-\ell} & \text{for} \quad \frac{N}{2} < k \leq N \\ -\sum_{\ell=0}^{k} c_\ell c_{k-\ell} + (k+1)Q c_k & \text{for} \quad k \leq \frac{N}{2} \end{cases} . \tag{2.42}$$

Thus, L_k concretely acts on this state

$$L_k|I^{(N/2)}\rangle = \begin{cases} 0 \qquad \text{for} \qquad N<k \\ \lambda_k|I^{(N/2)}\rangle \qquad \text{for} \quad \frac{N}{2} \leq k \leq N \\ \left(\lambda_k + \sum_{\ell=1}^{N/2-k} \ell\, c_{\ell+k}\frac{\partial}{\partial c_\ell}\right)|I^{(N/2)}\rangle \qquad \text{for} \quad 0 \leq k < \frac{N}{2} \end{cases} . \tag{2.43}$$

Note that $|I^{(N/2)}\rangle$ is a simultaneous eigenstate of $L_{N/2}, \cdots, L_{N-1}$ and $L_N$with fixed by $c_0, \cdots, c_{N/2-1}$ and $c_{N/2}$. On the other hand, since (2.43) contains differential equations, it is not an eigenstate of $L_0 \cdots L_{N/2-2}$ and $L_{N/2-1}$ but depends on the boundary condition or the asymptotic behavior. As discussed in [27], $|I^{(N/2)}\rangle$ is completely fixed by $N/2$ extra parameters characterized the asymptotic behavior of $|I^{(N/2)}\rangle$ in the small c_k limit for $k = 1, \cdots, N/2$, in addition to $c_0, \cdots, c_{N/2-1}$ and $c_{N/2}$.

Next, we show how the 2d parameters are related to the 4d parameters. Note that the 4d theory has N parameters while the two-dimensional theory depends on $(N+1)$ parameters characterized by c_k for $k = 0, \cdots, N/2$ and $N/2$ extra parameters. Therefore, there is a discrepancy between the number of parameters of the 4d and 2d theories. To explain this discrepancy, we start with the SW curve of the (A_1, D_N) theory for even N [26, 27]

$$x^2 = \frac{a^2}{z^2} + \sum_{k=1}^{\frac{N}{2}-1} \frac{u_k}{z^{\frac{N}{2}+2-k}} + \frac{m}{z^{\frac{N}{2}+2}} + \sum_{k=1}^{\frac{N}{2}-1} \frac{d_k}{z^{N+2-k}} + \frac{1}{z^{N+2}} \ , \tag{2.44}$$

where u_k, d_k, and m are respectively the VEVs of Coulomb branch operators, the relevant couplings and, a mass parameter. On the other hand, by inserting (2.43) into (2.35), the SW curve on the 2d theory is identified as the classical limit of the following

$$x^2 = -\frac{\langle a|T(z)|I^{(N/2)}\rangle}{\langle a|I^{(N/2)}\rangle} = -\frac{\Delta_a}{z^2} + \cdots + \frac{2c_{\frac{N}{2}}c_{\frac{N}{2}-1}}{z^{N+1}} + \frac{c_{\frac{N}{2}}^2}{z^{N+2}} \ . \tag{2.45}$$

Here, we make the change of variables as $z \to (c_{N/2})^{\frac{2}{N}} z$ and $x \to (c_{N/2})^{-\frac{2}{N}} x$ in (2.45) so that the coefficient of $1/z^{N+2}$ is 1, then the SW curve (2.45) corresponds to (2.44). To be easy to understand this relation, we define new parameters by

$$\gamma_k \equiv \frac{c_k}{(c_{\frac{N}{2}})^{\frac{2k}{N}}} \qquad \text{for} \quad 0 \le k < \frac{N}{2} \ . \tag{2.46}$$

Then, the relation between the 2d and 4d parameters is given by

$$d_k = \sum_{\ell=\frac{N}{2}-k}^{\frac{N}{2}} \frac{c_\ell c_{N-k-\ell}}{(c_{\frac{N}{2}})^{2-\frac{2k}{N}}} = \sum_{\ell=\frac{N}{2}-k}^{\frac{N}{2}} \gamma_\ell \gamma_{N-k-\ell} \ , \qquad m = \sum_{\ell=0}^{\frac{N}{2}} \frac{c_\ell c_{\frac{N}{2}-\ell}}{c_{\frac{N}{2}}} = \sum_{\ell=0}^{\frac{N}{2}} \gamma_\ell \gamma_{\frac{N}{2}-\ell} \ , \tag{2.47}$$

$$u_k = \sum_{\ell=0}^{\frac{N}{2}-k} \frac{c_\ell c_{\frac{N}{2}-k-\ell}}{(c_{\frac{N}{2}})^{1-\frac{2k}{N}}} - \sum_{\ell=1}^{k} \ell \frac{c_{\frac{N}{2}+\ell-k}}{(c_{\frac{N}{2}})^{1-\frac{2k}{N}}} \frac{\partial \mathcal{F}_{(A_1,D_N)}}{\partial c_\ell} = \sum_{\ell=0}^{\frac{N}{2}-k} \gamma_\ell \gamma_{\frac{N}{2}-k-\ell} - \sum_{\ell=1}^{k} \ell \gamma_{\ell+\frac{N}{2}-k} \frac{\partial}{\partial \gamma_\ell} \mathcal{F}_{(A_1,D_N)} \ , \tag{2.48}$$

where $\mathcal{F}_{(A_1,D_N)}$ is the prepotential of the (A_1, D_N) theory as following

$$\mathcal{F}_{(A_1,D_N)} \equiv \lim_{\epsilon_i \to 0} \left(-\epsilon_1 \epsilon_2 \log \mathcal{Z}_{(A_1,D_N)} \right) \ . \tag{2.49}$$

Note that all the 4d parameters is independent of $c_{N/2}$ when we take $\vec{\gamma} \equiv (\gamma_0, \cdots, \gamma_{N/2-1})$ and $c_{N/2}$ as independent variables, but we need to prove the $c_{N/2}$ independence of $\frac{\partial}{\partial \gamma_\ell} \mathcal{F}_{(A_1,D_N)}$. We will explain this in subsection 3.1.

Recall that we consider the case of even N. The colliding limit construction is not known for odd N on this state, and therefore the actions of $L_0, \cdots, L_{\frac{N-1}{2}}$ have not been specified by the parameters c_k. We will discuss the action of Virasoro and Heisenberg generators on $|I^{(N/2)}\rangle$ for odd N in subsection 3.2.

3 $U(2)$-version of the generalized AGT correspondence

In this section, we will show the extension of the generalized AGT correspondence as reviewed in subsection 2.3 to $U(2)$-version. Then one obtains the Nekrasov-type formula for the (A_1, D_N) theory coupled to the $U(2)$ gauge group. In subsection 2.2, the $U(2)$-version of the original AGT correspondence has been obtained by the direct sum of the Virasoro primary vertex operator and the chiral vertex operator for extra Heisenberg algebra in (2.27). Our strategy is to introduce an irregular state of $Vir \oplus H$ and to find the Nekrasov-type formula for the (A_1, D_N) theory as a sum over a pair of two Young diagrams.

In the next subsection 3.1, we introduce the irregular state of $Vir \oplus H$ for rank even N and construct this by the colliding limit construction. Moreover, We will discuss the case of odd N in subsection 3.2.

3.1 In the case of $(A_1, D_{\mathbf{even}})$

We first show that an irregular state corresponding to the $U(2)$-version of the generalized AGT can be constructed by the colliding limit construction when N is even. Application to this state, we then propose the Nekrasov-type formula for the (A_1, D_{even}) theories labeled a pair of two Young diagrams.

3.1.1 Irregular state of $Vir \oplus H$

Recall that the original AGT correspondence is extended to the $U(2)$-version when the algebra in the 2d theory is $Vir \oplus H$. Here, We extend the generalized AGT correspondence to the $U(2)$-version. The same argument as in subsection 2.3, an irregular state $|\widehat{I}^{(N/2)}\rangle$ is a state in the highest module of $Vir \oplus H$. In particular, the irregular state for even N is constructed by colliding limit construction of the regular states (2.27). To see that, we start with the state

$$|\phi^H_{\frac{N}{2}}(z_1, \cdots, z_{\frac{N}{2}})\rangle \equiv \, :\left(\prod_{i=1}^{\frac{N}{2}} V^H_{\alpha_i}(z_i)\right) V^H_{\alpha_0}(0) : |0\rangle \ . \tag{3.1}$$

The action of $J_>(y) \equiv \sum_{k\geq 1} y^{-k-1} a_k$ on it is given by

$$J_>(y)|\phi^H_{\frac{N}{2}}(z_1, \cdots, z_{\frac{N}{2}})\rangle = -\sum_{i=0}^{\frac{N}{2}} \frac{i(Q-\alpha_i)z_i}{y(y-z_i)} |\phi^H_{\frac{N}{2}}(z_1, \cdots, z_{\frac{N}{2}})\rangle \ . \tag{3.2}$$

Similarly to (2.41), we take the colliding limit $z_i \to 0$ and $\alpha_i \to \infty$ with c_k kept fixed, and then we define the irregular state of Heisenberg algebra as

$$|I_H^{(N/2)}\rangle \equiv \lim_{\text{colliding limit}} |\phi^H_{\frac{N}{2}}(z_1, \cdots, z_{\frac{N}{2}})\rangle \ . \tag{3.3}$$

We see that the irregular state of Heisenberg algebra is characterized by

$$a_k|I_H^{(N/2)}\rangle = \begin{cases} 0 \qquad \text{for} \qquad \frac{N}{2} < k \\ -ic_k|I_H^{(N/2)}\rangle \qquad \text{for} \qquad 1 \le k \le \frac{N}{2} \end{cases} . \tag{3.4}$$

Since we now consider the direct sum of Viraso and Heisenberg algebras, the irregular state $|\widehat{I}^{(N/2)}\rangle$ is decomposed as

$$|\widehat{I}^{(N/2)}\rangle = |I^{(N/2)}\rangle \otimes |I_H^{(N/2)}\rangle \ , \tag{3.5}$$

where $|I^{(N/2)}\rangle$ is satisfied with (2.43). Hence the new irregular state is defined as follows:

$$L_k|\widehat{I}^{(N/2)}\rangle = \begin{cases} 0 \qquad \text{for} \qquad N < k \\ \lambda_k|\widehat{I}^{(N/2)}\rangle \qquad \text{for} \quad \frac{N}{2} \le k \le N \\ \left(\lambda_k + \sum_{\ell=1}^{\frac{N}{2}-k} \ell\, c_{\ell+k} \frac{\partial}{\partial c_\ell}\right) |\widehat{I}^{(N/2)}\rangle \qquad \text{for} \quad 0 \le k < \frac{N}{2} \end{cases} , \tag{3.6}$$

$$a_k|\widehat{I}^{(N/2)}\rangle = \begin{cases} 0 \qquad \text{for} \quad \frac{N}{2} < k \\ -ic_k|\widehat{I}^{(N/2)}\rangle \qquad \text{for} \quad 1 \le k \le \frac{N}{2} \end{cases} , \tag{3.7}$$

where we again denote λ as

$$\lambda_k = \begin{cases} -\sum_{\ell=k-N/2}^{N/2} c_\ell c_{k-\ell} \quad \text{for} \quad \frac{N}{2} < k \le N \\ -\sum_{\ell=0}^{k} c_\ell c_{k-\ell} + (k+1)Qc_k \quad \text{for} \quad 0 \le k \le \frac{N}{2} \end{cases} . \tag{3.8}$$

3.1.2 Nekrasov-type formula

According to (2.5), the instanton part of the Nekrasov partition function is decomposed as the contributions from the gauge sector and the matter sector. Let us start with the non-conformally gauged AD theory described in Figure 3 for even N. Recall that the flavor symmetry $SU(2) \subset SU(2) \times U(1)$ of the (A_1, D_N) theories is gauged by the $SU(2)$ vector multiplet, and this theory is constructed by compactifying 6d $\mathcal{N} = 2$ A_1 type SCFT on a sphere $\mathcal{C}$ with two irregular punctures. Since the theory involved the (A_1, D_N) theories in the matter sector, the Nekrasov partition function of the theory cannot be evaluated by the supersymmetric localization technique. However, we can read off the contribution from the (A_1, D_N) theory at the fixed point labeled by (Y_1, Y_2) on the $U(2)$ instanton moduli space via $U(2)$-version of the generalized AGT correspondence.

To see that, similarly to the case of Lagrangian description, we first propose that the instanton part of the Nekrasov partition function can be decomposed as

$$\mathcal{Z}_{U(2)}^{2\times(A_1,D_N)} = \mathcal{Z}_{\rm pert}\sum_{Y_1,Y_2}\Lambda^{b_0(|Y_1|+|Y_2|)}\mathcal{Z}^{\rm vec}_{Y_1,Y_2}(a)\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}(a,m,\boldsymbol{d},\boldsymbol{u})\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}(a,\widetilde{m},\widetilde{\boldsymbol{d}},\widetilde{\boldsymbol{u}})\ , \tag{3.9}$$

where $b_0 = 4/N$, and we regard $\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}$ and $\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}$ as the contributions from two different (A_1, D_N) sectors, and m, $\boldsymbol{d} = (d_1,\cdots,d_{\frac{N}{2}-1})$, and $\boldsymbol{u} = (u_1,\cdots,u_{\frac{N}{2}-1})$ (or $\widetilde{m}$, $\widetilde{\boldsymbol{d}} = (\widetilde{d}_1,\cdots,\widetilde{d}_{\frac{N}{2}-1})$, and $\widetilde{\boldsymbol{u}} = (\widetilde{u}_1,\cdots,\widetilde{u}_{\frac{N}{2}-1})$) respectively the mass parameter, the relevant couplings, and the VEVs of Coulomb branch operators in the (A_1, D_N) theory. The same is true of the parameters with a tilde. Since the two (A_1, D_N) theories have different couplings to the $U(1)$ subgroup of the gauge group, $\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}$ and $\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}$ are not identical.

On the 2d side, the Nekrasov partition function of the theory is given by

$$\mathcal{Z}_{U(2)}^{2\times(A_1,D_N)} = \langle\widehat{I}^{(N/2)}|\widehat{I}^{(N/2)}\rangle\ . \tag{3.10}$$

As shown in [44], there exists a unique orthogonal basis $|a;Y_1,Y_2\rangle$, which is a linear combination of descendants of $|a\rangle$ at level $(|Y_1|+|Y_2|)$[10] such that

$$\frac{\langle a;Y_1,Y_2|\widehat{V}_\alpha(1)|b;W_1,W_2\rangle}{\langle a|\widehat{V}_\alpha(1)|b\rangle} = \mathcal{Z}^{\rm bifund}_{Y_1,Y_2;W_1,W_2}(a,b,\alpha)\ , \tag{3.11}$$

where the RHS is the contribution from the bi-fundamental hypermultiplet shown in (2.12), $\widehat{V}_\alpha(1)$ is the vertex operator reviewed in subsection 2.2, and Y_k and W_k are Young diagrams. In particular, $|a;\emptyset,\emptyset\rangle \equiv |a\rangle$ is the highest weight state satisfying $\langle a|a\rangle$=1, $L_0|a\rangle = \Delta(a)|a\rangle$ and $L_n|a\rangle = a_n|a\rangle = 0$ for $n > 0$, and we denote a few examples of $|a;Y_1,Y_2\rangle$ in Appendix A. Here, the conjugate $\langle a;Y_1,Y_2|$ is not the usual hermitian conjugate of $|a;Y_1,Y_2\rangle$, i.e., for $|a;Y_1,Y_2\rangle$ given by a linear combination of $L^{m_1}_{-k_1}\cdots L^{m_p}_{-k_p}a^{n_1}_{-\ell_1}\cdots a^{n_q}_{-\ell_q}|a\rangle$. The conjugate $\langle a;Y_1,Y_2|$ is obtained by replacing each of these states with $\langle a|a^{n_q}_{\ell_q}\cdots a^{n_1}_{\ell_1}L^{m_p}_{k_p}\cdots L^{m_1}_{k_1}$ without changing the coefficients of the linear combination as discussed in [44]. And also, recall that $\mathcal{Z}^{\rm vec}_{Y_1,Y_2}(a) = 1/\mathcal{Z}^{\rm bifund}_{Y_1,Y_2;Y_1,Y_2}(a,a,0)$, the orthogonal basis is satisfied with an equation as

$$\mathbf{1} = \sum_{Y_1,Y_2}\mathcal{Z}^{\rm vec}_{Y_1,Y_2}(a)\,|a;Y_1,Y_2\rangle\langle a;Y_1,Y_2|\ . \tag{3.12}$$

[10] The "level" is the sum of the levels of Virasoro and Heisenberg algebras. For instance, $L^{m_1}_{-k_1}\cdots L^{m_p}_{-k_p}a^{n_1}_{-\ell_1}\cdots a^{n_q}_{-\ell_q}|a\rangle$ has level $\sum_{i=1}^p m_ik_i + \sum_{i=1}^q n_i\ell_i$.

Then inserting (3.12) into the RHS of (3.10), one is given by

$$\mathcal{Z}_{U(2)}^{2\times(A_1,D_N)} = \sum_{Y_1,Y_2} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a)\,\langle \widehat{I}^{(N/2)}|a;Y_1,Y_2\rangle\langle a;Y_1,Y_2|\widehat{I}^{(N/2)}\rangle\ . \tag{3.13}$$

Comparing (3.9) with (3.13), we can interpret the expression (3.13) as a sum over the fixed points of torus action on the $U(2)$ instanton moduli space. Recall that $|\widehat{I}^{(N/2)}\rangle$ and $\langle\widehat{I}^{(N/2)}|$ correspond to each of the AD theories. Therefore, we regard the factor $\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}$ and $\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}$ as respectively

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2} \sim \frac{\langle a;Y_1,Y_2|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle}\ ,\qquad \widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2} \sim \frac{\langle\widehat{I}^{(N/2)}|a;Y_1,Y_2\rangle}{\langle\widehat{I}^{(N/2)}|a\rangle}\ , \tag{3.14}$$

where we define the perturbative part as $\mathcal{Z}_{\text{pert}} \equiv \langle\widehat{I}^{(N/2)}|a\rangle\langle a|\widehat{I}^{(N/2)}\rangle$ since the perturbative part makes the power series in Λ start with 1.

Let us focus on the left relation of (3.14). As discussed in subsection 2.3, the LHS of (3.14) is independent of $c_{N/2}$, and then the $c_{N/2}$ dependence of the RHS needs to be canceled by a constant of proportionality. To see that dependence, let us consider the form

$$\langle a;Y_1,Y_2|(L_0-\Delta_{c_0})|\widehat{I}^{(N/2)}\rangle = (\Delta_a-\Delta_{c_0}+|Y_1|+|Y_2|)\,\langle a;Y_1,Y_2|\widehat{I}^{(N/2)}\rangle\ . \tag{3.15}$$

On the other hand, from (2.43), the above expression is identified in terms of $\vec{\gamma}$ and $c_{N/2}$ as

$$\frac{Nc_{\frac{N}{2}}}{2}\,\frac{\partial}{\partial c_{\frac{N}{2}}}\bigg|_{\vec{\gamma}}\langle a;Y_1,Y_2|\widehat{I}^{(N/2)}\rangle + \cdots\ , \tag{3.16}$$

where "$\cdots$" stands for the derivative with respect to $\vec{\gamma}$. The equivalence between the above two equations implies that

$$\langle a;Y_1,Y_2|\widehat{I}^{(N/2)}\rangle = (c_{\frac{N}{2}})^{2\frac{\Delta_a-\Delta_{c_0}+|Y_1|+|Y_2|}{N}} f_{Y_1,Y_2}(\vec{\gamma})\ , \tag{3.17}$$

where $f_{Y_1,Y_2}(\vec{\gamma})$ is independent of $c_N/2$. To see that all 4d parameters are indeed independent of $c_{N/2}$, as discussed in subsection 2.3, let us consider a function

$$\log\langle a|\widehat{I}^{(N/2)}\rangle = \frac{2(\Delta_a-\Delta_{c_0})}{N}\log c_{\frac{N}{2}} + \log f_{\emptyset,\emptyset}(\vec{\gamma})\ , \tag{3.18}$$

where we set $Y_1 = Y_2 = \emptyset$ in (3.17). This implies that $\frac{\partial}{\partial\gamma_\ell}\log\langle a|\widehat{I}^{(N/2)}\rangle = \frac{\partial}{\partial\gamma_\ell}\log f_{\emptyset,\emptyset}(\vec{\gamma})$ is independent of $c_{N/2}$. Note that the $c_{N/2}$-independence of all the 4d parameters follows from that of $\frac{\partial}{\partial\gamma_\ell}\log\langle a|\widehat{I}^{(N/2)}\rangle$.

The above results imply that the $c_{N/2}$ dependence of the RHS of (3.14) is canceled by the factor $c_{N/2}^{-2\frac{|Y_1|+|Y_2|}{N}}$, and therefore the left relation of (3.14) is explicitly given by

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2} = (\zeta c_{N/2})^{-2\frac{|Y_1|+|Y_2|}{N}} \frac{\langle a; Y_1, Y_2 | \widehat{I}^{(N/2)} \rangle}{\langle a | \widehat{I}^{(N/2)} \rangle} \ , \tag{3.19}$$

where ζ is a possible numerical constant independent of all variables.

The same argument as the above, we can also show the right relation of (3.14). Recall that the conjugate $\langle a; Y_1, Y_2|$ is not the usual hermitian conjugate of $|a; Y_1, Y_2\rangle$. Note that the factor $\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}$ is obtained from $\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}$ by replacement as follows:

$$c_k \longrightarrow -\widetilde{c}_k^* \ , \qquad Q \to -Q^* \tag{3.20}$$

Therefore the right relation of (3.14) is given by

$$\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2} = (-\zeta \widetilde{c}^*_{N/2})^{-2\frac{|Y_1|+|Y_2|}{N}} \frac{\langle \widehat{I}^{(N/2)} | a; Y_1, Y_2 \rangle}{\langle \widehat{I}^{(N/2)} | a \rangle} \ . \tag{3.21}$$

3.1.3 Identification of a dynamical scale

Here, we show how the 4d dynamical scale Λ is identified with the 2d parameters. Inserting (3.19) and (3.21) into (3.13), one is given as

$$\mathcal{Z}^{2\times(A_1,D_N)}_{Y_1,Y_2} = \mathcal{Z}_{\text{pert}} \sum_{Y_1,Y_2} (-\zeta^2 c_{N/2} \widetilde{c}^*_{N/2})^{2\frac{|Y_1|+|Y_2|}{N}} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}(a, m, \boldsymbol{d}, \boldsymbol{u}) \widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}(a, \widetilde{m}, \widetilde{\boldsymbol{d}}, \widetilde{\boldsymbol{u}}) \ , \tag{3.22}$$

where $\mathcal{Z}_{\text{pert}} \equiv \langle \widehat{I}^{(N/2)} | a \rangle \langle a | \widehat{I}^{(N/2)} \rangle$. Comparing (3.9) with (3.22), the 4d dynamical scale is identified with

$$\Lambda^2 = -\zeta^2 c_{N/2} \widetilde{c}^*_{N/2} \ . \tag{3.23}$$

Recall that the $c_{N/2}$ (or $\widetilde{c}^*_{N/2}$) independence of the (A_1, D_N) theory is related to conformal invariance. Since the $U(2)$ gauge coupling breaks the conformal symmetry of the two (A_1, D_N) theories via the dynamical scale, it is natural that Λ depends on $c_{N/2}$ and $\widetilde{c}^*_{N/2}$.

We show that the above proposals (3.19) and (3.21) are consistent with the SW curve. The SW curve of the theory is written as [26]

$$x^2 = \widetilde{\Lambda}^2 z^{N-2} + \cdots + \frac{\widetilde{\Lambda}^2}{z^{N+2}} \ , \tag{3.24}$$

where $\widetilde{\Lambda}$ is equivalent to the dynamical scale Λ up to a numerical factor. While on the 2d theory, one can be evaluated as follows:

$$x^2 = -\frac{\langle \widehat{I}^{(N/2)}|T(z)|\widehat{I}^{(N/2)}\rangle}{\langle \widehat{I}^{(N/2)}|\widehat{I}^{(N/2)}\rangle} = (\widetilde{c}^*_{N/2})^2 z^{N-2} + \cdots + \frac{(c_{N/2})^2}{z^{N+2}} \ . \tag{3.25}$$

We here change the variables in (3.25) as $z \to z\,(-c_{N/2}/\widetilde{c}^*_{N/2})^{\frac{1}{N}}$ and $x \to x\,(-c_{N/2}/\widetilde{c}^*_{N/2})^{-\frac{1}{N}}$ so that the coefficient of z^{N-2} is equivalent to that of $1/z^{N+2}$, then these coefficients are $(-c_{N/2}\widetilde{c}^*_{N/2})$. Comparing this with (3.24), we see that $\widetilde{\Lambda}^2 = -c_{N/2}\widetilde{c}^*_{N/2}$. This relation is consistent with (3.23) up to a numerical factor ζ.

3.1.4 Consistency with a free fundamental hypermultiplet

Here, we show that our proposals are consistent. Recall that the (A_1, D_2) theory is a theory of a free hypermultiplet in the doublet of $U(2)$. Therefore, we can check the consistency of our proposals to compare the Nekrasov formula for the (A_1, D_2) theory with that for (anti-)fundamental hypermultiplet.

We first consider (3.19). When N is two, the irregular state $|\widehat{I}^{(1)}\rangle$ is characterized by

$$L_k|\widehat{I}^{(1)}\rangle = \quad 0 \qquad \text{for} \qquad k > 2 \ , \tag{3.26}$$

$$L_2|\widehat{I}^{(1)}\rangle = -c_1^2|\widehat{I}^{(1)}\rangle \ , \tag{3.27}$$

$$L_1|\widehat{I}^{(1)}\rangle = 2(Q - c_0)c_1|\widehat{I}^{(1)}\rangle \ , \tag{3.28}$$

$$L_0|\widehat{I}^{(1)}\rangle = \left(\Delta_{c_0} + c_1\frac{\partial}{\partial c_1}\right)|\widehat{I}^{(1)}\rangle \ , \tag{3.29}$$

and

$$a_k|\widehat{I}^{(1)}\rangle = \quad 0 \qquad \text{for} \qquad k > 1 \ , \tag{3.30}$$

$$a_1|\widehat{I}^{(1)}\rangle = -ic_1|\widehat{I}^{(1)}\rangle \ . \tag{3.31}$$

By computing the ratio $\langle a; Y_1, Y_2|\widehat{I}^{(1)}\rangle/\langle a|\widehat{I}^{(1)}\rangle$, we then find that [11]

$$\left(-\frac{c_1}{2}\right)^{-|Y_1|-|Y_2|}\frac{\langle a; Y_1, Y_2|\widehat{I}^{(1)}\rangle}{\langle a|\widehat{I}^{(1)}\rangle} = \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, m) \ , \tag{3.32}$$

where $\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}$ is the contribution of a fundamental hypermultiplet as appeared in subsection 2.1, We see that (3.32) is entirely consistent with (3.19) for $\zeta = -1/2$. The mass parameter

[11] We checked this statement for $|Y_1| + |Y_2| \geq 6$

m is related by

$$m = c_0 - \frac{Q}{2} \ . \tag{3.33}$$

The above relation is also consistent with (2.47) up to a numerical factor.

The same argument as the above, we can check the consistency of an anti-fundamental hypermultiplet. Let us consider (3.21); we then find that

$$\left(\frac{\widetilde{c}_1^*}{2}\right)^{-|Y_1|-|Y_2|} \frac{\langle \widehat{I}^{(1)}|a; Y_1, Y_2\rangle}{\langle \widehat{I}^{(1)}|a\rangle} = \mathcal{Z}^{\text{anti-fund}}_{Y_1,Y_2}(a, \widetilde{m}) \ , \tag{3.34}$$

and the mass parameter $\widetilde{m}$ is also relate to $\widetilde{c}_0$ by

$$\widetilde{m} = \left(\widetilde{c}_0 - \frac{Q}{2}\right)^* \ . \tag{3.35}$$

This is also consistent with our proposal for $\zeta = -1/2$.

3.2 In the case of $(A_1, D_{\mathbf{odd}})$

In this subsection, we extend the $U(2)$-version generalized AGT correspondence to the case of the (A_1, D_N) theories for odd N. Recall that these theories are also obtained by compactifying 6d $\mathcal{N} = (2, 0)$ SCFT of type A_1, therefore, similar to the case of even N, we assume that the Nekrasov formula for the (A_1, D_N) theories for odd N is identified with

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2} \sim \frac{\langle a; Y_1, Y_2|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} \ . \tag{3.36}$$

Here, the irregular state $|\widehat{I}^{(N/2)}\rangle$ for odd N is decomposed as $|\widehat{I}^{(N/2)}\rangle = |I^{(N/2)}\rangle \otimes |I_H^{(N/2)}\rangle$, where $|I^{(N/2)}\rangle$ is the irregular state of Virasoro algebra satisfied with (2.33), and $|I_H^{(N/2)}\rangle$ is that of Heisenberg algebra. For odd N, the irregular state $|I^{(N/2)}\rangle$ cannot be obtained by the colliding limit construction, hence an explicit expression for the action of $L_1, \cdots, L_{\frac{N-1}{2}}$ on $|I^{(N/2)}\rangle$ has not been identified. Moreover, it is difficult to find the action of $a_1, \cdots, a_{\frac{N-1}{2}}$ on $|I_H^{(N/2)}\rangle$, and therefore one cannot compute

$$\frac{\langle a|a_{m_\ell}^{q_\ell} \cdots a_1^{q_1} L_{n_k}^{p_k} \cdots L_{n_1}^{p_1}|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} \ , \tag{3.37}$$

for $n_i > 0$ and $m_i > 0$ without understanding these actions. This means that computation of (3.36) looks hard since $\langle a; Y_1, Y_2|$ is a linear combination of $\langle a|a_{m_\ell}^{q_\ell} \cdots a_1^{q_1} L_{n_k}^{p_k} \cdots L_{n_1}^{p_1}$. However, we will discuss that (3.37) can be computed if we focus on the classical limit $\epsilon_i \to 0$ and turn off the relevant couplings and the VEV of the Coulomb branch operators in the (A_1, D_N) theory below.

3.2.1 Matrix elements in the classical lmit

Here, we show that subalgebra $\{L_{n>0}\}$ reduces to the commutative algebra in the classical limit $\epsilon_i \to 0$. Then we propose the conjecture that the matrix elements can be evaluated by the product of follows:

$$\mathfrak{a}_m \equiv \frac{\langle a|a_m|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} , \qquad \mathfrak{b}_n \equiv \frac{\langle a|L_n|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} , \tag{3.38}$$

for $n, m > 0$.

To see that, we focus on the four-dimensional scaling dimension. Recall that the SW curve of the (A_1, D_N) theory is written as (2.35). Since the SW 1-form xdz has four-dimensional scaling dimension 1, we see that the stress-energy tensor and the coordinates (z, x) in (2.35) have four-dimensional scaling dimension $\Delta_{4\mathrm{d}}(z) = -2/N$ and $\Delta_{4\mathrm{d}}(T(z)) = \Delta_{4\mathrm{d}}(x^2) = 2(1+2/N)$, and therefore Virasoro generators associated with four-dimensional scaling dimension

$$\Delta_{4\mathrm{d}}(L_n) = 2\left(1 - \frac{n}{N}\right) . \tag{3.39}$$

Recall that we set $\epsilon_1\epsilon_2$. When we need to recover the full ϵ_i-dependence, we rescale all quantity of dimension $\Delta_{4\mathrm{d}}$ in two-dimensional theory by $(\epsilon_1\epsilon_2)^{\Delta_{4\mathrm{d}}/2}$. Therefore, the Virasoro generators L_n are replaced by $(\epsilon_1\epsilon_2)^{1-\frac{n}{N}}$, and one can rewrite Virasoro algebra (2.28) as following:

$$[L_n, L_m] = (n-m)(\epsilon_1\epsilon_2)L_{n+m} , \tag{3.40}$$

for $m, n > 0$. This implies that the subalgebra $\{L_{n>0}\}$ reduces to the commutative algebra in the classical limit. Therefore, in the classical limit, we can regard $\{L_{n>0}\}$ and $\{a_{n>0}\}$ as commutative, and then they have a simultaneous eigenstate. We propose that the irregular state is asymptotically equivalent to a simultaneous eigenstate in the classical limit.[12] Then the matrix elements (3.37) are evaluated by (3.38) as

$$\frac{\langle a|a_{m_\ell}^{q_\ell}\cdots a_{m_1}^{q_1}L_{n_k}^{p_k}\cdots L_{n_1}^{p_1}|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} = \left(\prod_{i=1}^{\ell}(\mathfrak{a}_{m_i})^{q_i}\right)\left(\prod_{j=1}^{k}(\mathfrak{b}_{n_j})^{p_j}\right) . \tag{3.41}$$

where $\mathfrak{b}_\mathfrak{n}$ for $\left\lfloor \frac{N}{2} \right\rfloor \le n \le N$ are eigenvalues λ_n in (2.33).

[12]When $N = 4$, despite the action of L_1 on the irregular state $|\widehat{I}^{(2)}\rangle$ involves $c_2\frac{\partial}{\partial c_1}$, $|\widehat{I}^{(2)}\rangle$ approaches to a simultaneous eigenstate in the classical limit, and then the eigenvalue of L_1 regarded as the VEV of the Coulomb branch operator u_1 in section 4.

3.2.2 Nekrasov-type formula for $(A_1, D_{\mathbf{odd}})$

Similarly to the case of even N, we must remove an unphysical degree of freedom from the RHS in (3.36). To see that, let us first consider evaluating the four-dimensional scaling dimensions of $\{\mathfrak{a}_\mathfrak{m}\}$ and $\{\mathfrak{b}_\mathfrak{n}\}$. From (3.39), we see that

$$\Delta_{\rm 4d}\left(\mathfrak{b}_n\right) = 2\left(1-\frac{n}{N}\right) \ , \tag{3.42}$$

Note that $\mathfrak{b}_\mathfrak{n}$ for $n > N$ have negative scaling dimensions; therefore, we see that these parameters stand for irrelevant couplings. However, the prepotential related to the Nekrasov partition function in (1.1) is defined in the infrared, so such irrelevant couplings must be zero. Indeed, this is consistent with (2.33).

The 4d scaling dimension of $\{\mathfrak{a}_\mathfrak{m}\}$ are evaluated by the highest module $|a; Y_1, Y_2\rangle$ of $Vir \oplus H$ as shown in Appendix A [45]. For instance, $\mathcal{Z}^{(A_1,D_N)}_{\square,\emptyset}$ is evaluated by (A.2) and (3.38) as

$$\mathcal{Z}^{(A_1,D_N)}_{\square,\emptyset} \sim -i(\epsilon_1+\epsilon_2+2a)\mathfrak{a}_1 - \mathfrak{b}_1 \ . \tag{3.43}$$

Here, the 4d scaling dimensions of two terms in (3.43) must be equivalent. Recall that the highest weight a of $|a\rangle$ is the VEV of the scalar in $SU(2)$ vector multiplet in the four-dimensional theory, which has 4d scaling dimension one. Also, we see that Ω-background parameters have 4d scaling dimension one, i.e. $\Delta_{\rm 4d}(a) = \Delta_{\rm 4d}(\epsilon_1) = \Delta_{\rm 4d}(\epsilon_2) = 1$. Thus, one obtains

$$\Delta_{\rm 4d}(\mathfrak{a}_1) = \Delta_{\rm 4d}(\mathfrak{b}_1) - 1 = 1 - \frac{2}{N} \ . \tag{3.44}$$

Similarly, one obtains for $\mathcal{Z}^{(A_1,D_N)}_{\square\square,\emptyset}$ and $\mathcal{Z}^{(A_1,D_N)}_{\square\square\square,\emptyset}$ follows:

$$\Delta_{\rm 4d}(\mathfrak{a}_2) = 2\Delta_{\rm 4d}(\mathfrak{b}_1) - 3 = 1 - \frac{4}{N} \ , \tag{3.45}$$

$$\Delta_{\rm 4d}(\mathfrak{a}_3) = 3\Delta_{\rm 4d}(\mathfrak{b}_1) - 5 = 1 - \frac{6}{N} \ . \tag{3.46}$$

We can find the 4d scaling dimension of $\mathfrak{a}_\mathfrak{m}$ for all m by the same analysis for $\mathcal{Z}^{(A_1,D_N)}_{Y_1,\emptyset}$ with $Y_1 = [1, \cdots, 1]$. The basis $|a; Y_1, \emptyset\rangle$ are written as

$$\begin{aligned} |a;\, Y_1 = [1,\cdots,1],\, \emptyset\rangle = \Bigg(& \mathcal{N}(Y_1)\,\epsilon_1^{|Y_1|-1}\left(\prod_{j=1}^{|Y_1|}(2a+j\epsilon_1+\epsilon_2)\right) a_{-|Y_1|} \\ & + \ (-L_{-1})^{|Y_1|} \ + \ \cdots \Bigg)|a\rangle \ , \end{aligned} \tag{3.47}$$

where a numerical factor $\mathcal{N}(Y_1)$ is independent of ϵ_1 and ϵ_2.[13] We then obtain the factor $\mathcal{Z}_{[1,\cdots,1],\emptyset}$ as

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1=[1,\cdots,1],\emptyset} \sim \mathcal{N}(Y_1)\, \epsilon_1^{|Y_1|-1} \prod_{j=1}^{|Y_1|}(2a+j\epsilon_1+\epsilon_2)\mathfrak{a}_{|Y_1|} + (-\mathfrak{b}_1)^{|Y_1|} + \cdots \ . \tag{3.51}$$

This implies that $\{\mathfrak{a}_{\mathfrak{m}}\}$ have 4d scaling dimension:

$$\Delta_{\text{4d}}(\mathfrak{a}_m) = m\Delta_{\text{4d}}(\mathfrak{b}_1) - 2m + 1 = 1 - \frac{2m}{N} \ . \tag{3.52}$$

Note that the SW curve of the (A_1, D_N) theory is described by (3.38) as

$$\begin{aligned} x^2 = \frac{1}{z^{N+2}} - \Bigg(&\frac{\mathfrak{b}_{N-1}}{(-\mathfrak{b}_N)^{\frac{N-1}{N}}}\frac{1}{z^{N+1}} + \frac{\mathfrak{b}_{N-2}}{(-\mathfrak{b}_N)^{\frac{N-2}{N}}}\frac{1}{z^N} + \cdots \\ &+ \frac{\mathfrak{b}_{\frac{N+1}{2}}}{(-\mathfrak{b}_N)^{\frac{N+1}{2N}}}\frac{1}{z^{\frac{N+5}{2}}} + \cdots + \frac{\mathfrak{b}_1}{(-\mathfrak{b}_N)^{\frac{1}{N}}}\frac{1}{z^3}\Bigg) + \frac{a^2}{z^2} \ , \end{aligned} \tag{3.53}$$

where we made the change of variables as $x \to (-\mathfrak{b}_N)^{-\frac{1}{N}}x$ and $z \to (-\mathfrak{b}_N)^{\frac{1}{N}}z$ in (3.53) so that the coefficient of $1/z^{N+2}$ is 1. Since the (A_1, D_N) theory has no exactly marginal coupling and $\Delta_{4d}(L_0) = \Delta_{4d}(\mathfrak{b}_N) = 0$, the parameter $\mathfrak{b}_N$ has no counterpart in four-dimensional theory. This means that $\mathfrak{b}_N$ is an unphysical degree of freedom in the four-dimensional theory. Since the VEVs of the Coulomb branch operators have 4d scaling dimension larger than one, we see that $\mathfrak{b}_1, \cdots, \mathfrak{b}_{\frac{N-1}{2}}$ are identified with the VEVs of the Coulomb branch operators. On the other hand, the relevant couplings have 4d scaling dimension smaller than one, and therefore $\mathfrak{b}_{\frac{N+1}{2}}, \cdots, \mathfrak{b}_{N-1}$ are identified with the relevant couplings.

---

[13] As shown in [45], The Virasoro generators $L_{n>0}$ are rewritten by the free boson representation as

$$L_n = \sum_{k\neq 0,n} c_k c_{k-n} + i(nQ-2a)c_n \ , \tag{3.48}$$

where $[c_n, c_m]\frac{n}{2}\delta_{n+m,0}$. Then the orthogonal basis $|a; Y_1, \emptyset\rangle$ are obtained as

$$|a; Y_1, \emptyset\rangle = \Omega_{Y_1}(a)\mathbf{J}^{(-\epsilon_2^2)}_{Y_1}(x)|a\rangle \ , \tag{3.49}$$

where $\Omega_{Y_1}(a) \equiv (-\epsilon_1)^{|Y_1|}\prod_{(j,k)\in Y_1}(2a+j\epsilon_1+k\epsilon_2)$, and $\mathbf{J}^{(-\epsilon_2^2)}_{Y_1}(x)$ is the normalized Jack polynomial. We here recovered the full ϵ_i-dependence. The variables $x \equiv (x_1, x_2, \cdots)$ are related to

$$a_{-n} - c_{-n} = -i\epsilon_1 p_n(x) \ , \tag{3.50}$$

where $p_n(x) \equiv \sum_{i=1}^{|Y_1|} x_i^n$.

Let us now turn off all the relevant couplings and the VEVs of the Coulomb branch operators in the (A_1, D_N) theory. This means that

$$\mathfrak{b}_n = 0\ , \qquad \text{for} \qquad n \neq N\ , \tag{3.54}$$

and this is equivalent to the condition that $\mathfrak{b}_n = 0$ unless $\Delta_{4d}(\mathfrak{b}_n) = 0$. We assume that $\mathfrak{a}_m$ is independent of a; therefore, $\mathfrak{a}_m$ is equal to zero unless $\Delta_{4d}(\mathfrak{a}_m) = 0$. According to (3.52), $\Delta_{4d}(\mathfrak{a}_m) \neq 0$ for odd N, and therefore one obtains

$$\mathfrak{a}_m = 0\ , \tag{3.55}$$

for all m. We conclude that the matrix elements (3.37) reduce to

$$\frac{\langle a|a_{m_\ell}^{q_\ell}\cdots a_{m_1}^{q_1}L_{n_k}^{p_k}\cdots L_{n_1}^{p_1}|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} = \begin{cases} 1 \qquad \text{for} \quad \ell = k = 0 \\ \delta_{n_1,N}(\mathfrak{b}_N)^{p_1} \qquad \text{for} \quad \ell = 0,\ \ k = 1 \\ 0 \qquad \text{for the others} \end{cases}, \tag{3.56}$$

when all the relevant couplings and the VEVs of the Coulomb branch operators in the (A_1, D_N) theory are turned off. To compute the Nekrasov partition function of the (A_2, A_5) theory, we will use this statement in section (5).

In the remaining part, we explicitly identified the Nekrasov formula $\mathcal{Z}_{Y_1,Y_2}^{(A_1,D_N)}$. Recall that $\mathfrak{b}_N$ is the unphysical degree of freedom in the four-dimensional theory. Hence, the unphysical parameter $\mathfrak{b}_N$ must be removed from the RHS on (3.36). The result (3.56) is implies that $\langle a; Y_1, Y_2|\widehat{I}^{(N/2)}\rangle/\langle a|\widehat{I}^{(N/2)}\rangle$ is proportional to $(\mathfrak{b}_N)^{\frac{|Y_1|+|Y_2|}{N}}$, and therefore the Nekrasov formula for the (A_1, D_N) theory for odd N is identified with

$$\mathcal{Z}_{Y_1,Y_2}^{(A_1,D_N)}(a) = (\xi\mathfrak{b}_N)^{-\frac{|Y_1|+|Y_2|}{N}}\ \frac{\langle a; Y_1, Y_2|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle}\ , \tag{3.57}$$

where ξ is a possible numerical factor related to the rescaling of the dynamical scale.

3.2.3 The a independence of $\mathfrak{a}_m$

Here, we give supporting evidence for the a-independence of $\mathfrak{a}_m$. Let us consider the theory described in Figure 3. According to the $U(2)$-version of the generalized AGT correspondence, the Nekrasov partition function for the $U(2)$ gauge group is given as $\mathcal{Z}_{U(2)}(a) = \langle\widehat{I}^{(N/2)}|\widehat{I}^{(N/2)}\rangle$, and by inserting (3.12) into this, one obtains

$$\mathcal{Z}_{U(2)}(a) = \sum_{Y_1,Y_2} \Lambda^{b_0(|Y_1|+|Y_2|)}\mathcal{Z}_{Y_1,Y_2}^{\text{vec}}(a)\langle\widehat{I}^{(N/2)}|a; Y_1, Y_2\rangle\langle a; Y_1, Y_2|\widehat{I}^{(N/2)}\rangle\ . \tag{3.58}$$

where $\langle\widehat{I}^{(N/2)}|$ and $|\widehat{I}^{(N/2)}\rangle$ correspond to two (A_1, D_N) sectors. On the other hand, the relation between the partition functions for $U(2)$ and $SU(2)$ gauge group is

$$\mathcal{Z}_{U(2)}(a) = \mathcal{Z}_{U(1)}\,\mathcal{Z}_{SU(2)}(a)\ , \tag{3.59}$$

where the $U(1)$-factor $\mathcal{Z}_{U(1)}$ is independent of a since the VEV of the scalar field in the $SU(2)$ vector multiplet and a is neutral under the $U(1)$ gauge group. the $SU(2)$-version of the generalized AGT correspondence implies $\mathcal{Z}_{SU(2)}(a) = \langle I^{(N/2)}|I^{(N/2)}\rangle$. Furthermore, in analogy with [44], the $U(1)$-factor is identified with $\mathcal{Z}_{U(1)} = \langle I_H^{(N/2)}|I_H^{(N/2)}\rangle$. Below, we set all the parameters in two AD theories to be equal since $\langle\widehat{I}^{(N/2)}|$ is the conjugate of the state $|\widehat{I}^{(N/2)}\rangle$. Then $\mathcal{Z}_{U(2)}(a)$ is written as

$$\mathcal{Z}_{U(2)}(a) = \left||\widehat{I}^{(N/2)}\rangle\right|^2 = \left||I^{(N/2)}\rangle\right|^2\,\left||I_H^{(N/2)}\rangle\right|^2\ , \tag{3.60}$$

Note that $\mathcal{Z}_{SU(2)}$ and $\mathcal{Z}_{U(1)}$ can be expanded in power of the dynamical scale Λ as

$$\mathcal{Z}_{SU(2)}(a) = \sum_{k=0}\Lambda^{b_0 k} Z_k^{SU(2)}(a)\ , \qquad \mathcal{Z}_{U(1)} = \sum_{\ell=0}\Lambda^{b_0\ell} Z_\ell^{U(1)}\ , \tag{3.61}$$

where $b_0 \equiv 4/N$ is the cofficient of the one-loop beta function. Since the identification $\mathcal{Z}_{U(1)} = \left||I_H^{(N/2)}\rangle\right|^2$ implies that the expansion coefficients $Z_\ell^{U(1)}$ are related to the state $|I_H^{(N/2)}\rangle$, we claim that $Z_\ell^{U(1)}$ are provided by $\langle a_H|I_H^{(N/2)}\rangle$ and

$$\mathfrak{a}_m \equiv \frac{\langle a|a_m|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} = \frac{\langle a_H|a_m|I_H^{(N/2)}\rangle}{\langle a_H|I_H^{(N/2)}\rangle}\ , \tag{3.62}$$

where $|a_H\rangle$ is the highest weight state of Heisenberg algebra that we have used in (2.30). Using (3.61) and (3.59), one can obtain

$$\mathcal{Z}_{U(2)}(a) = \sum_{k,\ell=0}\Lambda^{b_0(k+\ell)} Z_k^{SU(2)}(a)\ Z_\ell^{U(1)}\ . \tag{3.63}$$

By comparing this with (3.58) order by order, we can read off $Z_k^{SU(2)}(a)$ and $Z_k^{U(1)}$. Let us perform this comparison at $\mathcal{O}(\Lambda^0)$, $\mathcal{O}(\Lambda^{b_0})$, and $\mathcal{O}(\Lambda^{2b_0})$ below.

We first compare the terms of $\mathcal{O}(\Lambda^0)$, then we will find

$$Z_0^{SU(2)}(a)\,Z_0^{U(1)} = |\langle a_V|I^{(N/2)}\rangle|^2\,|\langle a_H|I_H^{(N/2)}\rangle|^2\ , \tag{3.64}$$

where we used (2.30) and (3.5). Since the identification $Z_0^{SU(2)}(a) = |\langle a_V|I^{(N/2)}\rangle|^2$ has been already discussed by [26, 27], we see that the identification

$$Z_0^{U(1)} = |\langle a_H|I_H^{(N/2)}\rangle|^2\ . \tag{3.65}$$

Note that the a-independence of $|\langle a_H|I_H^{(N/2)}\rangle|$ follows from that of $\mathcal{Z}_0^{U(1)}$.

We compare the terms of higher orders of Λ. To simplify this comparison, we set $\epsilon_1 = 1/\epsilon_2 = i$ below. For the terms of $\mathcal{O}(\Lambda^{b_0})$, the equation of this comparison is written as

$$
\begin{aligned}
&Z_1^{SU(2)}(a)\, Z_0^{U(1)} + Z_0^{SU(2)}(a)\, Z_1^{U(1)} \\
&\quad = \mathcal{Z}^{\text{vec}}_{\square,\emptyset}\langle \widehat{I}^{(N/2)}|a;\square,\emptyset\rangle\langle a;\square,\emptyset|\widehat{I}^{(N/2)}\rangle + \mathcal{Z}^{\text{vec}}_{\emptyset,\square}\langle \widehat{I}^{(N/2)}|a;\emptyset,\square\rangle\langle a;\emptyset,\square|\widehat{I}^{(N/2)}\rangle\ .
\end{aligned}
\tag{3.66}
$$

Using the identification of $Z_0^{SU(2)}$, (3.65) and the orthogonal basis $|a;Y_1,Y_2\rangle$ shown in [44], one can obtain the RHS as

$$
-\frac{1}{2a^2}\left|\langle a_V|L_1|I^{(N/2)}\rangle\right|^2 \mathcal{Z}_0^{U(1)} + 2\mathcal{Z}_0^{SU(2)}(a)\left|\langle a_H|a_1|I_H^{(N/2)}\rangle\right|^2\ . \tag{3.67}
$$

If we identify $Z_1^{SU(2)}$ and $Z_1^{U(1)}$ as

$$
Z_1^{SU(2)}(a) = -\frac{1}{2a^2}|\langle a_V|L_1|I^{(N/2)}\rangle|^2\ , \qquad Z_1^{U(1)} = 2|\langle a_H|a_1|I_H^{(N/2)}\rangle|^2\ , \tag{3.68}
$$

it is a solution of the equation (3.66). Hence, under the identification, we see that the a-independence of $|\langle a_H|a_1|I_H^{(N/2)}\rangle|$ follows from that of $\mathcal{Z}_0^{U(1)}$.

For the terms of $\mathcal{O}(\Lambda^{2b_0})$, one obtains

$$
\begin{aligned}
&Z_2^{SU(2)}(a)\, Z_0^{U(1)} + Z_1^{SU(2)}(a)\, Z_1^{U(1)} + Z_0^{SU(2)}(a)\, Z_2^{U(2)} \\
&\quad = \mathcal{Z}^{\text{vec}}_{\square\square,\emptyset}|\langle a;\square\square,\emptyset|\widehat{I}^{(N/2)}\rangle|^2 + \mathcal{Z}^{\text{vec}}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset}|\langle a;\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset|\widehat{I}^{(N/2)}\rangle|^2 + \mathcal{Z}^{\text{vec}}_{\square,\square}|\langle a;\square,\square|\widehat{I}^{(N/2)}\rangle|^2 \\
&\quad + \mathcal{Z}^{\text{vec}}_{\emptyset,\square\square}|\langle a;\emptyset,\square\square|\widehat{I}^{(N/2)}\rangle|^2 + \mathcal{Z}^{\text{vec}}_{\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}}|\langle a;\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}|\widehat{I}^{(N/2)}\rangle|^2\ .
\end{aligned}
\tag{3.69}
$$

Using the above identifications, we see that natural identification

$$
\begin{aligned}
Z_2^{SU(2)} &= \frac{1}{(1+4a^2)^2}\Bigg[4(-1+2a^2)^2\,|\langle a_V|L_2|I^{(N/2)}\rangle|^2 + \frac{-1+8a^2}{4a^4}\frac{|\langle a_V|L_1|I^{(N/2)}\rangle|^4}{|\langle a_V|I^{(N/2)}\rangle|^4} \\
&\qquad - 3\left(\frac{\langle I^{(N/2)}|L_{-2}|a_V\rangle\langle a_V|L_1|I^{(N/2)}\rangle^2}{\langle a_V|I^{(N/2)}\rangle} + \frac{\langle I^{(N/2)}|(L_{-1})^2|a_V\rangle\langle a_V|L_2^{(N/2)}\rangle}{\langle I^{(N/2)}|a_V\rangle}\right)\Bigg]
\end{aligned}
\tag{3.70}
$$

$$
Z_2^{U(1)} = |\langle a_H|a_2|I_H^{(N/2)}\rangle|^2 + 2\frac{|\langle a_H|a_1|I_H^{(N/2)}\rangle|^4}{|\langle a_H|I_H^{(N/2)}\rangle|^2}\ . \tag{3.71}
$$

solves the equation (3.69). Under the above identifications, since $Z_2^{U(2)}$, $|\langle a_H|a_1|I_H^{(N/2)}\rangle|$, and $|\langle a_H|I_H^{(N/2)}\rangle|$ are independent of a, $|\langle a_H|a_2|I_H^{(N/2)}\rangle|$ is also independent of a. Using the same procedure for the higher order, one can argue that

$$
|\mathfrak{a}_m| = \left|\frac{\langle a_H|a_m|I_H^{(N/2)}\rangle}{\langle a_H|I_H^{(N/2)}\rangle}\right| \tag{3.72}
$$

is independent of a for all m. Indeed, we have checked this argument up to $m = 4$. The above discussion is only about $|\mathfrak{a}_m|$; however, it helps to explain the a-independence of $\mathfrak{a}_m$ for all m.

4 The (A_3, A_3) theory

In this section, we evaluate the instanton part of the Nekrasov partition function of the (A_3, A_3) theory by application to our proposals as discussed in subsection 3.1 and study S-duality for this theory. This theory cannot be engineered by compactifying 6d $\mathcal{N} = (2, 0)$ theory of type A_1, while it is known to be constructed by "type A_3". The (A_3, A_3) theory is known to be the $SU(2)$ superconformal theory with two copies of the (A_1, D_4) theory and a fundamental hypermultiplet described by the quiver diagram in Figure 2 for $(p, q, r) = (2, 4, 4)$.

In the following subsection, we apply (3.19) to this theory. And then, by using the result of this computation, we study the S-duality of this theory in subsection 4.2.

4.1 Application to the (A_3, A_3) theory

In this subsection, we apply our proposal, as discussed in subsection 3.1, to the (A_3, A_3) theory to compute the instanton part of the Nekrasov partition function of this theory.

4.1.1 Partition function

To apply our proposals to the theory, let us replace $SU(2)$ gauge group in Figure 10 with $U(2)$. There is a difference between $\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}$ and $\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}$ which is how these theories couple to the $U(1)$-part of the gauge group, but there give the same contribution to the partition function of $SU(2)$ gauge theory. Therefore we here focus on the case of two AD theories corresponding to $\mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}$. Then the Nekrasov partition function of the theory is evaluated as

$$\mathcal{Z}_{U(2)} = \mathcal{Z}^{U(2)}_{\text{pert}} \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, M) \prod_{i=1}^{2} \mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}(a, m_i, d_i, u_i) \ , \tag{4.1}$$

where $\mathcal{Z}^{\text{vec}}_{Y_1,Y_2}$ and $\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}$ are regarded as the contributions of the vector multiplet and fundamental hypermultiplet, as reviewed in subsection 2.1. The factor $\mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}$ can be regarded as the contribution from the (A_1, D_4) sector, and and $\mathcal{Z}^{U(2)}_{\text{pert}}$ is a perturbative part that makes series in q start with 1. Since the (A_3, A_3) theory has a vanishing beta function, this theory is superconformal, and the instanton expansion is denoted by an exponential of marginal gauge coupling q instead of a dynamical scale. The parameters m_i, d_i, and u_i are respectively a mass, relevant coupling with dimension 1/2 and the VEV of Coulomb branch operator with dimension 3/2 of each of the (A_1, D_4) theories. As discussed in subsection

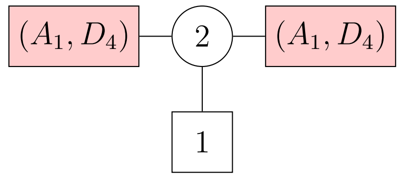

Figure 10: The quiver diagram of the (A_3, A_3) theory, where $(p, q, r) = (2, 4, 4)$ in Figure 2.

3.1, $\mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}$ is represented by inner product $\langle a; Y_1, Y_2|\widehat{I}^{(2)}\rangle$ as

$$\mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}(a, m, d, u) = (\zeta c_2)^{-\frac{|Y_1|+|Y_2|}{2}} \frac{\langle a; Y_1, Y_2|\widehat{I}^{(2)}\rangle}{\langle a|\widehat{I}^{(2)}\rangle} \ . \tag{4.2}$$

Here, we set the numerical factor as $\zeta = -1/2$ to be the straightforward expression below. The variable of the numerical factor ζ is related to the rescaling of the marginal coupling q. The action of L_k and a_k on the irregular state $|\widehat{I}^{(2)}\rangle$ is given as

$$L_k|\widehat{I}^{(2)}\rangle = \quad 0 \qquad \text{for} \qquad k > 4 \ , \tag{4.3}$$

$$L_4|\widehat{I}^{(2)}\rangle = -c_2^2|\widehat{I}^{(2)}\rangle \ , \tag{4.4}$$

$$L_3|\widehat{I}^{(2)}\rangle = -2c_1c_2|\widehat{I}^{(2)}\rangle \ , \tag{4.5}$$

$$L_2|\widehat{I}^{(2)}\rangle = -(c_1^2 + c_2(2c_0 - 3Q))|\widehat{I}^{(2)}\rangle \ , \tag{4.6}$$

$$L_1|\widehat{I}^{(2)}\rangle = \left(c_2\frac{\partial}{\partial c_1} - 2c_1(c_0 - Q)\right)|\widehat{I}^{(2)}\rangle \ , \tag{4.7}$$

$$L_0|\widehat{I}^{(2)}\rangle = \left(\Delta_{c_0} + c_1\frac{\partial}{\partial c_1} + 2c_2\frac{\partial}{\partial c_2}\right)|\widehat{I}^{(2)}\rangle \ , \tag{4.8}$$

and

$$a_k|\widehat{I}^{(2)}\rangle = \begin{cases} -ic_k|\widehat{I}^{(2)}\rangle \quad \text{for} \quad k = 1, 2 \\ 0 \quad \text{for} \quad k > 2 \end{cases} \ . \tag{4.9}$$

According to (2.47) and (2.48), the 4d parameters are identified by the 2d parameters as follows:

$$m = 2c_0 + \frac{c_1^2}{c_2} \ , \qquad d = \frac{2c_1}{\sqrt{c_2}} \ , \qquad u = \frac{2c_0c_1}{\sqrt{c_2}} - \sqrt{c_2}\frac{\partial \mathcal{F}_{(A_1,D_4)}}{\partial c_1} \ , \tag{4.10}$$

where $\mathcal{F}_{(A_1,D_4)} \equiv \lim_{\epsilon_i \to \infty}(-\epsilon_1\epsilon_2 \log \mathcal{Z}_{(A_1,D4)})$ is the prepotential of the (A_1, D_4) theory.

4.1.2 Prepotential

The prepotential of this theory is defined by the classical limit of the Nekrasov partition function as

$$\mathcal{F}_{U(2)} \equiv \lim_{\epsilon_i \to 0} \left(-\epsilon_1 \epsilon_2 \log \mathcal{Z}_{U(2)} \right) \ , \tag{4.11}$$

and $\mathcal{Z}_{U(2)}$ is decomposed into the contributions of $SU(2)$ gauge theory and $U(1)$ factor. We see that the Nekrasov partition function of the (A_3, A_3) theory is obtained by

$$\mathcal{Z}_{(A_3,A_3)} = \frac{\mathcal{Z}_{U(2)}}{\mathcal{Z}_{U(1)}} \ , \tag{4.12}$$

and the prepotential is also defined as

$$\mathcal{F}_{(A_3,A_3)} \equiv \lim_{\epsilon_i \to 0} \left(-\epsilon_1 \epsilon_2 \log \mathcal{Z}_{(A_3,A_3)} \right) \ . \tag{4.13}$$

The prepotential further splits into the perturbative part and instanton part as

$$\mathcal{F}_{(A_3,A_3)} = \mathcal{F}^{(A_3,A_3)}_{\text{pert}} + \mathcal{F}^{(A_3,A_3)}_{\text{inst}} \ , \tag{4.14}$$

and the instanton part is generally expanded by the exponential of marginal coupling q as

$$\mathcal{F}^{(A_3,A_3)}_{\text{inst}} = \sum_{k=1}^{\infty} \mathcal{F}_k q^k \ . \tag{4.15}$$

Below, we will evaluate the expansion coefficients of the above expression $\mathcal{F}_k$.

Note that (4.15) is identical to the instanton part of (4.11) as

$$\mathcal{F}^{U(2)}_{\text{inst}} \equiv \lim_{\epsilon_i \to 0} \left(-\epsilon_1 \epsilon_2 \log \frac{\mathcal{Z}^{U(2)}}{\mathcal{Z}^{U(2)}_{\text{pert}}} \right) \ , \tag{4.16}$$

up to $U(1)$ factor. The problem of computing the coefficients $\mathcal{F}_k$ is eventually equivalent to that of calculating the matrix elements as

$$\langle a | L^{m_p}_{k_p} \cdots L^{m_1}_{k_1} a^{n_q}_{\ell_q} \cdots a^{n_1}_{\ell_1} | \widehat{I}^{(2)} \rangle \ . \tag{4.17}$$

Using (4.3)–(4.9), this further reduces to computing

$$\langle a | L^k_1 | \widehat{I}^{(2)} \rangle = \left(c_2 \frac{\partial}{\partial c_1} - 2 c_1 (c_0 - Q) \right)^k \langle a | \widehat{I}^{(2)} \rangle \ , \tag{4.18}$$

where $\langle a|\widehat{I}^{(2)}\rangle$ is the Nekrasov partition function of (A_1, D_4) theory, whose $SU(2)$ flavor symmetry is not gauged. The above equation means that we need to know the c_1-dependence of $\langle a|\widehat{I}^{(2)}\rangle$ to compute (4.1). The c_1 expansion of $\langle a|\widehat{I}^{(2)}\rangle$ is studied in [33]. However, in the classical limit, we do not need to know it. Indeed, we here recover the full ϵ_i-dependence by $c_k \to c_k/\sqrt{\epsilon_1\epsilon_2}$, then we see that (4.18) is eventually given as

$$\langle a|L_1^k|\widehat{I}^{(2)}\rangle = (c_2)^{\frac{k}{2}}(-u)^k\langle a|\widehat{I}^{(2)}\rangle \ . \tag{4.19}$$

We can evaluate $\mathcal{F}_{\text{inst}}^{(A_3,A_3)}$ by eliminating the a-independence and the perturbative part from $\mathcal{Z}_{U(2)}$ and using the result in (4.19).

4.2 S-duality

Here, we will evaluate the instanton part of the prepotential of this theory. Then we show that this instanton part is related to the instanton part of the SU(2 SQCD with four flavors, and we study the S-duality of the (A_3, A_3) theory via that of the $SU(2)$ SQCD with four flavors by using discovered a prepotential relation.

4.2.1 On the UV gauge coupling

To study the S-duality of the theory, let us first focus on the case in which all the dimensionful 4d parameters are turned off except for a (and Ω-background parameters ϵ_i). Since $\mathcal{F}_{U(1)} \equiv \lim_{\epsilon_i\to 0}\left(-\epsilon_1\epsilon_2\log\mathcal{Z}_{U(1)}\right)$ has scaling dimension 2, it must be proportional to a^2 in this case. Since the VEV of the scalar field in the $SU(2)$ vector multiplet a is neutral under the $U(1)$, the $U(1)$-factor $\mathcal{Z}_{U(1)}$ is independent of a. Therefore, this implies that $\mathcal{F}_{U(1)} = 0$. By using the above discussions, the prepotential of the (A_3, A_3) theory, in this case, is given as

$$\mathcal{F}_{\text{inst}}^{(A_3,A_3)}(q;a) = \left(\frac{1}{4}q^2 + \frac{13}{128}q^4 + \frac{23}{384}q^6 + \cdots\right)a^2 \ . \tag{4.20}$$

Remarkably, this expression is similar to the instanton part of the prepotential of the $SU(2)$ gauge theory with four flavors whose all mass parameters are turned off (2.15). We find the following relation between (4.20) and (2.15):[14]

$$2\mathcal{F}_{\text{inst}}^{(A_3,A_3)}(q;a) = \mathcal{F}_{\text{inst}}^{N_f=4}(q^2,a) \ . \tag{4.21}$$

[14] We checked this statement up to order $\mathcal{O}(q^8)$.

We assume that the perturbative part also satisfies with $2\mathcal{F}^{(A_3,A_3)}_{\text{pert}}(q;a) = \mathcal{F}^{N_f=4}_{\text{pert}}(q^2;a)$, then one can extend the relation (4.21) to the "full" prepotential:

$$2\mathcal{F}_{(A_3,A_3)}(q;a) = \mathcal{F}_{N_f=4}(q^2;a) \ . \tag{4.22}$$

The perturbative part is generally written as

$$\mathcal{F}_{\text{pert}} = (\log q + X)a^2 \ , \tag{4.23}$$

where X is a constant and is related to the one-loop part. Therefore we need to prove a relation $2X^{(A_3,A_3)} = X^{N_f=4}$ so that we obtain (4.22), but we leave for future work. The relation (4.22) means that one can study the S-duality of the (A_3, A_3) theory via that of the $SU(2)$ gauge theory with four flavors. To see that, let us first denote the prepotentials of two theories by the IR gauge coupling as

$$\mathcal{F}_{(A_3,A_3)}(q;a) = (\log q_{\text{IR}})\, a^2 \ , \qquad \mathcal{F}_{N_f=4}(q;a) = (\log \widetilde{q}_{\text{IR}})\, a^2 \ , \tag{4.24}$$

where q_{IR} and $\widetilde{q}_{\text{IR}}$ are a function of the UV gauge coupling q. As discussed in subsection 2.1, the $SU(2)$ gauge theory with four flavors is invariant under $PSL(2,\mathbb{Z})$. Its actions on the IR and UV gauge coupling are written as (2.19) and (2.20). Similarly, the (A_3, A_3) theory is known to be invariant under $PSL(2,\mathbb{Z})$ [34, 35, 37]. However, unlike the $SU(2)$ gauge theory with four flavors, $PSL(2,\mathbb{Z})$-action has only been studied in the IR language. Below, by using the surprising relation (4.22), we will discuss the action of $PSL(2,\mathbb{Z})$ on the UV gauge coupling via the $SU(2)$ gauge theory with four flavors.

The relation (4.22) implies that the prepotential $\mathcal{F}_{(A_3,A_3)}(q;a)$ is obtained from $\mathcal{F}_{N_f=4}(q;a)$ by the non-trivial replacement

$$q \longrightarrow q^2 \ , \qquad \widetilde{q}_{\text{IR}} \longrightarrow q_{\text{IR}}^2 \ . \tag{4.25}$$

Hence the IR gauge coupling q_{IR} is related to q by

$$q^2 = \frac{\theta_2(q_{\text{IR}}^2)^4}{\theta_3(q_{\text{IR}}^2)^4} \ , \tag{4.26}$$

this means that

$$\tau_{\text{IR}} \equiv \frac{2}{\pi i} \log q_{\text{IR}} = \frac{2\theta_{\text{IR}}}{\pi} + \frac{16\pi i}{g_{\text{IR}}^2} \tag{4.27}$$

is the modulus of the elliptic curve, which corresponds to the double cover of a sphere with four punctures whose cross ratio is q^2, and then the elliptic curve is identified as the SW curve of the theory. The curve has the T and S transformations of the S-duality group identified as

$$T : \tau_{\rm IR} \to \tau_{\rm IR} + 1 \ , \qquad S : \tau_{\rm IR} \to -\frac{1}{\tau_{\rm IR}} \ . \tag{4.28}$$

Recall that the S-transformation for the $\mathcal{N}=4$, $SU(2)$ SYM and the $\mathcal{N}=2$ $SU(2)$ SQCD represented the exchanging the minimal magnetic monopole with the W-boson and the quark, with 1/2 of the electric charge of the W-boson, while the T-transformation for the two theories exchanges this monopole with the dyonic particle whose electric charge is equivalent to the quark. Therefore, (4.27) means that the S and T-transformation exchanges the minimal magnetic monopole with the electric particle and dyonic particle whose electric charge is 1/4 of that of the W-boson, respectively.

From (4.26), we see that the action of $PSL(2,\mathbb{Z})$ on the UV gauge coupling is given as

$$T: \ q^2 \to \frac{q^2}{q^2-1} \ , \qquad S: \ q^2 \to 1-q^2 \ . \tag{4.29}$$

We have checked that (4.20) combined with the classical part $\mathcal{F}^{(A_3,A_3)}_{\rm cl} = (\log q)a^2$ is indeed invariant under the T-transformation up to $\mathcal{O}(q^8)$.

Let us now turn on all the mass parameters. Then the instanton part of the prepotential is a function of three mass parameters in addition to q and a; one obtains

$$\begin{aligned}
&\mathcal{F}^{(A_3,A_3)}_{\rm inst}(q;a,M,m_1,m_2) \\
&\qquad \sim \frac{1}{4}\left(a^2 + M^2 m_1 m_2 a^{-2}\right) q^2 \\
&\qquad\quad + \frac{1}{128}\Big[13a^2 + (M^4 + 2M^2(m_1^2 + 8m_1m_2 + m_2^2) + m_1^2m_2^2)a^{-2} \\
&\qquad\quad - (3M^4(m_1^2+m_2^2) + 6M^2m_1^2m_2^2)a^{-4} + 5M^4m_1^2m_2^2a^{-6}\Big]q^4 + \cdots \ ,
\end{aligned} \tag{4.30}$$

where "$\sim$" means that the LHS in (4.30) is equivalent to the RHS up to a-independent terms. Comparing this with (2.21), we find the relation with massive deformations as

$$2\mathcal{F}^{(A_3,A_3)}_{\rm inst}(q;a,M,m_1,m_2) = \mathcal{F}^{N_f=4}_{\rm inst}(q^2,a,-M,-M,m_1,m_2) \ . \tag{4.31}$$

Note that two of the four mass parameters of the $SU(2)$ SQCD with four flavors are identified with each of the mass parameters of two (A_1,D_4) sectors in the (A_3,A_3) theory, and the other two mass parameters of the former are identified with the mass parameter of a fundamental hypermultiplet in the latter.

4.2.2 On all the parameters

In the previous section, we found the S-duality transformations (4.29) when we turned off all dimensionful parameters except for a and ϵ_i. Here, let us turn on all the parameters in this theory, and we extend (4.29) to the S-duality transformation for all the parameters. However, since S-transformation is a strong-weak transformation, it looks hard to find by using order-by-order computations. Thus, we here focus on the T-transformation. We computed the prepotential for generic values of all the parameters, and then we found that this prepotential is invariant under the T-transformation in (4.29) follows:[15]

$$q \to \frac{iq}{\sqrt{1-q^2}} \ , \quad d_1 \to \frac{d_1 + qd_2}{\sqrt{1-q^2}} \ , \quad d_2 \to id_2 \ , \quad m_2 \to -m_2 \ , \quad u_2 \to -iu_2 \ , \tag{4.33}$$

where m_1 and m_2 are kept fixed. Note that the above transformations can be interpreted as the extension of the T-transformation in (4.29) for all the parameters. This transformation is consistent with a symmetry of the SW curve.

The SW curve of this theory is written as [34]

$$0 = x^4 + \mathsf{q}x^2z^2 + z^4 + c_{3,0}x^3 + c_{0,3}z^3 + c_{2,0}x^2 + mxz + c_{0,2}z^2 + c_{1,0}x + c_{0,1}z + c_{0,0} \ , \tag{4.34}$$

where q is a non-trivial function of q_{IR}, and the SW 1-form is $\lambda = xdz$. As shown in [34], in the weak coupling limit $\mathsf{q} \to \infty$, the parameters m and c_{ij} in (4.34) are identified by u_i, d_i, m_i, M and u as[16]

$$\begin{aligned} &c_{3,0} = \mathsf{q}^{\frac{1}{4}}d_1 \ , \qquad c_{0,3} = \mathsf{q}^{\frac{1}{4}}d_2 \ , \qquad c_{2,0} = \mathsf{q}^{\frac{1}{2}}m_1 \ , \qquad c_{0,2} = \mathsf{q}^{\frac{1}{2}}m_2 \ , \\ &m = \mathsf{q}M \ , \qquad c_{1,0} = \mathsf{q}^{\frac{3}{4}}u_1 \ , \qquad c_{0,1} = \mathsf{q}^{\frac{3}{4}}u_2 \ , \qquad c_{0,0} = \mathsf{q}u \ . \end{aligned} \tag{4.35}$$

Note that the curve (4.34) is invariant under the transformation $\mathsf{q} \to -\mathsf{q}$ combined with the follows:

$$\begin{aligned} &c_{3,0} \to -e^{\frac{3\pi i}{4}}c_{3,0} \ , \quad c_{0,3} \to -e^{-\frac{3\pi i}{4}}c_{0,3} \ , \quad c_{2,0} \to -ic_{2,0} \ , \quad c_{0,2} \to ic_{0,2} \ , \\ &m \to -m \ , \quad c_{1,0} \to -e^{\frac{\pi i}{4}}c_{1,0} \ , \quad c_{0,1} \to -e^{-\frac{\pi i}{4}}c_{0,1} \ , \quad c_{0,0} \to -c_{0,0} \ . \end{aligned} \tag{4.36}$$

[15] The instanton part of the prepotential is a function of all the parameters in this theory, and is expand in power of a as

$$\mathcal{F}^{(A_3,A_3)}_{\mathrm{inst}} = \sum_{k=-1}^{\infty} \mathcal{F}_{2k}(q, M, \{m_i\}, \{u_i\}, \{d_i\})a^{-2k} \ . \tag{4.32}$$

In appendix B, we show the first few terms $\mathcal{F}_{-2}, \mathcal{F}_2$ and $\mathcal{F}_4$. We find that the expansion coefficients $\mathcal{F}_{2k}$ except for $k \neq 0$ are invariant under (4.33). Since we here removed the $U(1)$-factor that is independent of a and do not explicitly know it, $\mathcal{F}_0$ cannot be evaluated without it.

[16]

Below, we show that this transformation is identical to the T-transformation. To see that, we consider the case that the hypermultiplet in the (A_3, A_3) theory is decoupled, i.e.$M \to \infty$. Then the theory reduces to the non-conformally gauged AD theory described in Figure 3, for $N = 4$. one needs to keep

$$\Lambda \equiv \frac{M}{\sqrt{\mathsf{q}}} \tag{4.37}$$

finite so that all the periods of the curve are finite in this limit, where Λ is identified with the dynamical scale of the resulting theory.[17] Note that, by standard argument, the gauge coupling of the CFT and the dynamical scale mass-deformed theory are related by

$$\frac{\Lambda}{M} = e^{i\theta_{\mathrm{IR}} - \frac{8\pi^2}{g_{\mathrm{IR}}^2}} . \tag{4.40}$$

This implies that

$$\mathsf{q} = e^{-2i\theta_{\mathrm{IR}} + \frac{16\pi^2}{g_{\mathrm{IR}}^2}} , \tag{4.41}$$

and therefore "$\mathsf{q} \to -\mathsf{q}$" is interpreted as the T-transformation in (4.29).

Note that in terms of u_i, d_i, m_i, M, and u, (4.35) reduces to

$$d_2 \to i d_2 , \qquad m_2 \to -m_2 , \qquad u_2 \to -i u_2 , \tag{4.42}$$

where d_1, u_1, m_1, M, and u are kept fixed. We see that the above expression is identical to (4.33) at the leading order of q. Thus, (4.33) is consistent at the level of the SW curve, and (4.35) is interpreted as a weak coupling limit of the T-transformation.

[17] In this limit $\mathsf{q}, M \to \infty$, the curve reduces to

$$X^2 = \widetilde{\Lambda}^2 Z^2 + \widetilde{\Lambda}^{\frac{3}{2}} C_{0,3} Z + \widetilde{\Lambda} C_{0,2} + \frac{\widetilde{\Lambda}^{\frac{1}{2}} C_{0,1}}{Z} + \frac{U}{Z^2} + \frac{\widetilde{\Lambda}^{\frac{1}{2}} C_{1,0}}{Z^3} + \frac{\widetilde{\Lambda} C_{2,0}}{Z^4} + \frac{\widetilde{\Lambda}^{\frac{3}{2}} C_{3,0}}{Z^5} + \frac{\widetilde{\Lambda}^2}{Z^6} , \tag{4.38}$$

where we take the new variables as

$$X \equiv i(\sqrt{z} x^{\frac{3}{2}} + \frac{1}{2}\sqrt{q}\Lambda\sqrt{x/z}), \qquad Z \equiv \sqrt{z/x} , \tag{4.39}$$

and $\widetilde{\Lambda} \equiv -\Lambda/2$. The SW 1-form is written as $\frac{3}{2i} X dZ$ up to exact terms. Indeed, this curve is identical to (3.24) for $N = 4$.

5 The (A_2, A_5) theory

This section will discuss the (A_2, A_5) theory. This theory cannot be engineered by compactifying 6d $\mathcal{N} = (2,0)$ theory of type A_1, but it is known to be constructed by "type A_2" [76]. The (A_2, A_5) theory is a superconformal theory with $SU(2)$ gauge group coupled to a fundamental hypermultiplet, one (A_1, D_3) theory, and one (A_1, D_6) theory described in Figure 2 for $(p, q, r) = (2, 3, 6)$. By applying our method to the theory, we compute the instanton part of the Nekrasov partition function, and study the S-duality of the theory via a prepotential relation.

In the next subsection, we apply our method to this theory to compute the partition function of this theory and study the S-duality of this theory. This theory consists of the three matter sectors: the (A_1, D_6) sector, the (A_1, D_3) sector, and a fundamental hypermultiplet. In subsection 5.2, we show that the SW curve of the (A_2, A_5) theory splits into the SW curve of each of the three sectors in the weak coupling limit. In subsection 5.3, we will discuss the S-duality from the SW curve of this theory.

5.1 Application to the (A_2, A_5) theory

In this subsection, we apply our method to the (A_2, A_5) theory described in Figure 11 and study the S-duality of this theory. We first consider the gauge group $U(2)$ instead of $SU(2)$ to apply our proposals to this theory. The same argument as discussed in section 4, we focus on the case of the AD theory corresponding to $\mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}$.

5.1.1 Computation of prepotential

The Nekrasov partition function of the theory is evaluated as

$$\mathcal{Z}_{U(2)} = \mathcal{Z}^{U(2)}_{\text{pert}} \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, M) \mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}(a, d, u) \mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}(a, m, \boldsymbol{d}, \boldsymbol{u}) \ , \tag{5.1}$$

where $\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}$ and $\mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}$ can be respectively regarded as the contribution from the (A_1, D_3) sector and the (A_1, D_6) sector, and $\mathcal{Z}^{U(2)}_{\text{pert}}$ is a perturbative part that makes series in q start with 1. q and a are the exponential of marginal gauge coupling and the VEV of the scalar in $SU(2)$ vector multiplet. The parameters d and u are respectively the relevant coupling and the VEV of the Coulomb branch operator in the (A_1, D_3) theory, and m, $\boldsymbol{d} = (d_1, d_2)$ and $\boldsymbol{u} = (u_1, u_2)$ are respectively a mass, the relevant couplings and, the VEV of Coulomb branch operators in the (A_1, D_6) theory. These parameters have a scale

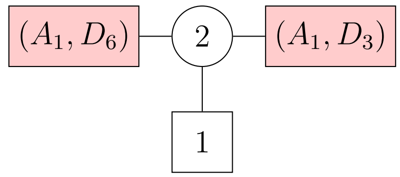

Figure 11: The quiver diagram of the (A_2, A_5) theory, where $p = 2, 3, 6$ in Figure 2.

dimension as

$$[q] = 0, \qquad [d_1] = \frac{1}{3}\ , \qquad [d] = [d_2] = \frac{2}{3}\ , \qquad [u] = [u_1] = \frac{4}{3}\ , \qquad [u_2] = \frac{5}{3}\ . \tag{5.2}$$

According to our proposal, as discussed in subsection 3.1, $\mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}$ is represented by inner product $\langle a; Y_1, Y_2 | \widehat{I}^{(3)} \rangle$ as

$$\mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}(a, m, \boldsymbol{d}, \boldsymbol{u}) = (\zeta c_3)^{-\frac{|Y_1|+|Y_2|}{3}} \frac{\langle a; Y_1, Y_2 | \widehat{I}^{(3)} \rangle}{\langle a | \widehat{I}^{(3)} \rangle}\ . \tag{5.3}$$

The action of L_k and a_k on the irregular state $|\widehat{I}^{(3)}\rangle$ is given as

$$L_k |I^{(3)}\rangle = 0 \quad \text{for} \quad k \geq 7\ , \tag{5.4}$$

$$L_6 |\widehat{I}^{(3)}\rangle = -c_3^2 |\widehat{I}^{(3)}\rangle\ , \tag{5.5}$$

$$L_5 |\widehat{I}^{(3)}\rangle = -2c_2 c_3 |\widehat{I}^{(3)}\rangle\ , \tag{5.6}$$

$$L_4 |\widehat{I}^{(3)}\rangle = -\left(c_2^2 + 2c_3 c_1\right) |\widehat{I}^{(3)}\rangle\ , \tag{5.7}$$

$$L_3 |\widehat{I}^{(3)}\rangle = -2\left(c_1 c_2 + c_3(c_0 - 2Q)\right) |\widehat{I}^{(3)}\rangle\ , \tag{5.8}$$

$$L_2 |\widehat{I}^{(3)}\rangle = \left(c_3 \frac{\partial}{\partial c_1} - c_2(2c_0 - 3Q) - c_1^2\right) |\widehat{I}^{(3)}\rangle\ , \tag{5.9}$$

$$L_1 |\widehat{I}^{(3)}\rangle = \left(2c_3 \frac{\partial}{\partial c_2} + c_2 \frac{\partial}{\partial c_1} - 2c_1(c_0 - Q)\right) |\widehat{I}^{(3)}\rangle\ , \tag{5.10}$$

and

$$a_k |\widehat{I}^{(3)}\rangle = \begin{cases} -ic_k |\widehat{I}^{(3)}\rangle & \text{for} \quad k = 1, 2, 3 \\ 0 \quad \text{for} \quad k > 3 \end{cases}\ . \tag{5.11}$$

Here from (2.47) and (2.48), the 4d parameters are identified by the 2d parameters as follows:

$$m = 2\left(c_0 + \frac{c_1 c_2}{c_3}\right)\ , \qquad d_1 = \frac{2c_2}{c_3^{2/3}}\ , \qquad d_2 = c_3^{-1/3}\left(2c_1 + \frac{c_2^2}{c_3}\right)\ ,$$

$$u_1 = c_3^{1/3}\left(\frac{2c_0c_2 + c_1^2}{c_3} - \frac{\partial\mathcal{F}_{(A_1,D_6)}}{\partial c_1}\right) , \qquad u_2 = c_3^{-1/3}\left(2c_0c_1 - \sum_{k=1}^{2} kc_{k+1}\frac{\partial\mathcal{F}_{(A_1,D_6)}}{\partial c_k}\right) , \tag{5.12}$$

where $\mathcal{F}_{(A_1,D_6)}$ is the prepotential of the (A_1, D_6) theory, which is defined by a classical limit of the Nekrasov partition function of the theory.

On the other hand, the contribution of the (A_1, D_3) sector can be evaluated in subsection 3.2, when the parameters d and u in the (A_1, D_3) theory are turned off. In this case $\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}$ is identified as

$$\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}(a) = (\xi\mathfrak{b}_3)^{-\frac{|Y_1|+|Y_2|}{3}}\,\frac{\langle a; Y_1, Y_2|\widehat{I}^{(3/2)}\rangle}{\langle a|\widehat{I}^{(3/2)}\rangle} . \tag{5.13}$$

Below, we consider this case so that we study the S-duality of the (A_2, A_5) theory via the Nekrasov partition function of the theory, and we set the numerical factor as $\zeta = 2\xi$ to be a simple expression in the following. Similar to the case of the (A_3, A_3) theory, the variables of the numerical factors ζ and ξ are related to the rescaling of the marginal coupling q.

Recall that we have considered the gauge group $U(2)$ instead of $SU(2)$. The partition function of the (A_2, A_5) theory is obtained by removing $U(1)$-factor from $\mathcal{Z}_{U(2)}$ as

$$\mathcal{Z}_{(A_2,A_5)} = \frac{\mathcal{Z}_{U(2)}}{\mathcal{Z}_{U(1)}} . \tag{5.14}$$

The prepotential is given by this partition function $\mathcal{Z}_{(A_2,A_5)}$ as

$$\mathcal{F}_{(A_2,A_5)} \equiv \lim_{\epsilon_i\to 0}\left(-\epsilon_1\epsilon_2\log\mathcal{Z}_{(A_2,A_5)}\right) , \tag{5.15}$$

and this further splits into the perturbative part and the instanton part as $\mathcal{F}_{(A_2,A_5)} = \mathcal{F}^{(A_2,A_5)}_{\text{pert}} + \mathcal{F}^{(A_2,A_5)}_{\text{inst}}$. Recall that since the VEV of the scalar field in the $SU(2)$ vector multiplet a is neutral under the $U(1)$, the $U(1)$-factor is independent of a, and therefore the instanton part is evaluated by removing the a-independent terms.

To study the S-duality of the (A_2, A_5) theory, let us turn off the couplings and the VEV of the Coulomb operators in the (A_1, D_6) sector, in addition to $d = u = 0$. Then the prepotential is a function of q, a, and the mass parameters. One obtains

$$\begin{aligned}\mathcal{F}^{(A_2,A_5)}_{\text{inst}}(q; a, m, M) \sim\ & \frac{1}{6}\left(a^2 + \frac{mM^3}{2}a^{-2}\right)q^3 \\ & + \frac{1}{192}\Bigg[13a^2 + \left(\frac{3}{4}m^2M^2 + 8mM^3 + 3M^4\right)a^{-2} \\ & - \left(\frac{9}{4}m^2M^4 + 3M^6\right)a^{-4} + \frac{5}{4}m^2M^6a^{-6}\Bigg]q^6 + \mathcal{O}(q^9) ,\end{aligned} \tag{5.16}$$

where "$\sim$" is means that the LHS and the RHS are equivalent up to the a-independent terms. Remarkably, this expression is similar to the instanton part of the prepotential of the $SU(2)$ SQCD with four flavors. This similarity has also appeared in the case of the (A_3, A_3) theory. Indeed, this instanton part corresponds to that of the $SU(2)$ SQCD with four flavors as following:[18]

$$3\mathcal{F}^{(A_2,A_5)}_{\text{inst}}(q; a, m, M) = \mathcal{F}^{N_f=4}_{\text{inst}}\left(q^3; a, \frac{m}{2}, M, M, M\right) \ . \tag{5.17}$$

Note that one of the four mass parameters of the $SU(2)$ SQCD with four flavors is identified with the mass parameter of the (A_1, D_6) sector, and the other three are identified with the mass parameter of a fundamental hypermultiplet.

In subsection 5.3, we will show that this relation of the mass parameters is consistent with the SW curve of the (A_2, A_5) theory. The same argument as discussed in section 4, we assume that the relation (5.17) can be extended to the full prepotential as[19]

$$3\mathcal{F}_{(A_2,A_5)}(q; a, m, M) = \mathcal{F}_{N_f=4}\left(q^3, a, \frac{m}{2}, M, M, M\right) \ . \tag{5.19}$$

5.1.2 S-duality

Here, we study the S-duality for the (A_2, A_5) theory. Let us turn off the mass parameters, i.e., $m = M = 0$. We then denote the prepotential of this theory as

$$\mathcal{F}_{(A_2,A_5)}(q; a) = (\log q_{\text{IR}})\, a^2 \ , \tag{5.20}$$

where q_{IR} is a non-trivial function of the marginal UV gauge coupling q. The prepotential relation implies that the prepotential $\mathcal{F}_{(A_2,A_5)}(q; a)$ is obtained from $\mathcal{F}_{N_f=4}(q; a)$ by the replacement $q \to q^3$.

Recall here that the prepotential of the $SU(2)$ SQCD with four flavors is written as $\mathcal{F}_{N_f=4}(q; a) = (\log \widetilde{q}_{\text{IR}})\, a^2$, where we again used $\widetilde{q}_{\text{IR}}$ instead of q_{IR} in (2.16) to distinguish q_{IR} in (5.20). Hence, in terms of the IR gauge coupling, the replacement is identical to $\widetilde{q}_{\text{IR}} \to q^3_{\text{IR}}$, and therefore the IR gauge coupling q_{IR} is related to q by

$$q^3 = \frac{\theta_2(q^3_{\text{IR}})^4}{\theta_3(q^3_{\text{IR}})^4} \ . \tag{5.21}$$

[18] We have checked this statement up to $\mathcal{O}(q^9)$.

[19] This means that we expect to satisfy the relation as

$$3\mathcal{F}^{(A_2,A_5)}_{\text{pert}}(q; a, m, M) = \mathcal{F}^{N_f=4}_{\text{pert}}(q^3; a, \frac{m}{2}, M, M, M) \tag{5.18}$$

This means that

$$\tau_{\rm IR} \equiv \frac{3}{\pi i}\log q_{\rm IR} = \frac{3\theta_{\rm IR}}{\pi} + \frac{24\pi i}{g_{\rm IR}^2} \tag{5.22}$$

is the modulus of the elliptic curve, which corresponds to the double cover of a sphere with four punctures whose cross ratio is q^3, and this curve has the $PSL(2,\mathbb{Z})$-action on the IR gauge coupling $\tau_{\rm IR}$ generated by

$$T : \tau_{\rm IR} \to \tau_{\rm IR} + 1\ , \qquad S : \tau_{\rm IR} \to -\frac{1}{\tau_{\rm IR}}\ . \tag{5.23}$$

As the same argument, (4.27) means that the S and T-transformation exchanges the minimal magnetic monopole with the electric particle and dyonic particle whose electric charge is 1/6 of that of the W-boson, respectively. The S-duality transformation (5.23) means, in terms of the UV gauge coupling q, that the $PSL(2,\mathbb{Z})$-action on the (A_2, A_5) theory is generated by

$$T : q^3 \to \frac{q^3}{q^3-1}\ , \quad S : q^3 \to 1 - q^3\ . \tag{5.24}$$

Let us now turn on the mass parameters and the VEV of the Coulomb branch operators u_1 and u_2. Then the prepotential $\mathcal{F}_{(A_2,A_5)}$ is a function of u_1 and u_2, in addition to q, a, and the mass parameters. We find that $\mathcal{F}_{(A_2,A_5)}$ is invariant under the follows:

$$q \to \frac{e^{\frac{\pi i}{3}}q}{(1-q^3)^{\frac{1}{3}}}\ , \qquad m \to -m\ , \qquad u_1 \to e^{\frac{2\pi i}{3}}u_1\ , \qquad u_2 \to e^{\frac{\pi i}{3}}u_2\ , \tag{5.25}$$

with M kept fixed. This transformation is interpreted as an extension of the T-transformation in (5.24). This consistency check will be shown in subsection 5.3

We expect this transformation to be further extended to non-vanishing all the parameters. The same argument as discussed in subsection 4.2, we expect that the T-transformation on the relevant couplings d, d_1, and d_2 involve a non-trivial q-dependence. In particular, we assume that the T-transformation on d and d_2 mix these parameters since these parameters have the same scale dimension.

However, $\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}$ for odd N discussed in subsection 3.2 can be evaluated only if all the dimensionful parameters in the (A_1, D_N) theory are turned off. Hence, we cannot explicitly check the case of the non-vanishing d and u for our method.

5.2 Three matter sectors

In the remaining section, we show that the surprising relation (5.19) is consistent with the SW curve of the theory. To see that, in this sub-section, we first discuss how the parameters of the (A_2, A_5) theory correspond to these of the three matter sectors in the (A_2, A_5) theory. The SW curve of the (A_2, A_5) theory is written as [25, 46]

$$\begin{aligned} 0 = x^3 + z^6 - \frac{1}{\mathsf{q}} x^2 z^2 - \mathsf{q} x z^4 + c_{20} x^2 + x(c_{11} z + c_{10}) \\ + c_{05} z^5 + c_{04} z^4 + c_{03} z^3 + c_{02} z^2 + c_{01} z - c_{00} \ , \end{aligned} \tag{5.26}$$

where the SW 1-form is $\lambda = x dz$ and q is a non-trivial function of q_{IR}. We show the scaling dimension of these parameters in (5.26). Since the period $\oint \lambda$ is the mass of a BPS state, the SW 1-from λ has the scaling dimension one. Thus, these parameters in (5.26) have the scaling dimension as

$$[x] = \frac{2}{3} \ , \qquad [z] = \frac{1}{3} \ , \qquad [c_{ij}] = 2 - \frac{2i + j}{3} \ , \qquad [\mathsf{q}] = 0 \ . \tag{5.27}$$

This means that c_{ij} with $0 < [c_{ij}] < 1$ are the relevant couplings, c_{ij} with $[c_{ij}] > 1$ are the VEV of Coulomb branch operators, and c_{11} and c_{03} are the mass parameters. In particular, c_{00} corresponds to the VEV of the scalar in $SU(2)$ vector multiplet. The same argument as discussed in [34], these parameters are renormalized so that the periods must be kept finite in the weak coupling limit $\mathsf{q} \to 0$. We can find the renormalized parameters as

$$C_{ij} \equiv \mathsf{q}^{\frac{[c_{ij}]}{2}} c_{ij} \quad \text{for} \quad i \neq j \ , \qquad C_{11} \equiv \mathsf{q} c_{11} \ , \qquad C_{00} \equiv \mathsf{q} c_{00} \ . \tag{5.28}$$

In terms of these parameters, the curve (5.26) reduces to

$$\begin{aligned} 0 = x^3 + z^6 - \frac{1}{\mathsf{q}} x^2 z^2 - \mathsf{q} x z^4 + \mathsf{q}^{-\frac{1}{3}} C_{20} x^2 + x(\mathsf{q}^{-1} C_{11} z + \mathsf{q}^{-\frac{2}{3}} C_{10}) \\ + \mathsf{q}^{-\frac{1}{6}} C_{05} z^5 + \mathsf{q}^{-\frac{1}{3}} C_{04} z^4 + \mathsf{q}^{-\frac{1}{2}} C_{03} z^3 + \mathsf{q}^{-\frac{2}{3}} C_{02} z^2 + \mathsf{q}^{-\frac{5}{6}} C_{01} z^1 - \mathsf{q}^{-1} C_{00} \ . \end{aligned} \tag{5.29}$$

Below, we show how c_{ij} are related to the parameters of the three matter sectors in (A_2, A_5) theory. To see that, let us consider the curve (5.29) splits into the three sectors in the weak coupling limit $\mathsf{q} \to 0$ with C_{ij} kept finite.

5.2.1 (A_1, D_3) sector

In the region $|z/x| \sim \mathsf{q}^{2/3}$, in terms of new set of variables $\widetilde{x} = \mathsf{q}^{-\frac{1}{3}} x$ and $\widetilde{z} = \mathsf{q}^{\frac{1}{3}} z$, the curve is described as

$$0 = \widetilde{x}^3 - \widetilde{x}^2 \widetilde{z}^2 + C_{20} \widetilde{x}^2 + \widetilde{x}(C_{11} \widetilde{z} + C_{10}) - C_{00} \ . \tag{5.30}$$

By shifting and rescaling the variables, the curve further reduces to

$$0 = X^2 + Z^4 + 2^{\frac{1}{3}} C_{20} Z^2 + 4\sqrt{C_{00} - \frac{C_{11}^2}{4}} Z - 2^{\frac{2}{3}} \left(C_{10} - \frac{C_{20}^2}{4} \right) , \tag{5.31}$$

where we defined $X \equiv 2^{\frac{1}{3}} i\big(\widetilde{x} + \frac{1}{2}(\widetilde{z}^2 - C_{20})\big)$ and $Z \equiv -2^{-\frac{1}{3}} i\widetilde{z}$. Note that the above expression coincides with the curve of (A_1, D_3) theory.[20] Here, we see that C_{20} and $(C_{10} - C_{20}^2/4)$ correspond to the relevant coupling and the VEV of the Coulomb branch operator in the (A_1, D_3) theory, respectively. In particular, $\sqrt{C_{00} - C_{11}^2/4}$ is a mass parameter associated with an $SU(2)$ subgroup of the $SU(3)$ flavor symmetry gauged by the $SU(2)$ vector multiplet.

5.2.2 (A_1, D_6) sector

In the region $|z/x| \sim \mathsf{q}^{-1/3}$, the SW curve reduces to

$$0 = -\widetilde{x}^2\widetilde{z}^2 + \widetilde{z}^6 + C_{11}\widetilde{x}\widetilde{z} + C_{05}\widetilde{z}^5 + C_{04}\widetilde{z}^4 + C_{03}\widetilde{z}^3 + C_{02}\widetilde{z}^2 + C_{01}\widetilde{z} - C_{00} , \tag{5.33}$$

where we defined $\widetilde{x} = \mathsf{q}^{-\frac{1}{6}} x$ and $\widetilde{z} = \mathsf{q}^{\frac{1}{6}} z$. By shifting $\widetilde{x} \to \widetilde{x} + C_{11}/(2\widetilde{z})$, the curve is written as

$$\widetilde{x}^2 = \widetilde{z}^4 + C_{05}\widetilde{z}^3 + C_{04}\widetilde{z}^2 + C_{03}\widetilde{z} + C_{02} + \frac{C_{01}}{\widetilde{z}} - \frac{C_{00} - \frac{C_{11}^2}{4}}{\widetilde{z}^2} , \tag{5.34}$$

where the SW curve is invariant under the above shift up to exact terms. Note that the above expression coincides with the curve of the (A_1, D_6) theory.[21] Hence we see that C_{05} and C_{04} are the relevant couplings, C_{02} and C_{01} are the VEVs of Coulomb branch operators, and C_{03} and $\sqrt{C_{00} - C_{11}^2/4}$ are mass parameters in the (A_1, D_6) theory. In particular, $\sqrt{C_{00} - C_{11}^2/4}$ is associated with an $SU(2)$ subgroup of the $SU(3)$ flavor symmetry gauged by the $SU(2)$ vector multiplet.

[20] The known expression of the curve of the (A_1, D_3) theory is written as [25, 26]

$$x^2 = z^4 + dz^2 + mz + u , \tag{5.32}$$

where the SW curve is $\lambda = x dz$, and d, m, and u are respectively the relevant coupling of dimension 2/3, a mass parameter and the VEV of a Coulomb branch operator of dimension 4/3 in the (A_1, D_3) theory.

[21] From (2.44), the (A_1, D_6) theory is written as

$$x^2 = \frac{a^2}{z^2} + \frac{u_2}{z^3} + \frac{u_1}{z^4} + \frac{m}{z^5} + \frac{d_2}{z^6} + \frac{d_1}{z^7} + \frac{1}{z^8} . \tag{5.35}$$

In terms of $w = 1/z$, the curve reduces to the expression of (5.34).

5.2.3 A fundamental hypermultiplet

In the region $|z/x| \sim 1$, the curve reduces to

$$0 = -x^2z^2 + C_{11}xz - C_{00} \ . \tag{5.36}$$

This curve describes the $SU(2)$ superconformal QCD in the weak coupling limit, as discussed in [34]. Here, C_{11} is the mass parameter of a fundamental hypermultiplet.

From the above discussions, we see that the curve of the (A_2, A_5) theory splits into the curves for three sectors, and then the parameters C_{ij} are identified by the parameters of the three sectors in the (A_2, A_5) theory as

$$\begin{gathered} d_1 = C_{05} \ , \qquad d_2 = C_{04} \ , \qquad m = -\frac{C_{03}}{6} \ , \qquad u_1 = C_{02} \ , \qquad u_2 = C_{01} \ , \\ d = C_{20} \ , \qquad u = C_{10} - \frac{C_{20}^2}{4} \ , \qquad M = -\frac{C_{11}}{12} \ . \end{gathered} \tag{5.37}$$

5.3 Consistency with the SW curve

We have found a surprising relation between the prepotential of the (A_2, A_5) theory and that of the $SU(2)$ SQCD with four flavors in subsection 5.1. In particular, one of the mass parameters of the latter is identified with the mass of a fundamental hypermultiplet in the (A_2, A_5) theory and the other mass parameters of the latter are identified with the mass of the (A_1, D_6) sector.

In this subsection, we show that the T-transformation (5.25) coincides with a symmetry of the curve. We rederive the relation of the mass parameters in (5.19) from the curve at the end of this subsection.

5.3.1 T-transformation

Here, we show that the SW curve of the (A_2, A_5) theory can be represented by that for the $SU(2)$ superconformal QCD, and a symmetry of the SW curve can be interpreted as a weak coupling limit of the T-transformation for all the parameters.

We see that the curve (5.26) is invariant under the following transformation:

$$\begin{aligned} &\mathsf{q} \to e^{\frac{2\pi i}{3}}\mathsf{q} \ , \qquad c_{10} \to e^{-\frac{4\pi i}{9}}c_{10} \ , \qquad c_{11} \to e^{-\frac{2\pi i}{3}}c_{11} \ , \qquad c_{20} \to e^{-\frac{2\pi i}{9}}c_{20} \ , \\ &c_{01} \to -e^{\frac{\pi i}{9}}c_{01} \ , \qquad c_{02} \to -e^{-\frac{\pi i}{9}}c_{02} \ , \qquad c_{03} \to e^{\frac{2\pi i}{3}}c_{03} \ , \\ &c_{04} \to e^{\frac{4\pi i}{9}}c_{04} \ , \qquad c_{05} \to e^{\frac{2\pi i}{9}}c_{05} \ , \qquad c_{00} \to -e^{\frac{\pi i}{3}}c_{00} \ , \end{aligned} \tag{5.38}$$

and at the same time, we take the change of variables as $x \to e^{-\frac{2\pi i}{9}} x$ and $z \to e^{\frac{2\pi i}{9}} z$. In the weak coupling limit $\mathsf{q} \to 0$, this transformation reduces to a transformation of the parameters in the three matter sectors as

$$\begin{aligned} \mathsf{q} \to e^{\frac{2\pi i}{3}} \mathsf{q} \ , \qquad d_1 \to -e^{\frac{2\pi i}{3}} d_1 \ , \qquad d_2 \to -e^{\frac{\pi i}{3}} d_2 \ , \qquad m \to -m \ , \\ u_1 \to e^{\frac{2\pi i}{3}} u_1 \ , \qquad u_2 \to e^{\frac{\pi i}{3}} u_2 \ , \end{aligned} \tag{5.39}$$

with the other parameters kept fixed. We can interpret this as the T-transformation for all the parameters. To see that, let us turn off c_{ij} except for c_{00}. Then the curve (5.26) is written as

$$0 = (x - \sqrt{\mathsf{q}} z^2)(x + \sqrt{\mathsf{q}} z^2) \left(x - \frac{z^2}{\mathsf{q}} \right) - c_{00} \ . \tag{5.40}$$

In terms of $w = x/z^2$ and $v = z^3$, the curve can further reduce to

$$v^2 = \frac{c_{00}}{(w^2 - \mathsf{q}) \left(w - \frac{1}{\mathsf{q}} \right)}. \tag{5.41}$$

Changing the variables preserved the SW 1-form up to exact term,[22] we find that the above expression is equivalent to the following curve:

$$y^2 = (\widetilde{x}^2 - \widetilde{u})^2 - f \widetilde{x}^4 \ , \tag{5.44}$$

where we defined $f \equiv 1 - \mathsf{q}$ and $\widetilde{u} \equiv \frac{2(1-f)^{\frac{1}{3}}}{\sqrt{1+\sqrt{f}}} c_{00}$, and the coordinates are defined by $\widetilde{x} \equiv i\sqrt{\widetilde{u}} w$ and $y \equiv \widetilde{u}^{\frac{3}{2}}/v$. Here, the SW 1-form is now written as $\frac{i\widetilde{u}}{3} \frac{d\widetilde{x}}{y}$ up to exact terms. The curve (5.44) is a known expression of the $SU(2)$ SQCD with four flavors. Since $\widetilde{u}$ is the VEV of the scalar in the $SU(2)$ vector multiplet, we note that c_{00} is indeed identified with it. The equivalence of the SW curves of the two theories, in this case, means that the

[22]Here, we consider the change of variables:

$$w \to \frac{w \mathsf{q}^{\frac{1}{2}} \sqrt{1 + \sqrt{f}} + \mathsf{q}^{\frac{1}{2}} \sqrt{\frac{1-\sqrt{f}}{1+\sqrt{f}}}}{w \sqrt{1 - \sqrt{f}} + 1} \ , \quad v \to \frac{\sqrt{1 + \sqrt{f}}}{2 \mathsf{q}^{\frac{1}{2}} \sqrt{f}} v \left(w \sqrt{1 - \sqrt{f}} + 1 \right)^2 \ . \tag{5.42}$$

Then the curve is reduces to

$$v^2 = \frac{\widetilde{u}}{(w^2 + 1) - f w^4} \ , \tag{5.43}$$

and the SW curve is now written as $\frac{1}{3} w dv$ up to the exact term.

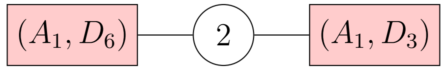

Figure 12: Decoupling a fundamental hypermultiplet from the (A_2, A_5) theory

(A_2, A_5) theory is indeed the $SU(2)$ superconformal theory.[23] This curve has an S-duality transformation [1, 77]:

$$\sqrt{1-f} \to -\sqrt{1-f} \ , \qquad \widetilde{u} \to \widetilde{u} \ . \tag{5.45}$$

In terms of q and c_{00}, this means that

$$\mathsf{q} \to e^{\frac{2\pi i}{3}}\mathsf{q} \ , \qquad c_{00} \to -e^{\frac{\pi i}{3}} c_{00} \ . \tag{5.46}$$

Note that (5.38) is an S-duality transformation since this coincides with the action of (5.38) on q and c_{00}. In particular, we show that this transformation is regarded as the T-transformation below.

We consider that the hypermultiplet in the (A_2, A_5) theory is decoupled, i.e., $C_{11} \to \infty$, and then the resulting theory is the non-conformally gauged AD theory described in Figure 12. To be finite periods, one needs to keep

$$\Lambda \equiv -\frac{1}{2}\sqrt{\mathsf{q}} C_{11} \tag{5.47}$$

finite in this limit $C_{11} \to \infty$ combined with weak coupling limit $\mathsf{q} \to 0$, where Λ is identical to the dynamical scale of the resulting theory.[24] Similarly to (4.40), the gauge coupling of

[23] The same argument of the (A_3, A_3) theory was discussed by [34].

[24] In the limit $C_{11} \to \infty$ and $\mathsf{q} \to 0$, the curve (5.26) reduces to

$$\begin{aligned} X^2 = \Lambda^{5/3} C_{05} Z^3 + \Lambda^{4/3} C_{04} Z^2 + \Lambda C_{03} Z + \Lambda^{2/3} C_{02} + \frac{\Lambda^{1/3} C_{01}}{Z} \\ - \frac{C_{00} + 2\Lambda^2}{Z^2} + \frac{\Lambda^{2/3} C_{10}}{Z^3} + \frac{\Lambda^{4/3} C_{20}}{Z^4} + \frac{\Lambda^2}{Z^5} \ , \end{aligned} \tag{5.48}$$

where we take the new variables as

$$X \equiv -\frac{\Lambda}{\sqrt{\mathsf{q}}} \left(\frac{x}{z^2}\right)^{\frac{1}{3}} + z^3 \left(\frac{x}{z^2}\right)^{\frac{2}{3}} \ , \qquad Z \equiv \left(\frac{z^2}{x}\right)^{\frac{1}{3}} \ , \tag{5.49}$$

and the SW 1-form is written as $\lambda = X dZ$ up to exact terms. Indeed, this curve is identical to that of the resulting theory

$$x^2 = -\frac{\langle \widehat{I}^{(3)} | T(z) | \widehat{I}^{(3/2)} \rangle}{\langle \widehat{I}^{(3)} | \widehat{I}^{(3/2)} \rangle} \ . \tag{5.50}$$

the (A_2, A_5) theory and the dynamical scale of the mass-deformed theory are related by

$$\frac{\Lambda}{C_{11}} = \exp\left(i\theta_{\rm IR} - \frac{8\pi^2}{g_{\rm IR}^2}\right) . \tag{5.51}$$

This implies from (5.47) that

$$\mathsf{q} \propto \exp\left(2i\theta_{\rm IR} - \frac{16\pi^2}{g_{\rm IR}^2}\right) . \tag{5.52}$$

This means that the action of (5.39) on q coincides with the T-transformation in (5.23). Therefore, (5.25) is interpreted as a weak coupling limit of the T-transformation of generic values of all parameters.

5.3.2 Relation of mass parameters

Here, we rederive the relation between the mass parameters in (5.17). Let us turn on the two mass parameters. Then the SW curve (5.40) is modified as

$$0 = v^2\,(w - \sqrt{\mathsf{q}})\,(w + \sqrt{\mathsf{q}})\left(w - \frac{1}{\mathsf{q}}\right) + v\,(c_{03} + c_{11}w) - c_{00} . \tag{5.53}$$

By shifting v as $v \to v - (c_{03} + c_{11}w)/(2P_3(w))$, the curve reduces to

$$v^2 = \frac{c_{00}}{P_3(w)} + \frac{(c_{03} + c_{11}w)^2}{4P_3(w)^2} , \tag{5.54}$$

where we defined $P_3(w) \equiv (w - \sqrt{\mathsf{q}})(w + \sqrt{\mathsf{q}})(w - 1/\mathsf{q})$, and the SW 1-form is now $\lambda = -\frac{1}{3}vdw$ up to exact terms. Note that (5.54) is an expression of the mass-deformed curve of the $SU(2)$ SQCD with four flavors.

Recall that the mass-deformed curve of the $SU(2)$ SQCD with four flavors is given as [13]

$$v^2 = \frac{U}{P_3(w)} + \frac{M_4(w)}{P_3(w)^2} , \tag{5.55}$$

where the SW curve is now $\lambda = vdw$. Here U is the VEV of the scalar in $SU(2)$ vector multiplet, and $M_4(w)$ is a fourth-order polynomial of w and related to the mass parameters. Since there is a constraint on $M_4(w)$ coefficients, the polynomial $M_4(w)$ has four independent coefficients.

Comparing (5.54) with (5.55), we see that $(c_{03}+c_{11}w)^2/4$ is identified with $M_4(w)$. Thus two mass parameters of the former are identified with the mass parameters of the latter.

To see this identification, let us focus on the residues of the SW 1-form. The residues of the SW 1-form are known to be identified with the linear combinations of the mass parameters. For $SU(2)$ SQCD with four flavors, these residues are obtained by $m_1, \cdots, m_4$ of fundamental hypermultiplets as

$$m_1 \pm m_2 \ , \qquad m_3 \pm m_4 \ . \tag{5.56}$$

We see that the SW 1-form has four simple poles at $w = \pm\sqrt{\mathsf{q}}$, $1/\mathsf{q}$, and ∞. Thus the residues of the SW 1-form at these simple poles are obtained as

$$-\frac{c_{03} \pm c_{11}\sqrt{\mathsf{q}}}{12(\mathsf{q} - 1/\sqrt{\mathsf{q}})} \ , \qquad -\frac{c_{03} + \frac{c_{11}}{\mathsf{q}}}{6\left(\frac{1}{\mathsf{q}} - \sqrt{\mathsf{q}}\right)\left(\frac{1}{\mathsf{q}} + \sqrt{\mathsf{q}}\right)} \ , \qquad 0 \ , \tag{5.57}$$

In the weak coupling limit $\mathsf{q} \to 0$, one reduces to

$$\frac{m \pm 2M}{2} \ , \qquad 2M \ , \qquad 0 \ . \tag{5.58}$$

Note that if two mass parameters of the former are identified as

$$m_1 = \frac{m}{2} \ , \qquad m_2 = m_3 = m_4 = M \ , \tag{5.59}$$

(5.58) coincides with (5.56). This means that the mass-deformed SW curve of the (A_2, A_5) theory is identified with that of the $SU(2)$ SQCD with four flavors when the mass parameters of the two theories are related by (5.59). The relation of mass parameters completely coincides with that for the prepotential (5.17). Therefore this is a non-trivial consistency check of our method.

6 Conclusion

In this thesis, we have proposed the Nekrasov formula for (A_1, D_N) theory as given in (3.19) and (3.57) by extending the generalized AGT correspondence to the case of $U(2)$ gauge group. This Nekrasov formula $\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}$ has been regarded as the contribution from the (A_1, D_N) theory to each fixed point on the $U(2)$ instanton moduli space. To that end, we have first defined the irregular state of Virasoro and Heisenberg algebras $(Vir \oplus H)$ in section 3. we have shown that the action of $Vir \oplus H$ on the irregular state for integer-rank has been completely determined as (3.6) and (3.7) by the extension of Gaiotto-Teschner's colliding limit construction [27]. For half-integer-rank, we have guessed that the subalgebra $\{L_{n>0}\}$ reduces to the commutative algebra in the classical limit $\epsilon_i \to 0$, and then the matrix elements have been evaluated by (3.38). In particular, in the case of the dimensionful parameters in the (A_1, D_N) theory turned off, we have shown that the non-vanishing eigenvalue in (3.38) was only $\mathfrak{b}_N$.

As we apply our formula to the (A_3, A_3) theory, which is the $SU(2)$ superconformal field theory coupled to two copies (A_1, D_4) theories and one fundamental hypermultiplet described in Figure 10, we can evaluate the Nekrasov partition function. In section 4, when we turned off some parameters, we found that the prepotential relates to that of the $SU(2)$ superconformal QCD in (4.20) or (4.31). From this relation, we have read off how the S-duality group acts on the UV gauge coupling via replacing of the gauge coupling of $SU(2)$ SQCD with four flavors as $q \to q^2$. And also, we have found the action of S-duality group on all parameters. That replacement is consistent with the replacement of the Schur index of the two theories as $\mathbf{q} \to \mathbf{q}^2$, however, $\mathbf{q}$ in the Schur index is different from the gauge coupling q and is a non-trivial function of q [38].

In section 5, we have applied our formula to the (A_2, A_5) theory, which is the $SU(2)$ superconformal field theory, including one each of (A_1, D_3), (A_1, D_6), and fundamental hypermultiplet in matter sector described in Figure 11. When all the relevant parameters and the VEVs of the Coulomb branch operators in this theory are turned off, we can evaluate the Nekrasov partition function of this theory, and then we found an analogous relation in the case of the (A_3, A_3) theory. From this relation, we have also read off the action of the S-duality group on the parameters, including the UV gauge coupling. Also, we have shown how the T-transformation acts on all the parameters at the level of the Seiberg-Witten curve,

The existence of the prepotential relation between the two conformally gauged AD theories and the $SU(2)$ with four flavors provide the non-trivial consistency check for our fourmula (3.19) and (3.57), since the curves of these theories have the same form.

The T-transformations of the (A_3, A_3) theory and the (A_2, A_5) theory correspond to

$$
\begin{aligned}
(A_3, A_3): \qquad & \theta_{\text{IR}} \to \theta_{\text{IR}} + \frac{2}{\pi} \ , \\
(A_2, A_5): \qquad & \theta_{\text{IR}} \to \theta_{\text{IR}} + \frac{3}{\pi} \ .
\end{aligned}
$$

The T-transformation of the $\mathcal{N} = 4$ SYMs exchanges the minimal magnetic monopole with the W-boson, while that of the $\mathcal{N} = 2$, $SU(2)$ with four flavors exchanges the monopole with the quark, of which electric charge is 1/2 of the W-boson [1, 3–6]. This means that the T-transformations exchange the monopole of minimal magnetic charge with the dyon whose electric charges are respectively half and 1/3 of that of fundamental quark for the (A_3, A_3) theory and the (A_2, A_5) theory, which are consistent with [35].

We have also applied our formula to the $SU(2)$ superconformal field, including three (A_1, D_3) theories in the matter sector, of which all the relevant couplings and the VEVs of the Coulomb branch operators were turned off. We have found that the prepotential of this theory vanishes. This is the same situation in the case of $\mathcal{N} = 4$ super Yang-Mills theories. Indeed, the Schur index of this superconformal theory is related to that of the $\mathcal{N} = 4$ $SU(2)$ SYM by changing variables, which has been studied in [78].

There are natural future directions. The most important direction is understanding the origin of the relations of the prepotentials (4.20) and (5.16). It would also be important to generalize our works to $SU(N)$ gauge theories coupled to Argyres-Douglas theories. To that end, we need a $U(N)$-version of the generalized AGT correspondence. The $SU(3)$-version of that has already been studied in [29]. The other important direction is the study of the Nekrasov-Shatashvili limit [79] of (A_3, A_3) and (A_2, A_5) deformed by Ω background, which combines our formula with the results [80–83]. It is also interesting to study the uplift of our formula to five dimensions. The AGT correspondence of five dimensions has been studied in [84–93].

Acknowledgements

The author is grateful to the many people who have supported and advised him.

Firstly, the author would like to express his gratitude to Professor Yuji Sugawara, who taught him many things as a graduate student. The author would also like to express his appreciation to Takahiro Nishinaka for introducing him to the subject of the AGT correspondence, and Takahiro Uetoko for consulting with him through the process of completing this thesis.

Secondly, the author would like to thank the colleagues in his laboratory at Ritsumeikan University.

Finally, the author would like to express his deep appreciation to his family. The author has had their support and encouragement.

A Special orthogonal Basis

The orthogonal basis $|a; Y_1, Y_2\rangle$ that have been used in section 3, is defined as the solutions to (3.11), which were first found in [44]. When we recovered the full ϵ_i-dependence, the first few examples of the orthogonal basis are written as follows:

$$|a;\emptyset,\emptyset\rangle = |a\rangle\ , \tag{A.1}$$

$$|a;\Box,\emptyset\rangle = \left(-i(\epsilon_1+\epsilon_2+2a)a_{-1} - L_{-1}\right)|a\rangle\ , \tag{A.2}$$

$$\begin{aligned}|a;{\scriptsize\begin{array}{c}\Box\\[-4pt]\Box\end{array}},\emptyset\rangle = \Big(& - i\epsilon_2(\epsilon_1+\epsilon_2+2a)(\epsilon_1+2\epsilon_2+2a)a_{-2}\\ &- (\epsilon_1+\epsilon_2+2a)\,(\epsilon_1+2\epsilon_2+2a)\,a_{-1}^2\\ &+ 2i(\epsilon_1+2\epsilon_2+2a)a_{-1}L_{-1} - \epsilon_2(\epsilon_1+\epsilon_2+2a)L_{-2} + L_{-1}^2\Big)|a\rangle\ ,\end{aligned} \tag{A.3}$$

$$\begin{aligned}|a;\Box\Box,\emptyset\rangle = \Big(& - i\epsilon_1(\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-2}\\ &- (\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-1}^2\\ &+ 2i(2\epsilon_1+\epsilon_2+2a)a_{-1}L_{-1} - \epsilon_1(\epsilon_1+\epsilon_2+2a)L_{-2} + L_{-1}^2\Big)|a\rangle\ ,\end{aligned} \tag{A.4}$$

$$\begin{aligned}|a;\Box,\Box\rangle = \Big(& - i\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)a_{-2} - (\epsilon_1^2+\epsilon_2^2+\epsilon_1\epsilon_2-4a^2)a_{-1}^2\\ &+ 2i(\epsilon_1+\epsilon_2)a_{-1}L_{-1} - \epsilon_1\epsilon_2L_{-2} + L_{-1}^2\Big)|a\rangle\ ,\end{aligned} \tag{A.5}$$

$$\begin{aligned}|a;{\scriptsize\begin{array}{c}\Box\\[-4pt]\Box\\[-4pt]\Box\end{array}},\emptyset\rangle = \Big(& - 2i\epsilon_2^2(\epsilon_1+\epsilon_2+2a)(\epsilon_1+2\epsilon_2+2a)(\epsilon_1+3\epsilon_2+2a)a_{-3}\\ &- 3\epsilon_2(\epsilon_1+\epsilon_2+2a)(\epsilon_1+2\epsilon_2+2a)(\epsilon_1+3\epsilon_2+2a)a_{-2}a_{-1}\\ &+ i(\epsilon_1+\epsilon_2+2a)(\epsilon_1+2\epsilon_2+2a)(\epsilon_1+3\epsilon_2+2a)a_{-1}^3\\ &+ 3i\epsilon_2(\epsilon_1+2\epsilon_2+2a)(\epsilon_1+3\epsilon_2+2a)a_{-2}L_{-1}\\ &+ 3(\epsilon_1+2\epsilon_2+2a)(\epsilon_1+3\epsilon_2+2a)a_{-1}^2L_{-1}\\ &+ 3i\epsilon_2(\epsilon_1+\epsilon_2+2a)(\epsilon_1+3\epsilon_2+2a)a_{-1}L_{-2}\\ &- 3i(\epsilon_1+3\epsilon_2+2a)a_{-1}L_{-1}^2 - \epsilon_2^2(\epsilon_1+4\epsilon_2+4a)(\epsilon_1+\epsilon_2+2a)L_{-3}\\ &+ \epsilon_2(3\epsilon_1+5\epsilon_2+6a)L_{-2}L_{-1} - L_{-1}^3\Big)|a\rangle\ ,\end{aligned} \tag{A.6}$$

$$
\begin{aligned}
|a;\ydiagram{2,1},\emptyset\rangle = \Big(& -i\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2+2a)(\epsilon_1+2\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-3} \\
& -(\epsilon_1+\epsilon_2)(\epsilon_1+\epsilon_2+2a)(\epsilon_1+2\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-2}a_{-1} \\
& +i(\epsilon_1+\epsilon_2+2a)(\epsilon_1+2\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-1}^3 \\
& +i(\epsilon_1+\epsilon_2)(\epsilon_1+2\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-2}L_{-1} \\
& +3(\epsilon_1+2\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-1}^2L_{-1} \\
& +i(\epsilon_1+\epsilon_2+2a)(\epsilon_1^2+5\epsilon_1\epsilon_2+\epsilon_2^2+2(\epsilon_1+\epsilon_2)a)a_{-1}L_{-2} \\
& -i(5\epsilon_1+5\epsilon_2+6a)a_{-1}L_{-1}^2-\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2+2a)^2L_{-3} \\
& +(\epsilon_1^2+3\epsilon_1\epsilon_2+\epsilon_2^2+2(\epsilon_1+\epsilon_2)a)L_{-2}L_{-1}-L_{-1}^3\Big)|a\rangle\ ,
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
|a;\ydiagram{3},\emptyset\rangle = \Big(& -2i\epsilon_1^2(\epsilon_1+\epsilon_2+2a)(3\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-3} \\
& -3\epsilon_1(\epsilon_1+\epsilon_2+2a)(3\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-2}a_{-1} \\
& +i(\epsilon_1+\epsilon_2+2a)(3\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-1}^3 \\
& +3i\epsilon_1(3\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-2}L_{-1} \\
& +3(3\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-1}^2L_{-1} \\
& +3i\epsilon_1(\epsilon_1+\epsilon_2+2a)(3\epsilon_1+\epsilon_2+2a)a_{-1}L_{-2} \\
& -3i(3\epsilon_1+\epsilon_2+2a)a_{-1}L_{-1}^2-\epsilon_1^2(\epsilon_1+\epsilon_2+2a)(4\epsilon_1+\epsilon_2+4a)L_{-3} \\
& +\epsilon_1(5\epsilon_1+3\epsilon_2+6a)L_{-2}L_{-1}-L_{-1}^3\Big)|a\rangle\ ,
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
|a;\ydiagram{1,1},\square\rangle = \Big(& -2i\epsilon_1\epsilon_2^2(\epsilon_1+\epsilon_2)(\epsilon_1+\epsilon_2+2a)a_{-3} \\
& -\epsilon_2(\epsilon_1+\epsilon_2+2a)(3\epsilon_1^2+3\epsilon_1\epsilon_2+2\epsilon_2^2-2\epsilon_2a-4a^2)a_{-2}a_{-1} \\
& +i(\epsilon_1+\epsilon_2+2a)(\epsilon_1^2+\epsilon_1\epsilon_2+2\epsilon_2^2-2\epsilon_2a-4a^2)a_{-1}^3 \\
& +i\epsilon_2(3\epsilon_1^2+7\epsilon_1\epsilon_2+2\epsilon_2^2+(4\epsilon_1+6\epsilon_2)a+4a^2)a_{-2}L_{-1} \\
& +(3\epsilon_1^2+7\epsilon_1\epsilon_2+6\epsilon_2^2+(4\epsilon_1+2\epsilon_2)a-4a^2)a_{-1}^2L_{-1} \\
& +i\epsilon_2(3\epsilon_1+\epsilon_2-2a)(\epsilon_1+\epsilon_2+2a)a_{-1}L_{-2} \\
& -i(3\epsilon_1+5\epsilon_2+2a)a_{-1}L_{-1}^2-\epsilon_1\epsilon_2^2(\epsilon_1+\epsilon_2+2a)L_{-3} \\
& +\epsilon_2(3\epsilon_1+\epsilon_2+2a)L_{-2}L_{-1}-L_{-1}^3\Big)|a\rangle\ ,
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
|a;\square\square,\square\rangle = \Big(& -2i\epsilon_1^2\epsilon_2(\epsilon_1+\epsilon_2)(\epsilon_1+\epsilon_2+2a)a_{-3} \\
& -\epsilon_1(\epsilon_1+\epsilon_2+2a)(2\epsilon_1^2+3\epsilon_1\epsilon_2+3\epsilon_2^2-2\epsilon_1 a-4a^2)a_{-2}a_{-1} \\
& +i(\epsilon_1+\epsilon_2+2a)(2\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2-2\epsilon_1 a-4a^2)a_{-1}^3 \\
& +i\epsilon_1(2\epsilon_1^2+7\epsilon_1\epsilon_2+3\epsilon_2^2+(6\epsilon_1+4\epsilon_2)a+4a^2)a_{-2}L_{-1} \\
& +(6\epsilon_1^2+7\epsilon_1\epsilon_2+3\epsilon_2^2+(2\epsilon_1+4\epsilon_2)a-4a^2)a_{-1}^2L_{-1} \\
& +i\epsilon_1(\epsilon_1+3\epsilon_2-2a)(\epsilon_1+\epsilon_2+2a)a_{-1}L_{-2} \\
& -i(5\epsilon_1+3\epsilon_2+2a)a_{-1}L_{-1}^2-\epsilon_1^2\epsilon_2(\epsilon_1+\epsilon_2+2a)L_{-3} \\
& +\epsilon_1(\epsilon_1+3\epsilon_2+2a)L_{-2}L_{-1}-L_{-1}^3\Big)|a\rangle \ .
\end{aligned}
\tag{A.10}
$$

It is straightforward to obtain $|a;Y_2,Y_1\rangle$ by using the relation $|P;Y_2,Y_1\rangle = |-P;Y_1,Y_2\rangle$.

B Deformed prepotential

Here, we list the first terms of instanton part of the prepotential in (4.32):

$$
\mathcal{F}_{-2} = q^2+\frac{13}{8}q^4+\frac{23}{6}q^6+\frac{2701}{256}q^8+\mathcal{O}(q^{10}) \ , \tag{B.1}
$$

$$
\begin{aligned}
\mathcal{F}_2 = {} & \frac{q}{2}Mu_1u_2+\frac{q^2}{16}\big(-4d_1^2d_2^2M^2-8d_1^2M^2m_2-d_1^2u_2^2 \\
& \qquad -8d_2^2M^2m_1-d_2^2u_1^2-16M^2m_1m_2-2m_1u_2^2-2m_2u_1^2\big) \\
& +\frac{q^3}{24}\big(12d_1^3d_2M^2-6d1^2d_2Mu_1+12d_1d_2^3M^2-6d_1d_2^2Mu_2+16d_1d_2M^3 \\
& \qquad +24d_1d_2M^2m_1+24d_1d_2M^2m_2+3d1d_2u_1^2+3d_1d_2u_2^2 \\
& \qquad -12d1Mm_2u_2-12d_2Mm_1u_1+12Mu_1u_2\big)+\mathcal{O}(q^4) \ ,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\mathcal{F}_4 = {} & \frac{3q^2}{64}\left(4d_1^2M^2u_2^2+4d_2^2M^2u_1^2+8M^2m_1u_2^2+8M^2m_2u_1^2+u_1^2u_2^2\right) \\
& +\frac{q^3}{48}\Big(-12d_1^2d_2^2Mu_1u_2+20d_1^2d_2M^3u_1-24d_1^2Mm_2u_1u_2+20d_1d_2^2M^3u_2 \\
& \qquad -18d_1d_2M^2u_1^2-18d_1d_2M^2u_2^2+40d_1M^3m_2u_2+5d_1Mu_2^3-24d_2^2Mm_1u_1u_2 \\
& \qquad +40d_2M^3m_1u_1+5d_2Mu_1^3-48Mm_1m_2u_1u_2\Big)+\mathcal{O}(q^4) \ .
\end{aligned}
\tag{B.3}
$$

It is straightforward to check that there are invariant under (4.33).

References

[1] N. Seiberg and E. Witten, "Monopoles, Duality and Chiral Symmetry Breaking in $\mathcal{N}=2$ Supersymmetric QCD," *Nucl. Phys. B* **431** (1994) 484–550, `arXiv:hep-th/9408099`.

[2] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys. B* **426** (1994) 19–52, `arXiv:hep-th/9407087`. [Erratum: Nucl.Phys.B 430, 485–486 (1994)].

[3] C. Montonen and D. I. Olive, "Magnetic Monopoles as Gauge Particles?," *Phys. Lett. B* **72** (1977) 117–120.

[4] P. Goddard, J. Nuyts, and D. I. Olive, "Gauge Theories and Magnetic Charge," *Nucl. Phys. B* **125** (1977) 1–28.

[5] E. Witten and D. I. Olive, "Supersymmetry Algebras That Include Topological Charges," *Phys. Lett. B* **78** (1978) 97–101.

[6] H. Osborn, "Topological Charges for N=4 Supersymmetric Gauge Theories and Monopoles of Spin 1," *Phys. Lett. B* **83** (1979) 321–326.

[7] T. J. Hollowood, "Calculating the prepotential by localization on the moduli space of instantons," *JHEP* **03** (2002) 038, `arXiv:hep-th/0201075`.

[8] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," *Adv. Theor. Math. Phys.* **7** no. 5, (2003) 831–864, `arXiv:hep-th/0206161`.

[9] N. Nekrasov and A. Okounkov, "Seiberg-Witten theory and random partitions," *Prog. Math.* **244** (2006) 525–596, `arXiv:hep-th/0306238`.

[10] H. Nakajima and K. Yoshioka, "Instanton counting on blowup. 1.," *Invent. Math.* **162** (2005) 313–355, `arXiv:math/0306198`.

[11] H. Nakajima and K. Yoshioka, "Lectures on instanton counting," in *CRM Workshop on Algebraic Structures and Moduli Spaces.* 11, 2003. `arXiv:math/0311058`.

[12] E. Witten, "Solutions of Four-Dimensional Field Theories via M Theory," *Nucl. Phys.* **B500** (1997) 3–42, `arXiv:hep-th/9703166 [hep-th]`. [,452(1997)].

[13] D. Gaiotto, "$\mathcal{N}=2$ Dualities," *JHEP* **08** (2012) 034, `arXiv:0904.2715 [hep-th]`.

[14] D. Gaiotto, G. W. Moore, and A. Neitzke, "Wall-Crossing, Hitchin Systems, and the WKB Approximation," arXiv:0907.3987 [hep-th].

[15] J. J. Heckman, D. R. Morrison, and C. Vafa, "On the Classification of 6D SCFTs and Generalized ADE Orbifolds," *JHEP* **05** (2014) 028, arXiv:1312.5746 [hep-th]. [Erratum: JHEP 06, 017 (2015)].

[16] M. Del Zotto, J. J. Heckman, A. Tomasiello, and C. Vafa, "6d Conformal Matter," *JHEP* **02** (2015) 054, arXiv:1407.6359 [hep-th].

[17] J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, "Atomic Classification of 6D SCFTs," *Fortsch. Phys.* **63** (2015) 468–530, arXiv:1502.05405 [hep-th].

[18] O. Chacaltana and J. Distler, "Tinkertoys for Gaiotto Duality," *JHEP* **11** (2010) 099, arXiv:1008.5203 [hep-th].

[19] O. Chacaltana and J. Distler, "Tinkertoys for the D_N series," *JHEP* **02** (2013) 110, arXiv:1106.5410 [hep-th].

[20] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-Dimensional Gauge Theories," *Lett. Math. Phys.* **91** (2010) 167–197, arXiv:0906.3219 [hep-th].

[21] D. Gaiotto, "Asymptotically free $\mathcal{N} = 2$ theories and irregular conformal blocks," *J. Phys. Conf. Ser.* **462** no. 1, (2013) 012014, arXiv:0908.0307 [hep-th].

[22] P. C. Argyres and M. R. Douglas, "New Phenomena in $SU(3)$ Supersymmetric Gauge Theory," *Nucl. Phys.* **B448** (1995) 93–126, arXiv:hep-th/9505062 [hep-th].

[23] P. C. Argyres, M. R. Plesser, N. Seiberg, and E. Witten, "New $\mathcal{N} = 2$ Superconformal Field Theories in Four-Dimensions," *Nucl. Phys.* **B461** (1996) 71–84, arXiv:hep-th/9511154 [hep-th].

[24] T. Eguchi, K. Hori, K. Ito, and S.-K. Yang, "Study of $\mathcal{N} = 2$ Superconformal Field Theories in Four-Dimensions," *Nucl. Phys.* **B471** (1996) 430–444, arXiv:hep-th/9603002 [hep-th].

[25] D. Xie, "General Argyres-Douglas Theory," *JHEP* **01** (2013) 100, arXiv:1204.2270 [hep-th].

[26] G. Bonelli, K. Maruyoshi, and A. Tanzini, "Wild Quiver Gauge Theories," *JHEP* **02** (2012) 031, `arXiv:1112.1691 [hep-th]`.

[27] D. Gaiotto and J. Teschner, "Irregular Singularities in Liouville Theory and Argyres-Douglas Type Gauge Theories, I," *JHEP* **12** (2012) 050, `arXiv:1203.1052 [hep-th]`.

[28] H. Kanno and M. Taki, "Generalized Whittaker states for instanton counting with fundamental hypermultiplets," *JHEP* **05** (2012) 052, `arXiv:1203.1427 [hep-th]`.

[29] H. Kanno, K. Maruyoshi, S. Shiba, and M. Taki, "W_3 Irregular States and Isolated $\mathcal{N}=2$ Superconformal Field Theories," *JHEP* **03** (2013) 147, `arXiv:1301.0721 [hep-th]`.

[30] T. Nishinaka and C. Rim, "Matrix models for irregular conformal blocks and Argyres-Douglas theories," *JHEP* **10** (2012) 138, `arXiv:1207.4480 [hep-th]`.

[31] H. Itoyama, T. Oota, and K. Yano, "Discrete Painleve System and the Double Scaling Limit of the Matrix Model for Irregular Conformal Block and Gauge Theory," `arXiv:1805.05057 [hep-th]`.

[32] H. Itoyama, T. Oota, and K. Yano, "Discrete Painlevé system for the partition function of $N_f = 2$ $SU(2)$ supersymmetric gauge theory and its double scaling limit," *J. Phys. A* **52** no. 41, (2019) 415401, `arXiv:1812.00811 [hep-th]`.

[33] T. Nishinaka and T. Uetoko, "Argyres-Douglas Theories and Liouville Irregular States," *JHEP* **09** (2019) 104, `arXiv:1905.03795 [hep-th]`.

[34] M. Buican, S. Giacomelli, T. Nishinaka, and C. Papageorgakis, "Argyres-Douglas Theories and S-Duality," *JHEP* **02** (2015) 185, `arXiv:1411.6026 [hep-th]`.

[35] S. Cecotti and M. Del Zotto, "Higher S-dualities and Shephard-Todd groups," *JHEP* **09** (2015) 035, `arXiv:1507.01799 [hep-th]`.

[36] S. Cecotti, M. Del Zotto, and S. Giacomelli, "More on the N=2 superconformal systems of type $D_p(G)$," *JHEP* **04** (2013) 153, `arXiv:1303.3149 [hep-th]`.

[37] M. Del Zotto, C. Vafa, and D. Xie, "Geometric engineering, mirror symmetry and $6\mathrm{d}_{(1,0)} \to 4\mathrm{d}_{(\mathcal{N}=2)}$," *JHEP* **11** (2015) 123, `arXiv:1504.08348 [hep-th]`.

[38] M. Buican, L. Li, and T. Nishinaka, "Peculiar Index Relations, 2D TQFT, and Universality of SUSY Enhancement," *JHEP* **01** (2020) 187, `arXiv:1907.01579 [hep-th]`.

[39] D. Xie and W. Yan, "Schur sector of Argyres-Douglas theory and W-algebra," *SciPost Phys.* **10** no. 3, (2021) 080, `arXiv:1904.09094 [hep-th]`.

[40] J. Choi and T. Nishinaka, "On the chiral algebra of Argyres-Douglas theories and S-duality," *JHEP* **04** (2018) 004, `arXiv:1711.07941 [hep-th]`.

[41] M. Buican and T. Nishinaka, "On the superconformal index of Argyres–Douglas theories," *J. Phys. A* **49** no. 1, (2016) 015401, `arXiv:1505.05884 [hep-th]`.

[42] T. Kimura, T. Nishinaka, Y. Sugawara, and T. Uetoko, "Argyres-Douglas theories, S-duality and AGT correspondence," *JHEP* **04** (2021) 205, `arXiv:2012.14099 [hep-th]`.

[43] T. Kimura and T. Nishinaka, "On the Nekrasov Partition Function of Gauged Argyres-Douglas Theories," *JHEP* **01** (2022) 30, `arXiv:2206.10937 [hep-th]`.

[44] V. A. Alba, V. A. Fateev, A. V. Litvinov, and G. M. Tarnopolskiy, "On combinatorial expansion of the conformal blocks arising from AGT conjecture," *Lett. Math. Phys.* **98** (2011) 33–64, `arXiv:1012.1312 [hep-th]`.

[45] V. Alba and A. Morozov, "Check of AGT Relation for Conformal Blocks on Sphere," *Nucl. Phys. B* **840** (2010) 441–468, `arXiv:0912.2535 [hep-th]`.

[46] S. Cecotti, A. Neitzke, and C. Vafa, "R-Twisting and 4D/2D Correspondences," `arXiv:1006.3435 [hep-th]`.

[47] A. Mironov and A. Morozov, "The Power of Nekrasov Functions," *Phys. Lett. B* **680** (2009) 188–194, `arXiv:0908.2190 [hep-th]`.

[48] A. Marshakov, A. Mironov, and A. Morozov, "Zamolodchikov asymptotic formula and instanton expansion in N=2 SUSY N(f) = 2N(c) QCD," *JHEP* **11** (2009) 048, `arXiv:0909.3338 [hep-th]`.

[49] A. Mironov and A. Morozov, "Proving AGT relations in the large-c limit," *Phys. Lett. B* **682** (2009) 118–124, `arXiv:0909.3531 [hep-th]`.

[50] V. A. Fateev and A. V. Litvinov, "On AGT conjecture," *JHEP* **02** (2010) 014, `arXiv:0912.0504 [hep-th]`.

[51] L. Hadasz, Z. Jaskolski, and P. Suchanek, "Proving the AGT relation for $N_f = 0,1,2$ antifundamentals," *JHEP* **06** (2010) 046, `arXiv:1004.1841 [hep-th]`.

[52] A. Marshakov, A. Mironov, and A. Morozov, "On non-conformal limit of the AGT relations," *Phys. Lett. B* **682** (2009) 125–129, `arXiv:0909.2052 [hep-th]`.

[53] C. Vafa, "Evidence for F theory," *Nucl. Phys. B* **469** (1996) 403–418, `arXiv:hep-th/9602022`.

[54] A. Sen, "F theory and orientifolds," *Nucl. Phys. B* **475** (1996) 562–578, `arXiv:hep-th/9605150`.

[55] T. Banks, M. R. Douglas, and N. Seiberg, "Probing F theory with branes," *Phys. Lett. B* **387** (1996) 278–281, `arXiv:hep-th/9605199`.

[56] K. Dasgupta and S. Mukhi, "F theory at constant coupling," *Phys. Lett. B* **385** (1996) 125–131, `arXiv:hep-th/9606044`.

[57] J. A. Minahan and D. Nemeschansky, "An N=2 superconformal fixed point with E(6) global symmetry," *Nucl. Phys. B* **482** (1996) 142–152, `arXiv:hep-th/9608047`.

[58] J. A. Minahan and D. Nemeschansky, "Superconformal fixed points with E(n) global symmetry," *Nucl. Phys. B* **489** (1997) 24–46, `arXiv:hep-th/9610076`.

[59] A. Klemm, W. Lerche, S. Yankielowicz, and S. Theisen, "Simple singularities and N=2 supersymmetric Yang-Mills theory," *Phys. Lett. B* **344** (1995) 169–175, `arXiv:hep-th/9411048`.

[60] A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. P. Warner, "Selfdual strings and N=2 supersymmetric field theory," *Nucl. Phys. B* **477** (1996) 746–766, `arXiv:hep-th/9604034`.

[61] N. J. Hitchin, "Stable bundles and integrable systems," *Duke Math. J.* **54** (1987) 91–114.

[62] M. R. Douglas and G. W. Moore, "D-branes, quivers, and ALE instantons," `arXiv:hep-th/9603167`.

[63] D. Gaiotto and J. Maldacena, "The Gravity duals of N=2 superconformal field theories," *JHEP* **10** (2012) 189, arXiv:0904.4466 [hep-th].

[64] F. Benini, S. Benvenuti, and Y. Tachikawa, "Webs of five-branes and N=2 superconformal field theories," *JHEP* **09** (2009) 052, arXiv:0906.0359 [hep-th].

[65] S. Cecotti and C. Vafa, "Classification of complete N=2 supersymmetric theories in 4 dimensions," arXiv:1103.5832 [hep-th].

[66] Y. Wang and D. Xie, "Classification of Argyres-Douglas theories from M5 branes," *Phys. Rev. D* **94** no. 6, (2016) 065012, arXiv:1509.00847 [hep-th].

[67] D. Xie and P. Zhao, "Central charges and RG flow of strongly-coupled N=2 theory," *JHEP* **03** (2013) 006, arXiv:1301.0210 [hep-th].

[68] A. D. Shapere and C. Vafa, "BPS structure of Argyres-Douglas superconformal theories," arXiv:hep-th/9910182.

[69] R. Flume and R. Poghossian, "An Algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential," *Int. J. Mod. Phys. A* **18** (2003) 2541, arXiv:hep-th/0208176.

[70] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini, "Multiinstanton calculus and equivariant cohomology," *JHEP* **05** (2003) 054, arXiv:hep-th/0211108.

[71] F. Fucito, J. F. Morales, and R. Poghossian, "Instantons on quivers and orientifolds," *JHEP* **10** (2004) 037, arXiv:hep-th/0408090.

[72] T. W. Grimm, A. Klemm, M. Marino, and M. Weiss, "Direct Integration of the Topological String," *JHEP* **08** (2007) 058, arXiv:hep-th/0702187.

[73] N. Wyllard, "A(N-1) conformal Toda field theory correlation functions from conformal N = 2 SU(N) quiver gauge theories," *JHEP* **11** (2009) 002, arXiv:0907.2189 [hep-th].

[74] A. Mironov and A. Morozov, "On AGT relation in the case of U(3)," *Nucl. Phys. B* **825** (2010) 1–37, arXiv:0908.2569 [hep-th].

[75] M. Taki, "On AGT Conjecture for Pure Super Yang-Mills and W-algebra," *JHEP* **05** (2011) 038, arXiv:0912.4789 [hep-th].

[76] C. Beem and W. Peelaers, "Argyres-Douglas Theories in Class S Without Irregularity," arXiv:2005.12282 [hep-th].

[77] P. C. Argyres, M. R. Plesser, and A. D. Shapere, "The Coulomb Phase of $\mathcal{N}=2$ Supersymmetric QCD," *Phys. Rev. Lett.* **75** (1995) 1699–1702, arXiv:hep-th/9505100 [hep-th].

[78] M. Buican and T. Nishinaka, "$\mathcal{N}=4$ SYM, Argyres-Douglas Theories, and an Exact Graded Vector Space Isomorphism," arXiv:2012.13209 [hep-th].

[79] N. A. Nekrasov and S. L. Shatashvili, "Quantization of Integrable Systems and Four Dimensional Gauge Theories," in *16th International Congress on Mathematical Physics*, pp. 265–289. 8, 2009. arXiv:0908.4052 [hep-th].

[80] K. Ito, S. Kanno, and T. Okubo, "Quantum periods and prepotential in $\mathcal{N}=2$ SU(2) SQCD," *JHEP* **08** (2017) 065, arXiv:1705.09120 [hep-th].

[81] K. Ito and T. Okubo, "Quantum periods for $\mathcal{N}=2$ $SU(2)$ SQCD around the superconformal point," *Nucl. Phys.* **B934** (2018) 356–379, arXiv:1804.04815 [hep-th].

[82] K. Ito, S. Koizumi, and T. Okubo, "Quantum Seiberg-Witten curve and Universality in Argyres-Douglas theories," *Phys. Lett. B* **792** (2019) 29–34, arXiv:1903.00168 [hep-th].

[83] K. Ito, S. Koizumi, and T. Okubo, "Quantum Seiberg-Witten periods for $\mathcal{N}=2$ $SU(N_c)$ SQCD around the superconformal point," *Nucl. Phys. B* **954** (2020) 115004, arXiv:2001.08891 [hep-th].

[84] H. Awata and Y. Yamada, "Five-dimensional AGT Conjecture and the Deformed Virasoro Algebra," *JHEP* **01** (2010) 125, arXiv:0910.4431 [hep-th].

[85] H. Awata and Y. Yamada, "Five-dimensional AGT Relation and the Deformed beta-ensemble," *Prog. Theor. Phys.* **124** (2010) 227–262, arXiv:1004.5122 [hep-th].

[86] S. Yanagida, "Five-dimensional SU(2) AGT conjecture and recursive formula of deformed Gaiotto state," *J. Math. Phys.* **51** (2010) 123506, arXiv:1005.0216 [math.QA].

[87] M. Taki, "On AGT-W Conjecture and q-Deformed W-Algebra," `arXiv:1403.7016 [hep-th]`.

[88] V. Mitev and E. Pomoni, "Toda 3-Point Functions From Topological Strings," *JHEP* **06** (2015) 049, `arXiv:1409.6313 [hep-th]`.

[89] M. Isachenkov, V. Mitev, and E. Pomoni, "Toda 3-Point Functions From Topological Strings II," *JHEP* **08** (2016) 066, `arXiv:1412.3395 [hep-th]`.

[90] H. Awata, H. Fujino, and Y. Ohkubo, "Crystallization of deformed Virasoro algebra, Ding-Iohara-Miki algebra and 5D AGT correspondence," *J. Math. Phys.* **58** no. 7, (2017) 071704, `arXiv:1512.08016 [math-ph]`.

[91] J.-E. Bourgine, M. Fukuda, Y. Matsuo, H. Zhang, and R.-D. Zhu, "Coherent states in quantum $\mathcal{W}_{1+\infty}$ algebra and qq-character for 5d Super Yang-Mills," *PTEP* **2016** no. 12, (2016) 123B05, `arXiv:1606.08020 [hep-th]`.

[92] S. Pasquetti, "Holomorphic blocks and the 5d AGT correspondence," *J. Phys. A* **50** no. 44, (2017) 443016, `arXiv:1608.02968 [hep-th]`.

[93] A. Neguţ, "The q-AGT-W relations via shuffle algebras," *Commun. Math. Phys.* **358** no. 1, (2018) 101–170, `arXiv:1608.08613 [math.RT]`.