Doctoral Thesis

Stability problems of stochastic differential equations by a symmetric stable process

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Stability problems of stochastic differential equations by a symmetric stable process
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Abstract

In this thesis, we consider a coefficient stability problem for one-dimensional stochastic differential equations driven by an α-stable process with $\alpha \in (1, 2)$. More precisely, we find an upper bound for the $L^{\alpha-1}(\Omega, \mathbb{P})$ distance between two solutions in terms of the $L^{\alpha}\left(\mathbb{R}, \mu_{x_0}^{\alpha}\right)$ distance of the equation coefficients for an appropriate measure $\mu_{x_0}^{\alpha}$ which characterizes symmetric stable laws and depends on the initial value of the stochastic differential equation.

The organization of the thesis is as follows. In Chapter 1, we introduce the background and the outline of this thesis. We also explain the meaning and implications of this study.

In Chapter 2, we describe the notations and basic definitions used in this thesis. We then recall a result proven by Kulik [10] who gives an upper bound of the density function of the solution of the stochastic differential equation.

In Chapter 3, we state the first of two main results of this thesis, which is a rate for the $L^{\alpha-1}(\Omega, \mathbb{P})$ distance between two solutions and give its proof. We obtain this result using the method introduced by Komatsu [11] which is used in the proof of uniqueness of solutions together with an upper bound for the transition density function of the solution of the stochastic differential equation obtained by Kulik in [9].

In Chapter 4, we present the second main result, which is a rate of uniform convergence of two solutions in probability when the difference between the coefficients converge to zero. We also introduce the definition of quasi-martingale and its properties and prove the second main result.

In Chapter 5, we present some accessory lemmas needed for proving our results.

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Chapter 1

1 Introduction

Consider a d-dimensional ordinary differential equation (ODE) defined by

$$\frac{dx_t}{dt} = b(x_t)dt, \quad x_0 = 0,$$

for $t \geq 0$, where the function $b : \mathbb{R}^d \to \mathbb{R}^d$. It is well-known that we can prove the existence and the uniqueness of a solution $(x_t)_{t\geq 0}$ of the ODE by using the Picard iteration when b is (globally) Lipschitz continuous (see [1], Theorem 6.1.3). However, when the function b is not Lipschitz continuous, there may be multiple solutions of the ODE, for example, when $d = 1$ and $b(x) = 2\mathrm{sign}(x)\sqrt{|x|}$. In the case of stochastic differential equations (SDEs), it is known that the uniqueness of solutions may be preserved even if the coefficients are not Lipschitz continuous.

Let $\sigma = (\sigma_{i,j})$, $b = (b_i)$, $X = (X_i)$ and $B = (B_i)$ for $i, j = 1, 2, \ldots, d$ be functions defined on $\mathbb{R}^d$, a $\mathbb{R}^d$-valued random variable and a $\mathbb{R}^d$-valued Brownian motion, respectively. We consider a d-dimensional stochastic differential equation following as

$$dX_i(t) = b_i(X_i(t))dt + \sum_{j=1}^{d} \sigma_{i,j}(X_i(t))dB_t^i, \qquad i = 1, 2, \ldots, d. \tag{1}$$

When $d = 1$ and b and σ are linear functions, the above SDE is well-known as the Black–Scholes model in finance.

The model is based on the assumption that stock prices are continuous with respect to time, and it has been applied in practice. However, stock prices are not continuous with respect to time and can jump. In order to deal with this, Lévy processes which include jumps are being applied to finance, and research on SDE's using these processes is being actively conducted. This is also applied in the engineering field when considering dynamical systems. The solutions of uniformly elliptic SDE's driven by Brownian motion do not possess heavy-tailed distribution, but SDE's driven by Lévy processes can have this property and have been actively studied in recent years. The term "heavy-tailed distribution" refers to a phenomenon in which the probability of a huge loss, such as in the record of insurance losses, is relatively high compared with Gaussian distributed models.

Let $T > 0$ and $Z := (Z_t)_{0\leq t\leq T}$ be a symmetric α-stable process with $\alpha \in (1, 2)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions. That is, Z is one of the stochastic processes with satisfy the properties which define Lévy processes.

In particular, their characteristic function (Fourier transform) is described by the formula

$$-\log\left(\mathbb{E}\left[e^{i\theta Z_t}\right]\right) = t|\theta|^{\alpha} \qquad \text{for any } t \geq 0 \text{ and } \theta \in \mathbb{R}.$$

We consider any solution $X := (X_t)_{0\leq t\leq T}$ to the following one-dimensional SDE

$$X_t = x_0 + \int_0^t \sigma(X_{s-})dZ_s, \tag{2}$$

where $x_0 \in \mathbb{R}$ and $\sigma : \mathbb{R} \to [0, \infty)$ and $X_{s-} := \lim_{u \uparrow s} X_u$. The infinitesimal generator A_α of Z is defined by

$$A_\alpha f(x) := \int_{\mathbb{R}\setminus\{0\}} \left\{ f(x+y) - f(x) - 1_{\{|y|\le 1\}} y f'(x) \right\} \frac{dy}{|y|^{1+\alpha}}, \text{ for any } f \in C_b^2(\mathbb{R}),\ x \in \mathbb{R}.$$

We briefly explain some known results in this area. In the case where the driving process is a Brownian motion, for example, Yamada and Watanabe [19] showed the pathwise uniqueness of solutions when the coefficient σ is $(1/2)$-Hölder continuous. The convergence of the Euler-Maruyama approximation for SDE (1) in the pathwise sense has been shown under non-Lipschitz condition for coefficients (see Kaneko-Nakao [7] and Yamada [18]). Here, the approximation is $X(n, \cdot) = (X_i(n, \cdot))$ for some time partition $(t_k)_{k=0,1,\dots,n}$ on $[0, T]$ defined by

$$X_i(n, 0) := X_i(0) \text{ and } X_i(n, t) := X_i(n, \eta_n(t)) + \int_{\eta_n(t)}^t b_i(X(n, \eta_n(s)))ds + \sum_{j=1}^d \int_{\eta_n(t)}^t \sigma_{i,j}(X(n, \eta_n(s)))dB_t^i$$

for any $t \in (0, T]$ and $i = 1, 2, \dots, d$, where $\eta_n(t) = t_k$ for any $t \in [t_{k-1}, t_k)$. In the case of a symmetric α-stable process, there are also many preceding studies. In the one-dimensional case, Komatsu [11] and Bass et.al [2] proved the pathwise uniqueness of solutions of SDE (2) if σ is $(1/\alpha)$-Hölder continuous. In the case of non-Lipshitz coefficient, Tsuchiya [16] obtained the pathwise uniqueness of solutions in the multi-dimensional case and Kulik [9] studied existence of the unique weak solution and the Gaussian boundedness of the density function of the solution. The convergence of the Euler-Maruyama approximation and the existence of strong solutions for the SDE (2) has been shown (see Hashimoto [4]).

We consider how the solution changes if the coefficients are altered. This situation is related to the so called stability problem. Let $X^{(n)} := \left(X_t^{(n)} \right)_{0 \le t \le T}$ be a solution of the following SDE

$$X_t^{(n)} = x_0^{(n)} + \int_0^t \sigma_n \left(X_{s-}^{(n)} \right) dZ_s \text{ for each } n \in \mathbb{N}, \tag{3}$$

where $x_0^{(n)} \in \mathbb{R}$ and σ_n is bounded non-negative γ-Hölder continuous. Stability problem for solutions of SDEs driven by a semimartingale with Lipschitz coefficients has been developed by Émery [3] (in the linear case) and Protter [15]. Kawabata and Yamada [8] also studied the stability problems in the case of SDEs driven by the Brownian motion with non-Lipschitz coefficients. Hashimoto [4] proved the convergence in the $L^\beta(\Omega, \mathbb{P})$-norm of the time-supremum distance between two solutions with $\beta \in (1, \alpha)$ when the sequence $(\sigma_n)_{n\in\mathbb{N}}$ uniformly converges to σ and satisfies Komatsu condition (see [4]), but the author did not obtain the rates of convergence. Hashimoto and Tsuchiya [5] got the rates of convergence in the case, $x_0 = x_0^{(n)}$:

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left| X_t - X_t^{(n)} \right|^{\alpha-1} \right] \le C \|\sigma - \sigma_n\|_\infty^p \text{ for some } p, C > 0. \tag{4}$$

The aim of this thesis is to extend the result (4) to the convergence in $L^\alpha \left(\mathbb{R}, \mu_{x_0}^\alpha \right)$-norm. We will prove the following inequality for $C > 0$ and an explicit value of p which depends on the problem parameters.

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left| X_t - X_t^{(n)} \right|^{\alpha-1} \right] \le |x_0 - x_0^{(n)}|^{\alpha-1} + C \|\sigma - \sigma_n\|_{L^\alpha\left(\mathbb{R}, \mu_{x_0}^\alpha\right)}^p. \tag{5}$$

Here, the finite measure $\mu_{x_0}^\alpha$ is defined as

$$\mu_{x_0}^\alpha(dy) := \left(|y - x_0|^{-1-\alpha} \wedge 1 \right) dy, \tag{6}$$

where the constant x_0 is the initial value of SDE (2).This finite measure $\mu^{\alpha}_{x_0}$ has features that it decreases with distance from the initial value x_0. The space $L^{\alpha}\left(\mathbb{R}, \mu^{\alpha}_{x_0}\right)$ is composed of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that the following norm is finite:

$$\|f\|_{L^{\alpha}\left(\mathbb{R},\mu^{\alpha}_{x_0}\right)} := \left(\int_{\mathbb{R}} |f(y)|^{\alpha} \mu^{\alpha}_{x_0}(dy)\right)^{\frac{1}{\alpha}}. \tag{7}$$

One defines similarly the spaces $L^p(\Omega, \mathbb{P})$ and $L^p(\mathbb{R}) := L^p(\mathbb{R}, \text{Leb})$ for any $p > 0$, where Leb is the Lebesgue measure.

We state two applications of the result (5). In the first application, we consider a Cauchy sequence of coefficients $(\sigma_n)_{n\in\mathbb{N}}$ in the norm $\|\cdot\|_{L^{\alpha}\left(\mathbb{R},\mu^{\alpha}_{x_0}\right)}$. Then we prove that there exists a subsequence limit of SDE (3) such that the limit is the unique solution of the SDE corresponding to the limit of the subsequence of coefficients. For more details, see Subsection 5.3.

In the second application, given in (5), we consider the case that the coefficient σ satisfies conditions that guarantee uniqueness of X but the exact value of σ is unknown on points distant from the initial value x_0. Under this situation, we may still approximate X using a coefficient σ_n which approximates σ near x_0. Result (5) ensures that the error estimation is small if the difference between σ_n and σ is small under the $L^{\alpha}\left(\mathbb{R}, \mu^{\alpha}_{x_0}\right)$-norm. For more details, see Example 3.1.

The method to prove (5) uses the pathwise uniqueness by Komatsu [11] and the estimate of the density of solutions to SDE (2) by Kulik [10]. We also give the rate of uniform convergence of solutions in probability. This is proven by using the properties of quasi-martingales given by Kurtz [12]. This doctoral thesis is in line with the content of Nakagawa [14].

Chapter 2

2 Notation and the density of the weak unique solution for SDE (2)

2.1 Notation

In this subsection, we explain the symbols used in this thesis.
We define minimum and maximum as $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$, respectively. We denote the gamma function by $\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx$, for any $z > 0$. The notation f^p stands for the p power of the function f. The uniform norm $\|\cdot\|_{\infty}$ for any real valued function f is denoted as $\|f\|_{\infty} = \sup_{x\in\mathbb{R}} |f(x)|$. The convolution of the functions f and g is denoted by $f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$, for any $x \in \mathbb{R}$. The Fourier transform of the function f is denoted by $\mathcal{F}(f)(\theta) = \int_{-\infty}^{\infty} e^{-i\theta x} f(x)dx$, for any $\theta \in \mathbb{R}$. The inverse Fourier transformation of the function f is denoted by $\mathcal{F}^{-1}(f)(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\theta x} f(x)dx$, for any $\theta \in \mathbb{R}$. The notation (P.V.) means the Cauchy principal value. A measure μ is absolutely continuous with respect to a measure μ' if $\mu'(A) = 0$ implies $\mu(A) = 0$ for every measurable set A. This is written as $\mu \ll \mu'$. Let $Z = (Z_t)_{0\le t\le T}$ be a symmetric α-stable process with $\alpha \in (1, 2)$. The jump size of Z at time t is defined by $\Delta Z_t = Z_t - Z_{t-}$ for any $t > 0$ and $\Delta Z_0 := 0$. The Poisson random measure associated to Z on $\mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R} \setminus \{0\})$ is denoted by $N(t, F) = \sum_{0\le s\le t} 1_F(\Delta Z_s)$ for $t \in [0, T]$ and $F \in \mathcal{B}(\mathbb{R}\setminus\{0\})$. The Lévy measure of Z on $\mathcal{B}(\mathbb{R}\setminus\{0\})$ is defined

as $\frac{c_\alpha}{|z|^{1+\alpha}}dz$, where the constant $c_\alpha := \pi^{-1}\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)$. The compensated Poisson random measure of Z is defined as $\widetilde{N}$.

2.2 The transition density function associated to the SDE (2)

In this subsection, we introduce results which Kulik [10] has proved. These results are about the transition density function of the solution of SDE (2). The following result shows that if the function σ^α is Hölder continuous, bounded and uniformly positive, then the solution of SDE (2) admits a transition density function.

Lemma 2.1. *([10], Theorems 2.2 and Proposition 4.1)*
Suppose that $(X_t)_{0\le t\le T}$ is the unique weak solution of the SDE

$$X_t = x_0 + \int_0^t \sigma(X_{s-})dZ_s,$$

where $Z := (Z_t)_{0\le t\le T}$ is an one-dimensional symmetric α-stable process with $\alpha \in (1,2)$, and σ^α satisfies the following conditions. There exist c_1, $c_2 > 0$ and $\gamma' \in (0,1)$ such that for any x, $y \in \mathbb{R}$,

$$c_1 < \sigma(x), \quad \|\sigma\|_\infty < \infty \ \text{ and } \ |\sigma^\alpha(x) - \sigma^\alpha(y)| < c_2|x-y|^{\gamma'}.$$

Then, for each $t \in (0,T]$, X_t has a transition density function $p_t(x_0,\cdot)$ and the function satisfies that there exists positive constants M_1, M_2 such that for any $y \in \mathbb{R}$,

$$M_1\widetilde{p}_t(x_0,y) \le p_t(x_0,y) \le M_2\widetilde{p}_t(x_0,y). \tag{8}$$

The function $\widetilde{p}_t$ is given by

$$\widetilde{p}_t(x_0,y) := \frac{1}{t^{\frac{1}{\alpha}}\sigma(y)} g^{(\alpha)}\left(\frac{y-x_0}{t^{\frac{1}{\alpha}}\sigma(y)}\right),$$

where $g^{(\alpha)}$ is the density function of Z_1. A function $G^{(\alpha)}$ denotes $G^{(\alpha)}(x) := \left(|x|^{-1-\alpha} \wedge 1\right)$ for any $x \in \mathbb{R}$.

Moreover, $g^{(\alpha)}$ and $G^{(\alpha)}$ satisfy the following properties. There exist constants K_1, $K_2 > 0$ such that for any $c_3 > 0$ and $x \in \mathbb{R}$,

$$G^{(\alpha)}(c_3 x) \le (c_3^{-1-\alpha} \vee 1)G^{(\alpha)}(x) \tag{9}$$

$$\text{and } \ K_1 G^{(\alpha)}(x) < g^{(\alpha)}(x) < K_2 G^{(\alpha)}(x). \tag{10}$$

The following upper bound for the density $p_t(x_0,\cdot)$ of X_t is useful for proving Theorem 3.1, 4.1.

Lemma 2.2. *There exists $K > 0$ such that for any $t \in (0,T]$,*

$$p_t(x_0,y) \le Kt^{-\frac{1}{\alpha}}\left(t^{1+\frac{1}{\alpha}} \vee 1\right) G^{(\alpha)}(y-x_0).$$

Proof. From (8) and (10), we obtain

$$p_t(x_0,y) \le \frac{M_2}{t^{\frac{1}{\alpha}}\sigma(y)} g^{(\alpha)}\left(\frac{y-x_0}{t^{\frac{1}{\alpha}}\sigma(y)}\right) \le \frac{M_2K_2}{t^{\frac{1}{\alpha}}\sigma(y)} G^{(\alpha)}\left(\frac{y-x_0}{t^{\frac{1}{\alpha}}\sigma(y)}\right).$$

Here, by the definition of $G^{(\alpha)}$, we have for all $x \in \mathbb{R}$ and $0 \le y \le z$,

$$G^{(\alpha)}(x) = G^{(\alpha)}(|x|) \text{ and } G^{(\alpha)}(y) \le G^{(\alpha)}(z).$$

Hence, from $c_1 < \sigma(x)$, we have

$$G^{(\alpha)}\left(\frac{y-x_0}{t^{\frac{1}{\alpha}}\sigma(y)}\right) = G^{(\alpha)}\left(\left|\frac{y-x_0}{t^{\frac{1}{\alpha}}\sigma(y)}\right|\right) \le G^{(\alpha)}\left(\left|\frac{y-x_0}{t^{\frac{1}{\alpha}}c_1}\right|\right) = G^{(\alpha)}\left(\frac{y-x_0}{t^{\frac{1}{\alpha}}c_1}\right). \tag{11}$$

It follows from (9), (11) and $c_1 < \sigma(x)$ that

$$p_t^0(x_0, y) \le K_2 c_1^{-1}\left(c_1^{1+\alpha} \vee 1\right) t^{-\frac{1}{\alpha}} \left(t^{1+\frac{1}{\alpha}} \vee 1\right) G^{(\alpha)}(y - x_0).$$

Hence the proof of Lemma 2.2 is completed. □

Chapter 3

3 The distance of two solutions in the $L^{\alpha-1}(\Omega, \mathbb{P})$-norm

In this Chapter, the first main result of this thesis is given. Hashimoto and Tsuchiya [5] obtained the distance between two solutions using the supremum norm. We give this distance in term of $L^\alpha\left(\mathbb{R}, \mu_{x_0}^\alpha\right)$-norm defined in (7).

Theorem 3.1. *Let $T > 0$ and Z be a one-dimensional symmetric α-stable process with $\alpha \in (1, 2)$. Consider two processes X and $\widetilde{X}$ that satisfy the following one-dimensional SDEs for $t \in [0, T]$ and x_0, $\widetilde{x}_0 \in \mathbb{R}$.*

$$X_t = x_0 + \int_0^t \sigma(X_{s-})dZ_s \tag{12}$$

$$\widetilde{X}_t = \widetilde{x}_0 + \int_0^t \widetilde{\sigma}(\widetilde{X}_{s-})dZ_s. \tag{13}$$

Suppose that σ and $\widetilde{\sigma}$ satisfy the following conditions. There exist constants ρ, $\widetilde{\rho}$, m_1, m_2, $m_3 > 0$, $\eta \in (0, 1]$ and $\gamma \in [1/\alpha, 1]$ such that for any $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| < \rho|x - y|^\eta, \quad |\widetilde{\sigma}(x) - \widetilde{\sigma}(y)| < \widetilde{\rho}|x - y|^\gamma,$$
$$m_1 < \sigma(x) < m_2, \quad \|\widetilde{\sigma}\|_\infty < m_3 \text{ and } \|\sigma - \widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R}, \mu_{x_0}^\alpha\right)} \le 1.$$

Then, there exists a positive constant C such that

$$\sup_{0 \le t \le T} \mathbb{E}\left[\left|X_t - \widetilde{X}_t\right|^{\alpha-1}\right] \le \begin{cases} |x_0 - \widetilde{x}_0|^{\alpha-1} + C\|\sigma - \widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R}, \mu_{x_0}^\alpha\right)}^{(\alpha\gamma-1)/\gamma} & \text{if } \gamma \in (1/\alpha, 1] \\ |x_0 - \widetilde{x}_0|^{\alpha-1} + C\left(\log \dfrac{1}{\|\sigma - \widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R}, \mu_{x_0}^\alpha\right)}}\right)^{-1} & \text{if } \gamma = 1/\alpha, \end{cases} \tag{14}$$

where the constant C depends on T, α, m_1, m_2, m_3, ρ, $\widetilde{\rho}$ and η.

Remark 3.2. We give two separate remarks:

(i) Theorem 3.1 holds even if the norm $\|\cdot\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}$ is replaced by $\|\cdot\|_{L^\alpha(\mathbb{R})}$. The reason is that the following inequality is satisfied: $\|f\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)} \leq \|f\|_{L^\alpha(\mathbb{R})}$ for any $f \in L^\alpha(\mathbb{R})$ and $x_0 \in \mathbb{R}$.

(ii) If $\gamma \in (1/\alpha, 1]$ and the assumption $\|\sigma - \widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)} \leq 1$ is replaced by $\|\sigma - \widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)} > 1$ in Theorem 3.1, then the power $(\alpha\gamma - 1)/\gamma$ in the inequality (14) is replaced by $\alpha - 1$.

When $\gamma = 1/\alpha$, the second term in the inequality (14) is changed. These results are obtained in the same way as in the proof of Theorem 3.1.

Note that in Theorem 3.1 the coefficient σ is required to have a positive lower bound and upper bound for the existence of the transition density function of X_t as stated in Lemma 2.1. The solution of SDE $(X_t)_{t\in[0,T]}$ fails the pathwise uniqueness property if $\eta < 1/\alpha$ (see [2]). Still, Theorem 3.1 holds for such solutions. The result in Theorem 3.1 can be slightly improved at the cost of higher complexity as stated in Section 5.4. In the following example, we show that the applicable class of diffusion coefficients has been significantly improved in comparison with [5].

Example 3.1. *We consider the SDE (2) and (3) with diffusion coefficients σ and σ_n, respectively. Suppose that σ and $(\sigma_n)_{n\in\mathbb{N}}$ satisfy the following conditions. There exist constants ρ, $\widetilde{\rho}$, m_1, m_2, $m_3 > 0$, $\eta \in (0, 1]$ and $\gamma \in [1/\alpha, 1]$ such that for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$|\sigma(x) - \sigma(y)| < \rho|x - y|^\eta, \quad |\sigma_n(x) - \sigma_n(y)| < \widetilde{\rho}|x - y|^\gamma,$$
$$m_1 < \sigma(x) < m_2 \ \textit{and} \ \|\sigma_n\|_\infty < m_3$$

$$|\sigma(x) - \sigma_n(x)| \begin{cases} \leq \frac{C}{n} & \textit{if } x \in D_{x_0,f(n)} := \{x \in \mathbb{R} \mid |x - x_0| \leq f(n)\} \\ = g(n, x) & \textit{if } x \notin D_{x_0,f(n)} \end{cases} \quad x \in \mathbb{R}, \tag{15}$$

where x_0 is the initial value of SDE (2), $f(n) > 1$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} f(n) = \infty$ and g is a some function. Then, for some constant C and each $n \in \mathbb{N}$, we have

$$\|\sigma - \sigma_n\|^\alpha_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)} \leq \frac{C}{n^\alpha} f(n) + \frac{C}{f^\alpha(n)} \sup_{f(n)\leq|x-x_0|} g^\alpha(n, x).$$

In fact, the result is obtained using the definition (6) and then estimating $\|\sigma - \sigma_n\|^\alpha_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}$ by dividing the region of integration according to $D_{x_0,f(n)}$ and its complementary set. For the first integral, one uses the inequality $|x - x_0|^{-1-\alpha} \wedge 1 \leq 1$ and (15). For the second integral, one notes that $|\sigma(x) - \sigma_n(x)| \leq \sup_{f(n)\leq|x-x_0|} |\sigma(x) - \sigma_n(x)|$ for any $x \in D_{x_0,f(n)}$ and then the remaining integral can be computed explicitly. Setting $\sup_{f(n)\leq|x-x_0|} g(n, x) = 1$ and $f(n) = n^p$ for each $n \in \mathbb{N}$ and some $p \in (0, \alpha)$, we obtain $\lim_{n\to\infty} \|\sigma - \sigma_n\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)} = 0$. However, in this case, the limit of the sup norm does not converge to 0:

$$\lim_{n\to\infty} \|\sigma - \sigma_n\|_\infty = 1.$$

This example shows that Theorem 3.1 may be useful for a bounded η-Hölder continuous function σ whose exact values are unknown on intervals distant from the initial value x_0.

We state two remarks in order to deepen the understanding of Theorem 3.1.

Remark 3.3. One may wonder if $\lim_{n\to\infty}\sup_{t\in[0,T]}\mathbb{E}[|X_t - X_t^{(n)}|^{\alpha-1}] = 0$ even if the equation satisfied by X may not satisfy the pathwise uniqueness property. If the Hölder coefficient of σ_n is not bounded in n, then the positive constant C in (14) may be unbounded in n since C depends on Hölder coefficient of σ_n, so that the right-hand side in (14) may not be converge to 0 as $n\to\infty$.

On the other hand, if $\lim_{n\to\infty}\|\sigma-\sigma_n\|_{L^\alpha\left(\mathbb{R},\mu_{x_0}^\alpha\right)} = 0$ and the Hölder coefficient of σ_n does not depend on n, then $\sigma = \lim_{n\to\infty}\sigma_n$ almost everywhere and σ is also γ-Hölder continuous with $\gamma\in[1/\alpha,1]$ (see (37) in Section 5.3). In this case, the equations associated to $(X_t)_{0\le t\le T}$ and $(X_t^{(n)})_{0\le t\le T}$ satisfy the pathwise uniqueness property.

Remark 3.4. In Theorem 3.1, if $\eta > 1/\alpha$, then the equations (12) and (13) satisfy the strong unique solution property. If $\eta < 1/\alpha$, (12) satisfies the weak unique solution property on a probability space (see [9]) and (13) satisfies the strong solution property on the probability space (see [4]). Thus, there always exists a probability space satisfying (14).

3.1 An approach by Komatsu

Before proving Theorem 3.1, we apply a variation of the method introduced by Komatsu ([11], proof of Theorem 1) in order to evaluate $\left|X_t - \widetilde{X}_t\right|^{\alpha-1}$.

Lemma 3.5. *For $\varepsilon > 0$, $\delta > 1$, we can choose a smooth function $\psi_{\delta,\varepsilon}$ which satisfies the following conditions,*

$$\psi_{\delta,\varepsilon}(x) = \begin{cases} \text{between } 0 \text{ and } 2(x\log\delta)^{-1} & \varepsilon\delta^{-1} < x < \varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

and $\int_{\varepsilon\delta^{-1}}^{\varepsilon}\psi_{\delta,\varepsilon}(y)dy = 1$. We define $u(x) = |x|^{\alpha-1}$ and $u_{\delta,\varepsilon} = u*\psi_{\delta,\varepsilon}$. Then, $u_{\delta,\varepsilon}\in C^2$ and for any $x\in\mathbb{R}$,

$$|x|^{\alpha-1} \le \varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(x), \tag{16}$$

$$u_{\delta,\varepsilon}(x) \le |x|^{\alpha-1} + \varepsilon^{\alpha-1}. \tag{17}$$

Proof. It follows from $\int_{\varepsilon\delta^{-1}}^{\varepsilon} 2(x\log\delta)^{-1}dx = 2$ that the above function $\psi_{\delta,\varepsilon}$ exists and that $u_{\delta,\varepsilon}$ is in C^2. First, we prove the inequality (16). Note that $u_{\delta,\varepsilon}$ is a nonnegative function since u and $\psi_{\delta,\varepsilon}$ are nonnegative functions. Since the support of $\psi_{\delta,\varepsilon}$ is contained in $[\varepsilon\delta^{-1},\varepsilon]$, we have

$$u_{\delta,\varepsilon}(x) = \int_{x-\varepsilon}^{x-\varepsilon\delta^{-1}} |y|^{\alpha-1}\psi_{\delta,\varepsilon}(x-y)dy.$$

The inequality (16) is proved by studying three different cases according to Case 1: $-\varepsilon\le x<\varepsilon$, Case 2: $\varepsilon\le x$ or Case 3: $x<-\varepsilon$.

Case 1: Since $|x|\le\varepsilon$, we have

$$|x|^{\alpha-1} \le \varepsilon^{\alpha-1} \le \varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(x).$$

The last inequality follows from $u_{\delta,\varepsilon}(x) \geq 0$ for all $x \in \mathbb{R}$.

Case 2: Since $0 \leq x - \varepsilon$, we obtain $|y| \geq |x - \varepsilon|$ for all $y \in [x - \varepsilon, x - \varepsilon\delta^{-1}]$. The inequality $|x|^p - |y|^p \leq |x + y|^p$ for any $x, y \in \mathbb{R}$ and $p \in (0, 1)$ and $\int_{\varepsilon\delta^{-1}}^{\varepsilon} \psi_{\delta,\varepsilon}(y)dy = 1$ imply that

$$u_{\delta,\varepsilon}(x) \geq |x - \varepsilon|^{\alpha-1} \int_{x-\varepsilon}^{x-\varepsilon\delta^{-1}} \psi_{\delta,\varepsilon}(x - y)dy \geq |x|^{\alpha-1} - \varepsilon^{\alpha-1}.$$

Hence we have

$$|x|^{\alpha-1} \leq \varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(x).$$

Case 3: Since $x - \varepsilon\delta^{-1} < 0$ then from the inequality $|y| \geq |x - \varepsilon\delta^{-1}|$ for all $y \in [x - \varepsilon, x - \varepsilon\delta^{-1}]$, we have

$$\begin{aligned} u_{\delta,\varepsilon}(x) &\geq \left|x - \varepsilon\delta^{-1}\right|^{\alpha-1} \int_{x-\varepsilon}^{x-\varepsilon\delta^{-1}} \psi_{\delta,\varepsilon}(x - y)dy \\ &= \left(|x| + \varepsilon\delta^{-1}\right)^{\alpha-1} \\ &\geq \left||x| - \varepsilon\delta^{-1}\right|^{\alpha-1}. \end{aligned}$$

The last inequality follows from $a + b \geq |a - b|$ for any $a, b \geq 0$. By using the inequality $|a - b|^p \geq |a^p - b^p|$ for any $a, b \geq 0$ and $p \in (0, 1)$ and $\varepsilon\delta^{-1} < \varepsilon$, we have

$$u_{\delta,\varepsilon}(x) \geq \left||x|^{\alpha-1} - \left(\varepsilon\delta^{-1}\right)^{\alpha-1}\right| \geq |x|^{\alpha-1} - \varepsilon^{\alpha-1}.$$

Thus, we obtain (16).

In order to show the inequality (17), we rewrite $u_{\delta,\varepsilon}(x)$ as

$$u_{\delta,\varepsilon}(x) = \int_{\mathbb{R}} u(x - y)\psi_{\delta,\varepsilon}(y)dy = \int_{\varepsilon\delta^{-1}}^{\varepsilon} |x - y|^{\alpha-1}\psi_{\delta,\varepsilon}(y)dy.$$

The last equation follows since the support of $\psi_{\delta,\varepsilon}$ is included in $[\varepsilon\delta^{-1}, \varepsilon]$. By the inequalities $|x-y| \leq |x|+|y|$ for any $x, y \in \mathbb{R}$ and $(a + b)^{\alpha-1} \leq a^{\alpha-1} + b^{\alpha-1}$ for any $a, b \geq 0$, we have (17) since

$$\begin{aligned} u_{\delta,\varepsilon}(x) &\leq \int_{\varepsilon\delta^{-1}}^{\varepsilon} (|x| + |y|)^{\alpha-1} \psi_{\delta,\varepsilon}(y)dy \\ &\leq \int_{\varepsilon\delta^{-1}}^{\varepsilon} \left(|x|^{\alpha-1} + |y|^{\alpha-1}\right) \psi_{\delta,\varepsilon}(y)dy \\ &\leq \left(|x|^{\alpha-1} + \varepsilon^{\alpha-1}\right) \int_{\varepsilon\delta^{-1}}^{\varepsilon} \psi_{\delta,\varepsilon}(y)dy \\ &\leq |x|^{\alpha-1} + \varepsilon^{\alpha-1}. \end{aligned}$$

The second to last inequality and the last one follows from $\psi_{\delta,\varepsilon}(x) \geq 0$ and $\int_{\varepsilon\delta^{-1}}^{\varepsilon} \psi_{\delta,\varepsilon}(y)dy = 1$, respectively. This concludes the statement. □

Definition 3.6. *A function g is called a tempered function (or a function of slow growth or a slowly increasing function) if g is a continuous function and for some $p \in \mathbb{N}$,*

$$|g(x)| = O(|x|^p) \ \ as \ \ |x| \to \infty.$$

The following lemma is used for proving Lemma 3.8.

Lemma 3.7. *For $\theta \neq 0$ and a tempered function g,*

$$\mathcal{F}(A_\alpha g)(\theta) = -\pi \left(\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1} |\theta|^\alpha \mathcal{F}(g)(\theta). \tag{18}$$

Note that the Fourier transform of tempered functions is finite (see [13], Definition 1, 2).

Proof. By the definition of $\mathcal{F}$ and A_α, we have

$$\mathcal{F}(A_\alpha g)(\theta) = \int_{-\infty}^{\infty} \mathrm{e}^{-i\theta x} \int_{\mathbb{R}\setminus\{0\}} \left\{ g(x+y) - g(x) - 1_{\{|y|\le 1\}} y g(x) \right\} \frac{dy}{|y|^{1+\alpha}} dx.$$

It follows from Fubini's theorem that

$$\begin{aligned}\mathcal{F}(A_\alpha g)(\theta) &= \int_{\mathbb{R}\setminus\{0\}} \int_{-\infty}^{\infty} \left\{ \frac{\mathrm{e}^{-i\theta x} g(x+y)}{|y|^{1+\alpha}} - \frac{\mathrm{e}^{-i\theta x} g(x)}{|y|^{1+\alpha}} - 1_{\{|y|\le 1\}} \frac{y g(x)}{|y|^{1+\alpha}} \right\} dx dy,\\ &= \int_{\mathbb{R}\setminus\{0\}} \left\{ \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i\theta x} g(x+y)}{|y|^{1+\alpha}} dx - \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i\theta x} g(x)}{|y|^{1+\alpha}} dx - 1_{\{|y|\le 1\}} \frac{y}{|y|^{1+\alpha}} \int_{-\infty}^{\infty} \mathrm{e}^{-i\theta x} g(x) dx \right\} dy.\end{aligned}$$

The integrals with respect to x in the last equation are finite since g is a tempered function. Furthermore, since $1_{\{|y|\le 1\}} \frac{y}{|y|^{1+\alpha}}$ is an odd function with respect to y, we have

$$\mathcal{F}(A_\alpha g)(\theta) = \int_{\mathbb{R}\setminus\{0\}} \left\{ \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i\theta x} g(x+y)}{|y|^{1+\alpha}} dx - \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i\theta x} g(x)}{|y|^{1+\alpha}} dx - 0 \right\} dy \text{ (P.V.)}.$$

Using the change of variables $w = x + y$ on $\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\theta x} g(x+y)}{|y|^{1+\alpha}} dx$, we have

$$\begin{aligned}\mathcal{F}(A_\alpha g)(\theta) &= \int_{\mathbb{R}\setminus\{0\}} \left\{ \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i\theta(w-y)} g(w)}{|y|^{1+\alpha}} dw - \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-i\theta x} g(x)}{|y|^{1+\alpha}} dx \right\} dy\\ &= \int_{\mathbb{R}\setminus\{0\}} \frac{\mathrm{e}^{i\theta y} - 1}{|y|^{1+\alpha}} dy \mathcal{F}(g)(\theta).\end{aligned}$$

Performing the change of variables $|\theta| y = z$, we get

$$\mathcal{F}(A_\alpha g)(\theta) = \int_{\mathbb{R}\setminus\{0\}} \frac{\mathrm{e}^{iz\mathrm{sgn}(\theta)} - 1}{|z|^{1+\alpha}} dz |\theta|^\alpha \mathcal{F}(g)(\theta).$$

Therefore, in order to obtain (18), we should prove only

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{iz} - 1}{|z|^{1+\alpha}} dz = \int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz} - 1}{|z|^{1+\alpha}} dz = -\pi \left(\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1}.$$

Using the change of variables $z = -z'$, we have

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{iz} - 1}{|z|^{1+\alpha}} dz = \int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz'} - 1}{|z'|^{1+\alpha}} dz'.$$

Hence, it is enough to prove that

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz}-1}{|z|^{1+\alpha}}dz = -\pi\left(\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1}. \tag{19}$$

From Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ for all $x \in \mathbb{R}$, we obtain

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz}-1}{|z|^{1+\alpha}}dz = \int_{\mathbb{R}\setminus\{0\}} \frac{\cos(z)-1}{|z|^{1+\alpha}}dz + i\int_{\mathbb{R}\setminus\{0\}} \frac{\sin(z)}{|z|^{1+\alpha}}dz.$$

Using the fact that $\frac{\cos(z)-1}{|z|^{1+\alpha}}$ is an even function and $\frac{\sin(z)}{|z|^{1+\alpha}}$ is an odd function, we have

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz}-1}{|z|^{1+\alpha}}dz = 2\int_0^\infty \frac{\cos(z)-1}{|z|^{1+\alpha}}dz + 0\ (P.V.).$$

Using the integration by parts formula, we have

$$2\int_0^\infty \frac{\cos(z)-1}{|z|^{1+\alpha}}dz = 2\lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{1}{\alpha}\left([(\cos(z)-1)z^{-\alpha}]_\varepsilon^R - \int_\varepsilon^R \frac{\sin(z)}{z^\alpha}dz\right).$$

Observe that first the term equals 0 on the right-hand side by L'Hôpital's theorem. Hence, we have

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz}-1}{|z|^{1+\alpha}}dz = 0 - \frac{2}{\alpha}\int_0^\infty \frac{\sin(z)}{z^\alpha}dz.$$

Here, $\int_0^\infty x^{s-1}\sin(ax)dx = \Gamma(s)a^{-s}\sin\left(\frac{s\pi}{2}\right)$ for any $a > 0$ and $0 < |\mathrm{Re}(s)| < 1$ ([6], P430). By setting $s = 1-\alpha$, we have

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz}-1}{|z|^{1+\alpha}}dz = -\frac{2}{\alpha}\Gamma(1-\alpha)\sin\left(\frac{(1-\alpha)\pi}{2}\right) = -\frac{2}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\alpha\pi}{2}\right).$$

From the reflection formula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(s\pi)}$ for $s > 0$, we have

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz}-1}{|z|^{1+\alpha}}dz = -\frac{2}{\alpha}\Gamma(\alpha)^{-1}\frac{\pi}{\sin(\alpha\pi)}\cos\left(\frac{\alpha\pi}{2}\right).$$

By $\sin(\alpha\pi) = 2\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)$ and $\Gamma(s+1) = s\Gamma(s)$ for any $s \in C$, we get

$$\int_{\mathbb{R}\setminus\{0\}} \frac{e^{-iz}-1}{|z|^{1+\alpha}}dz = -\pi\left(\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1}.$$

Thus (19) is satisfied. Thus, the equation (18) follows. □

The following result is useful to obtain an estimate of $A_\alpha u_{\delta,\varepsilon}$ since the function $\psi_{\delta,\varepsilon}$ is bounded for each $\varepsilon > 0$ and $\delta > 1$ and $\lim_{\theta\to 0} A_\alpha u_{\delta,\varepsilon}(\theta) = 0$. The property $\lim_{\theta\to 0} A_\alpha u_{\delta,\varepsilon}(\theta) = 0$ is not satisfied if we replace $u(x) = |x|^{\alpha-1}$ by a general power function $u(x) = |x|^\beta$ with $\beta \neq \alpha - 1$.

Lemma 3.8. *The function $u_{\delta,\varepsilon}$ satisfies*

$$A_\alpha u_{\delta,\varepsilon}(\theta) = C_\alpha \psi_{\delta,\varepsilon}(\theta) \text{ for each } \theta \neq 0,$$

where the constant $C_\alpha = -2\pi\alpha \cot\left(\frac{\alpha\pi}{2}\right)$.

Proof. Set $u^{(q)}_{\delta,\varepsilon}(x) = u^{(q)} * \psi_{\delta,\varepsilon}(x)$ and $u^{(q)}(x) = |x|^{\alpha-1}\mathrm{e}^{-q|x|}$ for $q \in (0,1)$. From the characteristic function of the one-dimensional gamma distribution, we have

$$\int_0^\infty \mathrm{e}^{i\theta x} x^{\alpha-1}\Gamma(\alpha)^{-1} q^\alpha \mathrm{e}^{-qx} dx = (1 - i\theta/q)^{-\alpha}.$$

Simplifying this, we have

$$\int_0^\infty \mathrm{e}^{i\theta x} u^{(q)}(x) dx = \Gamma(\alpha)(q - i\theta)^{-\alpha}. \tag{20}$$

Using changing of variables $x = -x'$ in (20), we obtain

$$\int_0^{-\infty} \mathrm{e}^{i(-\theta)y} u^{(q)}(-x')(-dx') = \Gamma(\alpha)(q - i\theta)^{-\alpha}.$$

Since $u^{(q)}(-x') = u^{(q)}(x')$, we have by substituting θ for $-\theta'$

$$\int_{-\infty}^0 \mathrm{e}^{i\theta' x'} u^{(q)}(x') dx' = \Gamma(\alpha)(q + i\theta')^{-\alpha}. \tag{21}$$

These equations (20) and (21) imply that

$$\mathcal{F}(u^{(q)})(\theta) = \Gamma(\alpha)\left\{(q + i\theta)^{-\alpha} + (q - i\theta)^{-\alpha}\right\}. \tag{22}$$

Since $u^{(q)}_{\delta,\varepsilon}$ is a tempered function, it follows from Lemma 3.7 that

$$\mathcal{F}(A_\alpha u^{(q)}_{\delta,\varepsilon})(\theta) = -\pi\left(\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1} |\theta|^\alpha \mathcal{F}(u^{(q)}_{\delta,\varepsilon})(\theta).$$

It follows from the convolution theorem, (22) and $\Gamma(s+1) = s\Gamma(s)$ for any $s > 0$ that

$$\begin{aligned}
\mathcal{F}(A_\alpha u^{(q)}_{\delta,\varepsilon})(\theta) &= -\pi\left(\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1} |\theta|^\alpha \mathcal{F}(u^{(q)})(\theta)\mathcal{F}(\psi_{\delta,\varepsilon})(\theta) \\
&= -\pi|\theta|^\alpha \left\{(q+i\theta)^{-\alpha} + (q-i\theta)^{-\alpha}\right\}\left(\alpha\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1} \mathcal{F}(\psi_{\delta,\varepsilon})(\theta) \\
&\to -\pi\{i^{-\alpha} + i^\alpha\}\left(\alpha\sin\left(\frac{\alpha\pi}{2}\right)\right)^{-1} \mathcal{F}(\psi_{\delta,\varepsilon})(\theta) \quad \text{as} \quad q \to +0.
\end{aligned}$$

Using the principal value of $i^{-\alpha}$ and i^α it follows

$$i^{-\alpha} + i^\alpha = \mathrm{e}^{-i\alpha\pi/2} + \mathrm{e}^{i\alpha\pi/2} = 2\cos\left(\frac{\alpha\pi}{2}\right).$$

Thus, since $u^{(q)}_{\delta,\varepsilon}$ is monotone increasing to $u_{\delta,\varepsilon}$ as $q \downarrow 0$, we have

$$A_\alpha u_{\delta,\varepsilon}(\theta) = \lim_{q\to+0} A_\alpha u^{(q)}_{\delta,\varepsilon}(\theta) = \lim_{q\to+0} \mathcal{F}^{-1}\left[\mathcal{F}(A_\alpha u^{(q)}_{\delta,\varepsilon})\right](\theta) = -2\pi\alpha\cot\left(\frac{\alpha\pi}{2}\right)\psi_{\delta,\varepsilon}(\theta).$$

Hence the proof of Lemma 3.8 is completed. □

3.2 Proof of Theorem 3.1

The key point in this proof is to use the fact that the solution X_t has a transition density for which an upper bound is available (see Lemma 2.1, 2.2).

Proof. We set $Y_t = X_t - \widetilde{X}_t$. By Lemma 3.5, we obtain

$$|Y_t|^{\alpha-1} \leq \varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y_t). \tag{23}$$

By using the Lévy-Itô decomposition ([1], Theorem 2.4.16), we have

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right) dZ_s \\ &= Y_0 + \int_0^t \int_{|z|\geq 1} \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right) zN(dz, ds) + \int_0^t \int_{|z|<1} \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right) z\widetilde{N}(dz, ds). \end{aligned}$$

Using the Itô's formula ([1], Theorem 4.4.7) and $N(dz, dt) = \widetilde{N}(dz, dt) + \frac{c_\alpha}{|z|^{1+\alpha}} dzdt$, we have

$$\begin{aligned} u_{\delta,\varepsilon}(Y_t) &= u_{\delta,\varepsilon}(Y_0) + \int_0^t \int_{|z|\geq 1} \left\{u_{\delta,\varepsilon}\left(Y_{s-} + \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right) z\right) - u_{\delta,\varepsilon}(Y_{s-})\right\} N(dz, ds) \\ &\quad + \int_0^t \int_{|z|<1} \left\{u_{\delta,\varepsilon}\left(Y_{s-} + \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right) z\right) - u_{\delta,\varepsilon}(Y_{s-})\right\} \widetilde{N}(dz, ds) \\ &\quad + \int_0^t \int_{|z|<1} \left\{u_{\delta,\varepsilon}\left(Y_{s-} + \left(\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right) z\right) - u_{\delta,\varepsilon}(Y_{s-}) - z(\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s))u'_{\delta,\varepsilon}(Y_{s-})\right\} \frac{c_\alpha dz}{|z|^{1+\alpha}} ds \\ &= u_{\delta,\varepsilon}(Y_0) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left\{u_{\delta,\varepsilon}\left(Y_{s-} + \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right) z\right) - u_{\delta,\varepsilon}(Y_{s-})\right\} \widetilde{N}(dz, ds) \\ &\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left\{u_{\delta,\varepsilon}\left(Y_s + \left(\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right) z\right) - u_{\delta,\varepsilon}(Y_s) - 1_{\{|z|\leq 1\}}(z)(\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s))zu'_{\delta,\varepsilon}(Y_s)\right\} \frac{c_\alpha dz}{|z|^{1+\alpha}} ds \\ &=: u_{\delta,\varepsilon}(Y_0) + M_t^{\delta,\varepsilon} + I_t^{\delta,\varepsilon}. \end{aligned} \tag{24}$$

The function $u_{\delta,\varepsilon} \in C^2$ which appeared in Lemma 3.5. Here, $M^{\delta,\varepsilon} := \left(M_t^{\delta,\varepsilon}\right)_{0\leq t\leq T}$ is a martingale (see Section 5.1). Note that the integral $I_t^{\delta,\varepsilon}$ is equal to 0 on the set $\tau_t := \{s \in [0, t] \mid \sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s) = 0\}$. Using the change of variables $y = (\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s))z$ in $I_t^{\delta,\varepsilon}$, we have

$$I_t^{\delta,\varepsilon} = \int_{[0,t]\setminus\tau_t} \int_{\mathbb{R}\setminus\{0\}} \frac{\left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^{1+\alpha}}{\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)} \left\{u_{\delta,\varepsilon}((Y_s) + y) - u_{\delta,\varepsilon}(Y_s) - y1_{\{|y|\leq|\sigma(X_s)-\widetilde{\sigma}(\widetilde{X}_s)|\}}(y)u'_{\delta,\varepsilon}(Y_s)\right\} \frac{c_\alpha dy}{|y|^{1+\alpha}} ds.$$

Here, since $y1_{\{|y|\leq|\sigma(X_s)-\widetilde{\sigma}(\widetilde{X}_s)|\}}(y)u'_{\delta,\varepsilon}(Y_{s-})\frac{1}{|y|^{1+\alpha}}$ is an odd function with respect to y, we obtain

$$\begin{aligned} \int_{\mathbb{R}\setminus\{0\}} y1_{\{|y|\leq|\sigma(X_s)-\widetilde{\sigma}(\widetilde{X}_s)|\}}(y)u'_{\delta,\varepsilon}(Y_{s-}) \frac{\left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^{1+\alpha}}{|y|^{1+\alpha}} dy &= \int_{\mathbb{R}\setminus\{0\}} y1_{\{|y|\leq 1\}}(y)u'_{\delta,\varepsilon}(Y_{s-}) \frac{\left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^{1+\alpha}}{|y|^{1+\alpha}} dy \\ &= 0 \text{ (P.V.)}. \end{aligned}$$

Hence, we observe, by the definition of A_α, that

$$I_t^{\delta,\varepsilon} = c_\alpha \int_{[0,t]\setminus\tau_t} \frac{\left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^{1+\alpha}}{\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)} A_\alpha u_{\delta,\varepsilon}(Y_s)\, ds$$
$$= c_\alpha C_\alpha \int_{[0,t]\setminus\tau_t} \frac{\left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^{1+\alpha}}{\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)} \psi_{\delta,\varepsilon}(Y_s) ds, \tag{25}$$

The last equation follows from Lemma 3.8. We observe, using the Jensen's inequality, that

$$I_t^{\delta,\varepsilon} \le c_\alpha \left|C_\alpha\right| \int_0^t \left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^\alpha \psi_{\delta,\varepsilon}(Y_s) ds =: J_t^{\delta,\varepsilon}. \tag{26}$$

The boundedness of the function $\psi_{\delta,\varepsilon}$ introduced in Lemma 3.5 yields

$$J_t^{\delta,\varepsilon} \le 2c_\alpha \left|C_\alpha\right| \int_0^t \left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^\alpha \frac{1_{[\varepsilon\delta^{-1},\varepsilon]}(Y_s)}{|Y_s| \log\delta} ds.$$

Here, by using the inequality $(a+b)^\alpha \le 2^{\alpha-1}a^\alpha + 2^{\alpha-1}b^\alpha$ for any $a, b \ge 0$ and γ-Hölder continuity of $\widetilde{\sigma}$, we obtain

$$\begin{aligned} \left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^\alpha &= \left|\sigma(X_s) - \widetilde{\sigma}(X_s) + \widetilde{\sigma}(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^\alpha \\ &\le 2^{\alpha-1} \left|\widetilde{\sigma}(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^\alpha + 2^{\alpha-1} \left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha \\ &\le 2^{\alpha-1}\widetilde{\rho}^\alpha \left|Y_s\right|^{\alpha\gamma} + 2^{\alpha-1} \left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha. \end{aligned}$$

Hence, we have

$$\begin{aligned} J_t^{\delta,\varepsilon} &\le \widehat{C}_\alpha \int_0^t |Y_s|^{\alpha\gamma-1} \frac{1_{[\varepsilon\delta^{-1},\varepsilon]}(Y_s)}{\log\delta} ds + \widehat{C}_\alpha \int_0^t \left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha \frac{1_{[\varepsilon\delta^{-1},\varepsilon]}(Y_s)}{|Y_s| \log\delta} ds \\ &\le \frac{\widehat{C}_\alpha t \varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \int_0^t \left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha ds, \end{aligned} \tag{27}$$

where $\widehat{C}_\alpha := 2^\alpha c_\alpha \left|C_{\alpha,\beta}\right| \max\{\widetilde{\rho}^\alpha, 1\}$. Note that $\alpha\gamma - 1 \ge 0$ since $\alpha \in (1,2)$ and $\gamma \in [1/\alpha, 1]$. Using (23), (26), (27) and (17), we obtain

$$\begin{aligned} u_{\delta,\varepsilon}(Y_t) &\le u_{\delta,\varepsilon}(Y_0) + M_t^{\delta,\varepsilon} + \frac{\widehat{C}_\alpha t \varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \int_0^t \left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha ds \\ &\le |x_0 - \widetilde{x}_0|^{\alpha-1} + \varepsilon^{\alpha-1} + M_t^{\delta,\varepsilon} + \frac{\widehat{C}_\alpha t \varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \int_0^t \left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha ds. \end{aligned} \tag{28}$$

It follows from (16) and (28) that

$$|Y_t|^{\alpha-1} \le |x_0 - \widetilde{x}_0|^{\alpha-1} + 2\varepsilon^{\alpha-1} + M_t^{\delta,\varepsilon} + \frac{\widehat{C}_\alpha t \varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \int_0^t \left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha ds,$$

By taking expectations and using Fubini's theorem, we obtain

$$\mathbb{E}\left[\left|X_t - \widetilde{X}_t\right|^{\alpha-1}\right] \le |x_0 - \widetilde{x}_0|^{\alpha-1} + 2\varepsilon^{\alpha-1} + \frac{\widehat{C}_\alpha t \varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \int_0^t \mathbb{E}\left[\left|\sigma(X_s) - \widetilde{\sigma}(X_s)\right|^\alpha\right] ds. \tag{29}$$

Note that σ^α is $\gamma(\alpha-1)$-Hölder continuous (see Lemma 5.2). Hence, from Lemma 2.1 we have that X_t has the transition density function $p_t(x_0,\cdot)$. Using Lemma 2.2, there exists $K>0$ such that

$$\begin{aligned}
\int_0^t \mathbb{E}\left[|\sigma(X_s)-\widetilde{\sigma}(X_s)|^\alpha\right] ds &\leq K\int_0^t \int_{\mathbb{R}} |\sigma(y)-\widetilde{\sigma}(y)|^\alpha\, s^{-\frac{1}{\alpha}}\left(s^{1+\frac{1}{\alpha}}\vee 1\right)\left(|y-x_0|^{-1-\alpha}\wedge 1\right) dyds \\
&= KD_\alpha(t)\int_{\mathbb{R}} |\sigma(y)-\widetilde{\sigma}(y)|^\alpha \left(|y-x_0|^{-1-\alpha}\wedge 1\right) dy \\
&= KD_\alpha(t)\,\|\sigma-\widetilde{\sigma}\|^\alpha_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}, \qquad (30)
\end{aligned}$$

where $D_\alpha(t)=\int_0^t s^{-\frac{1}{\alpha}}(s^{1+\frac{1}{\alpha}}\vee 1)ds$ and the norm is the one defined in (7). Therefore, we get the following inequality.

Here, we have introduced the notation $\lambda=\|\sigma-\widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}$ in order to simplify long expressions. Now, we use (30) and take the supremum for $t\in[0,T]$ in (29) so as to obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|X_t-\widetilde{X}_t\right|^{\alpha-1}\right]\leq |x_0-\widetilde{x}_0|^{\alpha-1}+2\varepsilon^{\alpha-1}+\frac{\widehat{C}_\alpha T\varepsilon^{\alpha\gamma-1}}{\log\delta}+\frac{\widehat{C}_\alpha KD_\alpha(T)}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)\lambda^\alpha. \qquad (31)$$

We consider the following two cases: case 1: $\gamma\in(1/\alpha,1]$ and case 2: $\gamma=1/\alpha$. The above expression proves that if we choose some appropriate $\varepsilon>0$ and $\delta>1$ then the left-hand side in (31) will go to zero for $x_0=\widetilde{x}_0$. In order to choose ε and δ optimally, we divide our study in the following two cases according to the possible values of γ.

Case 1: In this case, note that $\alpha\gamma-1>0$. We set

$$\delta=2,\ p>0,\ \varepsilon=\lambda^p \text{ and } \frac{1}{3}C=\max\left\{2,\ \frac{\widehat{C}_\alpha T}{\log 2},\ \frac{2\widehat{C}_\alpha KD_\alpha(T)}{\log 2}\right\}.$$

Here, p is a positive real parameter which will be specified later. Using the above choice in (31) as well as the inequalities $p(\alpha-1)\geq p(\alpha\gamma-1)$ and $\lambda\leq 1$, we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|X_t-\widetilde{X}_t\right|^{\alpha-1}\right]\leq |x_0-\widetilde{x}_0|^{\alpha-1}+\frac{1}{3}C\left(2\lambda^{p(\alpha\gamma-1)}+\lambda^{-p+\alpha}\right). \qquad (32)$$

Therefore in order to obtain the optimal rate, we choose $p=1/\gamma$ which satisfies the equation $p(\alpha\gamma-1)=-p+\alpha$

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|X_t-\widetilde{X}_t\right|^{\alpha-1}\right]\leq |x_0-\widetilde{x}_0|^{\alpha-1}+C\lambda^{\frac{\alpha\gamma-1}{\gamma}}.$$

This value of p gives a rate of convergence but it is not an optimal choice for fixed λ (see Section 5.4).

Case 2: We set $C_{\alpha,T}=\max\{2,\ \widehat{C}_\alpha T,\ \widehat{C}_\alpha KD_\alpha(T)\}$ and choose $\varepsilon=(\log\frac{1}{\lambda})^{-p}$ and $\delta=\lambda^{-q}$. Then we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|X_t-\widetilde{X}_t\right|^{\alpha-1}\right]\leq |x_0-\widetilde{x}_0|^{\alpha-1}+C_{\alpha,T}\left[\left(\log\frac{1}{\lambda}\right)^{-p(\alpha-1)}+\frac{1}{q}\left(\log\frac{1}{\lambda}\right)^{-1}+\frac{1}{q}\left(\log\frac{1}{\lambda}\right)^{p-1}\lambda^{\alpha-q}\right].$$

By choosing p as $p=\frac{1}{\alpha-1}$ and $q=\frac{\alpha}{2}$, we have

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left|X_t-\widetilde{X}_t\right|^{\alpha-1}\right]\leq |x_0-\widetilde{x}_0|^{\alpha-1}+C_{\alpha,T}\left(\log\frac{1}{\lambda}\right)^{-1}\left[1+\frac{2}{\alpha}+\frac{2\lambda^{\frac{\alpha}{2}}}{\alpha}\left(\log\frac{1}{\lambda}\right)^{\frac{1}{\alpha-1}}\right].$$

Note that

$$\sup_{x\in(0,1]} x^{\frac{\alpha}{2}} \left(\log \frac{1}{x}\right)^{\frac{1}{\alpha-1}} < \infty.$$

Thus, choosing $C = C_{\alpha,T} \left\{1 + \frac{2}{\alpha} + \frac{2}{\alpha} \sup_{x\in(0,1]} x^{\frac{\alpha}{2}} \left(\log \frac{1}{x}\right)^{\frac{1}{\alpha-1}}\right\}$, we get

$$\sup_{0\le t\le T} \mathbb{E}\left[\left|X_t - \widetilde{X}_t\right|^{\alpha-1}\right] \le |x_0 - \widetilde{x}_0|^{\alpha-1} + C\left(\log \frac{1}{\lambda}\right)^{-1}.$$

This concludes the proof. □

Chapter 4

4 The distance between two solutions in probability

In this Chapter, we give our second main result. We study a rate of the convergence in probability of the time-supremum difference between two solutions.The result is proven by introducing the concept of a quasi-martingale and their properties. Concretely, we use a lemma obtained by Kurtz [12].

Theorem 4.1. *Assume the same conditions of Theorem 3.1. Then, there exists a positive constant C such that*

$$\sup_{h\ge 0} h\mathbb{P}\left(\sup_{0\le t\le T} \left|X_t - \widetilde{X}_t\right|^{\alpha-1} > h\right) \le \begin{cases} |x_0 - \widetilde{x}_0|^{\alpha-1} + C\|\sigma - \widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}^{\frac{\alpha\gamma-1}{\gamma}} & \text{if } \gamma \in (1/\alpha, 1] \\ |x_0 - \widetilde{x}_0|^{\alpha-1} + C\left(\log \frac{1}{\|\sigma - \widetilde{\sigma}\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}}\right)^{-1} & \text{if } \gamma = 1/\alpha, \end{cases}$$

where the constant C depends on T, α, m_1, m_2, m_3, ρ, $\widetilde{\rho}$ and η.

4.1 Quasi-martingales and their properties

We introduce quasi-martingales in order to prove Theorem 4.1. Let $T \in [0, \infty]$ and Z be a càdlàg adapted process defined on $[0, T]$. A finite subdivision of $[0, T]$ is defined by $\Delta t = (t_0, t_1, \ldots, t_{n+1})$ such that $0 = t_0 < t_1 < \cdots < t_{n+1} = T$.

Definition 4.2. *The mean variation of X is defined by*

$$V_T(X) := \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{n} \left|\mathbb{E}\left[X_{t_i} - X_{t_{i+1}} | \mathcal{F}_{t_i}\right]\right|\right].$$

Definition 4.3. *A càdlàg adapted process Z is a quasi-martingale on* $[0, T]$ *if for each* $t \in [0, T]$, $\mathbb{E}[|Z_t|] < \infty$ *and* $V_T(Z) < \infty$.

Kurtz [12] proved the following lemma by using Rao's theorem ([15], Section III, Theorem 17).

Lemma 4.4. *([12], Lemma 5.3) Let Z be a càdlàg adapted process defined on* $[0, T]$. *Suppose that for each* $t \in [0, T]$, $\mathbb{E}[|Z_t|] < \infty$ *and* $V_t(Z) < \infty$. *Then, for each* $h > 0$*,*

$$h\mathbb{P}\left(\sup_{0\le t\le T} |Z_t| > h\right) \le V_T(Z) + \mathbb{E}\left[|Z_T|\right].$$

The inequality in Lemma 4.4 is useful for proving Theorem 4.1.

4.2 Proof of Theorem 4.1

Proof. We set $u_{\delta,\varepsilon}(Y) := \{u_{\delta,\varepsilon}(Y_t)\}_{0\le t\le T}$ with $Y = X = \widetilde{X}$. Note that $u_{\delta,\varepsilon}$ is a nonnegative function since u and $\psi_{\delta,\varepsilon}$ are nonnegative (see Lemma 3.5). Hence, for each $t \in [0, T]$, we have

$$\mathbb{E}\left[\left|u_{\delta,\varepsilon}(Y_t)\right|\right] = \mathbb{E}\left[u_{\delta,\varepsilon}(Y_t)\right] < \infty.$$

Note that from (28) and (30), the above expectation is finite. Since $M^{\delta,\varepsilon}$ is a martingale (see Section 5.1), it follows from (24) and the definition of the mean variation that

$$\begin{aligned}
V_T\left(u_{\delta,\varepsilon}(Y)\right) &= \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{n}\left|\mathbb{E}\left[\left(u_{\delta,\varepsilon}(Y_0) + M^{\delta,\varepsilon}_{t_{i+1}} + I^{\delta,\varepsilon}_{t_{i+1}}\right) - \left(u_{\delta,\varepsilon}(Y_0) + M^{\delta,\varepsilon}_{t_i} + I^{\delta,\varepsilon}_{t_i}\right)\middle|\mathcal{F}_{t_i}\right]\right|\right] \\
&= \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{n}\left|\mathbb{E}\left[I^{\delta,\varepsilon}_{t_{i+1}} - I^{\delta,\varepsilon}_{t_i}\middle|\mathcal{F}_{t_i}\right]\right|\right] \\
&\le \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{n}\mathbb{E}\left[\left|I^{\delta,\varepsilon}_{t_{i+1}} - I^{\delta,\varepsilon}_{t_i}\right|\middle|\mathcal{F}_{t_i}\right]\right].
\end{aligned}$$

The last inequality follows from Jensen's inequality. The tower property for conditional expectations, (25), Lemma 3.8, Jensen's inequality and (26) imply that

$$\begin{aligned}
V_T(u_{\delta,\varepsilon}(Y)) &\le \sup_{\Delta t}\sum_{i=0}^{n}\mathbb{E}\left[\left|I^{\delta,\varepsilon}_{t_{i+1}} - I^{\delta,\varepsilon}_{t_i}\right|\right] \\
&\le \sup_{\Delta t}\sum_{i=0}^{n}\left|C_{\alpha,\beta}\right|\mathbb{E}\left[\int_{t_i}^{t_{i+1}}\left|\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s)\right|^{\alpha}\psi_{\delta,\varepsilon}(Y_{s-})ds\right] \\
&= \left|C_{\alpha,\beta}\right|\mathbb{E}\left[J^{\delta,\varepsilon}_T\right].
\end{aligned}$$

By using (27), Fubini's theorem and (30), we have

$$\begin{aligned}
V_T(u_{\delta,\varepsilon}(Y)) &\le \frac{\widehat{C}_\alpha t\varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)\int_0^t \mathbb{E}\left[|\sigma(X_s) - \widetilde{\sigma}(X_s)|^{\alpha}\right]ds \\
&\le \frac{\widehat{C}_\alpha T\varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{\widehat{C}_\alpha K D_\alpha(T)}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)\|\sigma - \widetilde{\sigma}\|^{\alpha}_{L^{\alpha}\left(\mathbb{R},\mu^{\alpha}_{x_0}\right)} \qquad (33) \\
&< \infty.
\end{aligned}$$

Thus $u_{\delta,\varepsilon}(Y)$ is a quasi-martingale. Hence, using (23) and Lemma 4.4, we have that for each $h > 0$

$$h\mathbb{P}\left(\sup_{0\le t\le T}|Y_t|^{\alpha-1} > h\right) \le h\mathbb{P}\left(\sup_{0\le t\le T}\left(\varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y_T)\right) > h\right)$$
$$\le V_T\left(\varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y)\right) + \mathbb{E}\left[\left|\varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y_t)\right|\right].$$

Here, by the definition of mean variation and (23), we have

$$V_T\left(\varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y)\right) = V_T\left(u_{\delta,\varepsilon}(Y)\right) \text{ and } \mathbb{E}\left[\left|\varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y_T)\right|\right] = \mathbb{E}\left[\varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y_T)\right].$$

Therefore, using (33), (28), Fubini's theorem and (30), we get

$$\sup_{h\ge 0} h\mathbb{P}\left(\sup_{0\le t\le T}|Y_t|^{\alpha-1} > h\right) \le V_T\left(u_{\delta,\varepsilon}(Y)\right) + \mathbb{E}\left[\varepsilon^{\alpha-1} + u_{\delta,\varepsilon}(Y_T)\right]$$
$$\le |x_0 - \widetilde{x}_0|^{\alpha-1} + 2\varepsilon^{\alpha-1} + \frac{2\widehat{C}_\alpha T\varepsilon^{\alpha\gamma-1}}{\log\delta} + \frac{2\widehat{C}_\alpha K D_\alpha(T)}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)\|\sigma - \widetilde{\sigma}\|^{\alpha}_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}.$$

The remainder of the proof is carried out in the same way as shown in the proof of Theorem 3.1. □

Chapter 5

5 Appendices

In this Chapter, we prove some lemmas used in the proof of Theorem 3.1, 4.1 and we introduce some results concomitant with these theorems.

5.1 Proof of the martingale property for $M^{\delta,\varepsilon}$

In this subsection, we prove that $M^{\delta,\varepsilon}$ is a martingale. For this, we need to show that the function $u_{\delta,\varepsilon}$ introduced in Lemma 3.5 is Lipschitz continuous.

Lemma 5.1. *The function $u_{\delta,\varepsilon}$ is Lipschitz continuous.*

Proof. From the mean value theorem, we know that a differentiable function is Lipschitz if and only if its derivative is bounded. We show that $u'_{\delta,\varepsilon}$ is bounded. Since the support of $\psi_{\delta,\varepsilon}$ is $[\varepsilon\delta^{-1}, \varepsilon]$ and using Jensen's inequality, we have

$$\left|u'_{\delta,\varepsilon}(x)\right| = \left|(\alpha-1)\int_{\mathbb{R}} \operatorname{sgn}(x-y)|x-y|^{\alpha-2}\psi_{\delta,\varepsilon}(y)dy\right|$$
$$\le (\alpha-1)\int_{\varepsilon\delta^{-1}}^{\varepsilon} |x-y|^{\alpha-2}\psi_{\delta,\varepsilon}(y)dy.$$

Here, when $x > 2\varepsilon$ or $0 \ge x$, we observe that for any $y \in [\varepsilon\delta^{-1}, \varepsilon]$,

$$|x-y| \ge |y| \text{ so that } |x-y|^{\alpha-2} \le |y|^{\alpha-2}.$$

Hence we have that for any $x > 2\varepsilon$ or $0 > x$,

$$\left|u'_{\delta,\varepsilon}(x)\right| \le (\alpha-1)\int_{\varepsilon\delta^{-1}}^{\varepsilon} y^{\alpha-2}\psi_{\delta,\varepsilon}(y)dy \le (\alpha-1)\left(\frac{\delta}{\varepsilon}\right)^{2-\alpha}\int_{\varepsilon\delta^{-1}}^{\varepsilon}\psi_{\delta,\varepsilon}(y)dy = (\alpha-1)\left(\frac{\delta}{\varepsilon}\right)^{2-\alpha}.$$

The last equality follows from $\int_{\varepsilon\delta^{-1}}^{\varepsilon}\psi_{\delta,\varepsilon}(y)dy = 1$. When $x \in (0, 2\varepsilon]$, then we have from $\psi_{\delta,\varepsilon}(x) \le 2/(x\log\delta)$,

$$\begin{aligned}
\left|u'_{\delta,\varepsilon}(x)\right| &\le (\alpha-1)\int_{\varepsilon\delta^{-1}}^{\varepsilon}|x-y|^{\alpha-2}\frac{2}{y\log\delta}dy \\
&\le \frac{2(\alpha-1)\delta}{\varepsilon\log\delta}\int_{\varepsilon\delta^{-1}}^{\varepsilon}|x-y|^{\alpha-2}dy \\
&= \frac{2\delta}{\varepsilon\log\delta}\left(|x-\varepsilon|^{\alpha-1} - \left|x-\varepsilon\delta^{-1}\right|^{\alpha-1}\right) \\
&\le \frac{2\delta}{\varepsilon^{2-\alpha}\log\delta} \\
&< \infty.
\end{aligned}$$

Thus $u'_{\delta,\varepsilon}$ is bounded for each $\varepsilon > 0$ and $\delta > 1$. This concludes the proof. □

Now, we will prove that $M^{\delta,\varepsilon}$ is a martingale. We set $M_t^{\delta,\varepsilon,1}$ and $M_t^{\delta,\varepsilon,2}$ for each $t \in [0, T]$ as

$$\begin{aligned}
M_t^{\delta,\varepsilon,1} &= \int_0^t\int_{|z|\ge 1}\{u_{\delta,\varepsilon}\left(Y_{s-} + \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right)z\right) - u_{\delta,\varepsilon}(Y_{s-})\}\widetilde{N}(dz,ds), \\
M_t^{\delta,\varepsilon,2} &= \int_0^t\int_{|z|<1}\{u_{\delta,\varepsilon}\left(Y_{s-} + \left(\sigma(X_{s-}) - \widetilde{\sigma}(\widetilde{X}_{s-})\right)z\right) - u_{\delta,\varepsilon}(Y_{s-})\}\widetilde{N}(dz,ds).
\end{aligned}$$

Then, $M_t^{\delta,\varepsilon} = M_t^{\delta,\varepsilon,1} + M_t^{\delta,\varepsilon,2}$. We need to prove that $\left(M_t^{\delta,\varepsilon,1}\right)_{0\le t\le T}$ and $\left(M_t^{\delta,\varepsilon,2}\right)_{0\le t\le T}$ are martingales. First, we treat the term $\left(M_t^{\delta,\varepsilon,2}\right)_{0\le t\le T}$. Since $u_{\delta,\varepsilon}$ is Lipschitz continuous by Lemma 5.1, there exists a constant $U_{\delta,\varepsilon}$ such that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_{|z|<1}\left|u_{\delta,\varepsilon}\left(Y_s + (\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s))z\right) - u_{\delta,\varepsilon}(Y_{s-})\right|^2\frac{c_\alpha}{|z|^{1+\alpha}}dzds\right] \\
&\le U_{\delta,\varepsilon}^2\mathbb{E}\left[\int_0^t\int_{|z|<1}\left|(\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s))z\right|^2\frac{c_\alpha}{|z|^{1+\alpha}}dzds\right].
\end{aligned}$$

By using this fact and the boundedness of σ and $\widetilde{\sigma}$, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^t\int_{|z|<1}\left|u_{\delta,\varepsilon}\left(Y_s + (\sigma(X_s) - \widetilde{\sigma}(\widetilde{X}_s))z\right) - u_{\delta,\varepsilon}(Y_{s-})\right|^2\frac{c_\alpha}{|z|^{1+\alpha}}dzds\right] &\le U_{\delta,\varepsilon}^2(\sup_{x\in\mathbb{R}}|\sigma(x)|^2 + m_2^2)\int_0^t\int_{|z|<1}|z|^2\frac{c_\alpha}{|z|^{1+\alpha}}dzds \\
&\le U_{\delta,\varepsilon}^2(\sup_{x\in\mathbb{R}}|\sigma(x)|^2 + m_2^2)T\int_{|z|<1}|z|^2\frac{dz}{|z|^{1+\alpha}} \\
&= \frac{2}{\alpha}U_{\delta,\varepsilon}^2(\sup_{x\in\mathbb{R}}|\sigma(x)|^2 + m_2^2)T \\
&< \infty.
\end{aligned}$$

Hence, $\left(M_t^{\delta,\varepsilon,2}\right)_{0\le t\le T}$ is a $L^2(\Omega,\mathbb{P})$-martingale (see [1], Theorem 4.2.3 and P.231). Similarly, $\left(M_t^{\delta,\varepsilon,1}\right)_{0\le t\le T}$ is a $L^1(\Omega,\mathbb{P})$-martingale since

$$\begin{aligned}\mathbb{E}\left[\int_0^t\int_{|z|\ge 1}\left|u_{\delta,\varepsilon}\left(Y_s+(\sigma(X_s)-\widetilde{\sigma}(\widetilde{X}_s))z\right)-u_{\delta,\varepsilon}\left(Y_{s-}\right)\right|\frac{c_\alpha}{|z|^{1+\alpha}}dzds\right] &\le U_{\delta,\varepsilon}(\sup_{x\in\mathbb{R}}|\sigma(x)|+m_2)T\int_{|z|\ge 1}|z|\frac{dz}{|z|^{1+\alpha}}\\ &= U_{\delta,\varepsilon}(\sup_{x\in\mathbb{R}}|\sigma(x)|+m_2)T(\alpha+1)^{-1}\\ &< \infty.\end{aligned}$$

Therefore $M^{\delta,\varepsilon}$ is a martingale since it is the sum of two martingales.

5.2 Hölder continuity of σ^α

In this subsection, we prove that σ^α satisfies the Hölder continuity property stated in Lemma 2.1.

Lemma 5.2. *Fix $\eta\in(0,1]$ and $\rho>0$. If σ satisfies $\|\sigma\|_\infty<\infty$ and $|\sigma(x)-\sigma(y)|<\rho|x-y|^\eta$ for any $x,y\in\mathbb{R}$, then the function σ^α is $\eta(\alpha-1)$-Hölder continuous.*

Proof. By the triangle inequality and $\|\sigma\|_\infty<\infty$, we obtain

$$\begin{aligned}|\sigma^\alpha(x)-\sigma^\alpha(y)| &= \left|\sigma(x)\sigma^{\alpha-1}(x)-\sigma(x)\sigma^{\alpha-1}(y)+\sigma(x)\sigma^{\alpha-1}(y)-\sigma(y)\sigma^{\alpha-1}(y)\right|\\ &\le |\sigma(x)|\left|\sigma^{\alpha-1}(x)-\sigma^{\alpha-1}(y)\right|+\left|\sigma^{\alpha-1}(y)\right||\sigma(x)-\sigma(y)|\\ &\le \|\sigma\|_\infty\left|\sigma^{\alpha-1}(x)-\sigma^{\alpha-1}(y)\right|+\|\sigma\|_\infty^{\alpha-1}|\sigma(x)-\sigma(y)|\\ &\le \|\sigma\|_\infty|\sigma(x)-\sigma(y)|^{\alpha-1}+\|\sigma\|_\infty^{\alpha-1}|\sigma(x)-\sigma(y)|.\end{aligned}$$

This last inequality follows as $|a^{\alpha-1}-b^{\alpha-1}|\le|a-b|^{\alpha-1}$ for any $a,b\ge 0$. Here, since $|\sigma(x)-\sigma(y)|<\rho|x-y|^\eta$ and $|\sigma(x)|<m_2$ for any $x,y\in\mathbb{R}$, we have that any $x\ne y$

$$\begin{aligned}\frac{|\sigma(x)-\sigma(y)|}{|x-y|^{\eta(\alpha-1)}} &= \left(\frac{|\sigma(x)-\sigma(y)|}{|x-y|^\eta}\right)^{\alpha-1}|\sigma(x)-\sigma(y)|^{2-\alpha}\\ &\le \rho^{\alpha-1}(|\sigma(x)|+|\sigma(y)|)^{2-\alpha}\\ &\le 2\rho^{\alpha-1}\|\sigma\|_\infty^{2-\alpha}.\end{aligned}$$

Thus, σ is η-Hölder continuous and $\eta(\alpha-1)$-Hölder continuous. Hence, we get

$$\begin{aligned}|\sigma^\alpha(x)-\sigma^\alpha(y)| &\le \|\sigma\|_\infty\rho^{\alpha-1}|x-y|^{\eta(\alpha-1)}+2\rho^{\alpha-1}\|\sigma\|_\infty^{2-\alpha}|x-y|^{\eta(\alpha-1)}\\ &\le \max\left\{\|\sigma\|_\infty\rho^{\alpha-1},\ 2\rho^{\alpha-1}\|\sigma\|_\infty^{2-\alpha}\right\}|x-y|^{\eta(\alpha-1)}.\end{aligned}$$

We conclude the proof. □

5.3 The limit of subsequences of solutions $(X^{(n)}, \sigma_n)_{n\in\mathbb{N}}$

In this subsection, we consider that the subsequential limit of the solution of SDE (3) is the solution of SDE (2) which the coefficient is the subsequential limit of $(\sigma_n)_{n\in\mathbb{N}}$. Suppose that $x_0 = x_0^{(n)}$ and $(\sigma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm $\|\cdot\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}$ and satisfies following conditions.

$$0 < \inf_{n\in\mathbb{N}} \inf_{x\in\mathbb{R}} \sigma_n(x) \text{ and } \sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}} \sigma_n(x) < \infty. \tag{34}$$

Furthermore, there exist constants $\check{\rho} > 0$ and $\gamma \in [1/\alpha, 1]$ such that for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\sup_{n\in\mathbb{N}} |\sigma_n(x) - \sigma_n(y)| \leq \check{\rho}|x-y|^\gamma. \tag{35}$$

We prove the following corollary by using Theorem 3.1, 4.1.

Corollary 5.3. *Suppose that $(\sigma_n)_{n\in\mathbb{N}}$ satisfies (34), (35) and $(\sigma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm $\|\cdot\|_{L^\alpha\left(\mathbb{R},\mu^\alpha_{x_0}\right)}$. Then, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that the limit $X^\infty := \lim_{k\to\infty} X^{(n_k)}$ exists almost surely and it is the unique solution of SDE (2) which has the coefficient $\lim_{k\to\infty}\sigma_{n_k}$*

Proof. First, we confirm existence of the subsequential limit of the solution of SDE (3). Since $\mu^\alpha_{x_0}$ is a finite measure and $\mu^\alpha_{x_0} \ll \text{Leb} \ll \mu^\alpha_{x_0}$, there exists a subsequence $(m_k)_{k\in\mathbb{N}}$ such that the sequence $(\sigma_{m_k})_{k\in\mathbb{N}}$ converges pointwise to $\sigma_\infty := \lim_{k\to\infty}\sigma_{m_k}$ Lebesgue almost everywhere (see [17] A13.2 (e)). Note that the limit σ_∞ is also γ-Hölder continuous Lebesgue almost everywhere (see (37)). Using Theorem 4.1, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of subsequence $(m_k)_{k\in\mathbb{N}}$ such that the following limit $\lim_{k\to\infty} X^{(n_k)}$ exists almost surely. Note that this subsequence $(n_k)_{k\in\mathbb{N}}$ does not depend on $t \in [0,T]$. The limit X^∞ is càdlàg since $X^{(n_k)}$ converges in supremum norm by Theorem 4.1 (see [1], P.140).

We confirm that the limit X^∞ is the unique solution of SDE (2). We define $V = (V_t)_{0\leq t\leq T}$ as

$$V_t = x_0 + \int_0^t \sigma_\infty(X^\infty_{s-})dZ_s.$$

We prove that

$$\mathbb{P}\left(V_t = X^\infty_t, \text{ for each } t \in [0,T]\right) = 1.$$

From Theorem 3.1 and the bounded convergence theorem, we have

$$\mathbb{E}\left[\left|V_t - X^\infty_t\right|^{\alpha-1}\right] = \lim_{k\to\infty}\mathbb{E}\left[\left|V_t - X^{(n_k)}_t\right|^{\alpha-1}\right].$$

In the same way as shown in the proof of Theorem 3.1, we have

$$\mathbb{E}\left[\left|V_t - X^{(n_k)}_t\right|^{\alpha-1}\right] \leq 2\varepsilon^{\alpha-1} + \widehat{M}^{\delta,\varepsilon,k}_t + \widehat{J}^{\delta,\varepsilon,k}_t, \tag{36}$$

where $c_\alpha = \pi^{-1}\Gamma(\alpha+1)\sin\left(\frac{\alpha\pi}{2}\right)$, C_α appears in Lemma 3.8,

$$\widehat{M}^{\delta,\varepsilon,k}_t := \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left\{u_{\delta,\varepsilon}\left(V_{s-} - X^{(n_k)}_{s-} + \left(\sigma_\infty(X^\infty_s) - \sigma_{n_k}(X^{(n_k)}_s\right)z\right) - u_{\delta,\varepsilon}\left(V_{s-} - X^{(n_k)}_{s-}\right)\right\}\widetilde{N}(dz,ds) \text{ and}$$

$$\widehat{J}^{\delta,\varepsilon,k}_t := c_\alpha\,|C_\alpha| \int_0^t \left|\sigma_\infty(X^\infty_s) - \sigma_{n_k}(X^{(n_k)}_s)\right|^\alpha \psi_{\delta,\varepsilon}(V_s - X^{(n_k)}_s)ds.$$

By using the same arguments as in Subsection 5.1, it is shown that $\widehat{M}^{\delta,\varepsilon,k} := \left(\widehat{M}_t^{\delta,\varepsilon,k}\right)_{0\le t\le T}$ is a martingale. By the inequality $|x+y|^\alpha \le 2^{\alpha-1}(|x|^\alpha + |y|^\alpha)$ for any $x, y \in \mathbb{R}$, we have

$$\widehat{J}_t^{\delta,\varepsilon,k} \le 2^{\alpha-1}|C_\alpha| \int_0^t \left\{ \left|\sigma_\infty(X_s^\infty) - \sigma_\infty(X_s^{(n_k)})\right|^\alpha + \left|\sigma_\infty(X_s^{(n_k)}) - \sigma_{n_k}(X_s^{(n_k)})\right|^\alpha \right\} \psi_{\delta,\varepsilon}(V_s - X_s^{(n_k)})ds.$$

Here, since the sequence $(\sigma_{n_k})_{k\in\mathbb{N}}$ converges pointwise to σ_∞ Lebesgue almost everywhere, we have

$$\begin{aligned} |\sigma_\infty(x) - \sigma_\infty(y)| &\le \left|\sigma_\infty(x) - \sigma_{n_k}(x)\right| + \left|\sigma_{n_k}(x) - \sigma_{n_k}(y)\right| + \left|\sigma_{n_k}(y) - \sigma_\infty(y)\right| \\ &\le \left|\sigma_\infty(x) - \sigma_{n_k}(x)\right| + \check{\rho}\,|x-y|^\gamma + \left|\sigma_{n_k}(y) - \sigma_\infty(y)\right| \\ &\to \check{\rho}\,|x-y|^\gamma \quad \text{as} \quad k\to\infty \quad \text{for almost all } x,\ y. \end{aligned} \tag{37}$$

Hence, σ_∞ is γ-Hölder continuous Lebesgue almost everywhere. Here, since σ_∞ is a bounded function and $\gamma \ge 1/\alpha > (\alpha-1)/\alpha$, the function σ_∞ is also $((\alpha-1)/\alpha)$-Hölder continuous. Thus, we have

$$\begin{aligned} \widehat{J}_t^{\delta,\varepsilon,k} &\le 2^{\alpha-1}|C_\alpha| \int_0^t \left\{ \check{\rho}\left|X_s^\infty - X_s^{(n_k)}\right|^{\alpha-1} + \left|\sigma_\infty(X_s^{(n_k)}) - \sigma_{n_k}(X_s^{(n_k)})\right|^\alpha \right\} \psi_{\delta,\varepsilon}(V_s - X_s^{(n_k)})ds \\ &\le \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \int_0^t \left\{ \check{\rho}\left|X_s^\infty - X_s^{(n_k)}\right|^{\alpha-1} + \left|\sigma_\infty(X_s^{(n_k)}) - \sigma_{n_k}(X_s^{(n_k)})\right|^\alpha \right\} ds. \end{aligned} \tag{38}$$

The last inequality follows from the explicit upper bound for $\psi_{\delta,\varepsilon}$. By (36), (38) and Fubini's theorem, we have

$$\begin{aligned} \sup_{0\le t\le T} \mathbb{E}\left[\left|V_t - X_t^{(n_k)}\right|^{\alpha-1}\right] &\le 2\varepsilon^{\alpha-1} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \int_0^T \left\{ \check{\rho}\mathbb{E}\left[\left|X_s^\infty - X_s^{(n_k)}\right|^{\alpha-1}\right] + \mathbb{E}\left[\left|\sigma_\infty(X_s^{(n_k)}) - \sigma_{(n_k)}(X_s^{(n_k)})\right|^\alpha\right] \right\} ds \\ &\le 2\varepsilon^{\alpha-1} + \frac{\widehat{C}_\alpha}{\log\delta}\left(\frac{\delta}{\varepsilon}\right) \left\{ T\check{\rho} \sup_{0\le s\le T} \mathbb{E}\left[\left|X_s^\infty - X_s^{(n_k)}\right|^{\alpha-1}\right] + \int_0^T \mathbb{E}\left[\left|\sigma_\infty(X_s^{(n_k)}) - \sigma_{n_k}(X_s^{(n_k)})\right|^\alpha\right] \right\} ds. \end{aligned}$$

From Theorem 3.1 and the bounded convergence theorem, we obtain

$$\sup_{0\le s\le T} \mathbb{E}\left[\left|X_s^\infty - X_s^{(n_k)}\right|^{\alpha-1}\right] \to 0 \quad \text{as} \quad k\to\infty.$$

By the assumption on σ_n, Lemma 5.2 and Lemma 2.1, $X_t^{(n)}$ has a transition density function for each $t \in (0,T]$ and each $n \in \mathbb{N}$. This density has an upper bound as stated in Lemma 2.2, so that we have by (30),

$$\int_0^T \mathbb{E}\left[\left|\sigma_\infty(X_s^{n_k}) - \sigma_{n_k}(X_s^{n_k})\right|^\alpha\right] ds \to 0 \text{ as } k\to\infty.$$

Hence, we have for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \sup_{0\le t\le T} \mathbb{E}\left[\left|V_t - X_t^{(n_k)}\right|^{\alpha-1}\right] \le 2\varepsilon^{\alpha-1}.$$

Therefore, we get

$$\lim_{k\to\infty} \sup_{0\le t\le T} \mathbb{E}\left[\left|V_t - X_t^{(n_k)}\right|^{\alpha-1}\right] = 0.$$

Thus, we have

$$\mathbb{P}\left(V_t = X_t^\infty\right) = 1 \text{ for each } t \in [0,T].$$

Here, since each sample path of V and X^∞ is càdlàg, we have (see [15], Section I, Theorem 2)

$$\mathbb{P}\left(V_t = X_t^\infty, \text{ for each } t \in [0,T]\right) = 1.$$

This concludes the proof. □

5.4 A more precise estimate for Theorem 3.1 and 4.1

In this subsection, we give a more precise estimate for the result on Theorem 3.1 and 4.1 for $\gamma \in (1/\alpha, 1]$. We recall (32), which states that for $p > 0$,

$$\sup_{0\le t\le T} \mathbb{E}\left[\left|X_t - \widetilde{X}_t\right|^{\alpha-1}\right] \le |x_0 - \widetilde{x}_0|^{\alpha-1} + \frac{1}{3}C\left(2\lambda^{p(\alpha\gamma-1)} + \lambda^{-p+\alpha}\right).$$

We consider the function $g(x) = 2\lambda^{x(\alpha\gamma-1)} + \lambda^{-x+\alpha}$. First, we find the critical points of g,

$$\begin{aligned} g'(x) = 2\lambda^{x(\alpha\gamma-1)}(\alpha\gamma-1)\log\lambda - \lambda^{-x+\alpha}\log\lambda &= 0 \\ 2\lambda^{x(\alpha\gamma-1)}(\alpha\gamma-1) &= \lambda^{-x+\alpha} \\ \log(2(\alpha\gamma-1)) + x(\alpha\gamma-1)\log\lambda &= (-x+\alpha)\log\lambda \\ x &= \frac{1}{\gamma} - \frac{\log\left(2(\alpha\gamma-1)\right)}{\log\lambda}. \end{aligned}$$

Second, since $\lim_{x\to\pm\infty} g(x) = +\infty$ and there is only one critical point, the function g takes its minimum value at $x = \frac{1}{\gamma} - \frac{\log(2(\alpha\gamma-1))}{\log\lambda}$.

Therefore, we get the inequality

$$\begin{aligned} \sup_{0\le t\le T} \mathbb{E}\left[\left|X_t - \widetilde{X}_t\right|^{\alpha-1}\right] &\le |x_0 - \widetilde{x}_0|^{\alpha-1} + \frac{1}{3}C\left(2\lambda^{\frac{\alpha\gamma-1}{\gamma} - \frac{(\alpha\gamma-1)\log(2(\alpha\gamma-1))}{\log\lambda}} + \lambda^{\frac{\alpha\gamma-1}{\gamma} + \frac{\log(2(\alpha\gamma-1))}{\log\lambda}}\right) \\ &< |x_0 - \widetilde{x}_0|^{\alpha-1} + Cg\left(\frac{1}{\gamma}\right) \\ &= |x_0 - \widetilde{x}_0|^{\alpha-1} + C\lambda^{\frac{\alpha\gamma-1}{\gamma}}. \end{aligned}$$

In the same way, we have for any $h > 0$,

$$\begin{aligned} h\mathbb{P}\left(\sup_{0\le t\le T}\left|X_t - \widetilde{X}_t\right|^{\alpha-1} > h\right) &\le |x_0 - \widetilde{x}_0|^{\alpha-1} + \frac{1}{3}C\left(2\lambda^{\frac{\alpha\gamma-1}{\gamma} - \frac{(\alpha\gamma-1)\log(2(\alpha\gamma-1))}{\log\lambda}} + \lambda^{\frac{\alpha\gamma-1}{\gamma} + \frac{\log(2(\alpha\gamma-1))}{\log\lambda}}\right) \\ &< |x_0 - \widetilde{x}_0|^{\alpha-1} + C\lambda^{\frac{\alpha\beta}{2(1+\beta)-\alpha}}. \end{aligned}$$

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