Doctoral Thesis

A Numerical Scheme for Expectations of Functionals Related to First Hitting Time of a Diffusion Process and Related Study on Stochastic Differential Equations

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（拡散過程が境界に到達する初期時刻を含む期待値の数値解析スキームおよび関連する確率微分方程式の研究）

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Abstract

In many fields of applied science and mathematics, prediction of the time a random event takes place is very important. The prediction is mostly based on a numerical calculation of mathematical expectation. Further, the random time is often modeled as the time a stochastic process reaches to a threshold. In financial engineering, this is the case in the pricing of barrier options.

In the present paper, we propose a numerical scheme to calculate expectations with first hitting time to a given boundary, in view of the application to the pricing of options with non-linear barriers. In financial practice, the numerical calculations are in most case based on discretization of the process in question, together with Monte-Carlo simulation. The path-dependence nature of the problem, however, make the procedure rather slow. Aiming to resolve the problem, the author, together with Y. Imamura and Y. Ishigaki, proposed a new numerical scheme in [18], which is based on "symmetrization" of a diffusion process. Chapter 2 of the present paper is taken from [18].

The scheme is extended in [1] to multi-dimensional settings, where the boundary is yet a hyperplane. Chapter 3 is devoted to an extension of the scheme to a more general boundary. which is taken from [12].

Chapter 4 is taken from [26], where a new construction of a unique strong solution under non-Lipshitz coefficient condition is discussed.

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The third part of the paper is motivated by stimulating discussions with Professor Toshio Yamada. I would like to express my sincere appreciation to him as well. When he and I had a dinner at a Western-style restaurant in Kyoto, he recommended me to go to doctoral course. Without his back up at that time, I would not be what I am now. He made many round trip between Tokyo and Kyoto for me and gave me a lot of support. He not only provided me advice on mathematics but also provided my mental support to continue study. I also remember that we had a meal at a French restaurant every time after having a seminar. I learned a lot from him about the history of mathematics and French cuisine while drinking delicious food and wine. These are my lifetime treasure, too.

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Toshiki Okumura

To Ayami and Minami

Contents

Chapter 1

Introduction

The present thesis consists of three parts. Two of them are based on Imamura-Ishigaki-Okumura [18] and Hishida-Ishigaki-Okumura [12]. The other one is based on Okumura [26].

The first part (Chapter 2 and Chapter 3) consists of researches regarding to a numerical scheme for the pricing of a barrier option. In financial practice, the pricing and hedging of barrier type derivatives becomes more and more important. In the Black-Scholes environment, some analytic formulas are available (see [24]). If the underlying process is a diffusion process which is more complicated than a Geometric Brownian Motion, it will not be able, basically, to rely on anymore an analytic formula. Instead one should resort to some numerical analysis. There is a problem, however, arising from the path-dependence of the pay-off function. As E. Gobet [7] pointed out, the weak convergence rate against the time-discretization gets worse compared with the standard path-independent pay-off cases due to the failure in the observation of hitting between two time steps. He showed that the weak order of Euler-Maruyama approximation is $\frac{1}{2}$, which is much slower than the standard case where the order is 1.

In Chapter 2, introduced is a new numerical scheme where the pricing (and hedging) of barrier options are reduced to that of plain (path-independent) ones. The scheme is based on an observation made by [3] which we will refer to as "arithmetic put-call symmetry" (APCS). In the Black-Scholes economy, it is well-recognized that the reduction is possible due to the reflection principle (see [16]). The put-call symmetry is an extension of the reflection principle, with which a semi-static hedge is still possible.

There are two keys in our scheme;

1. For a given diffusion X and a real number K, we can find another diffusion $\tilde{X}$ which satisfies the APCS at K (see section 2.1.3). We call this procedure "symmetrization".

2. For $T > 0$, the expectations $\mathbf{E}[f(X_T)1_{\{\tau>T\}}]$ and $\mathbf{E}[f(\tilde{X}_T)1_{\{\tilde{\tau}>T\}}]$ coincides, where τ and $\tilde{\tau}$ are the first hitting time at K of X and $\tilde{X}$, respectively.

We do not anymore regard the equation for semi-static hedging but just a relation to calculate the expectation of the diffusion with a Dirichlet boundary condition in terms of those without boundary conditions. In other words, the pricing is reduced to path-independent ones, where many stable techniques are available.

Presented will be some numerical results of applying (path-independent) Euler-Maruyama (EM) approximation to our scheme, comparing them with the path-dependent EM under Constant Elasticity of Volatility (CEV) models ([4]) including as a special case the Black-Scholes (BS) model, and stochastic volatility models of Heston's ([11]) and (λ)-SABR ([8, 10]).

Chapter 2 consists of two parts. In the first part, the discussion of our new scheme is concentrated on one-dimensional diffusion models, while the latter part deals with applications to the stochastic volatility models. Mathematically, the first part is somehow *self-contained*, while one may think the latter part to be dependent on the result in [1]. The fact is that we have found, in advance of [1], through numerical experiments how it should be applied to stochastic volatility models (see [18]).

In anyway, the main aim of Chapter 2 is to introduce the new scheme and to report numerical results which show the effectiveness of the scheme. In order to ensure the consistency of the experiments, we present detailed descriptions.

In Section 2.1, we recall the APCS and how it is applied to the pricing and semi-static hedging of barrier options. In section 2.1.2 we give a sufficient condition shown by [3] under which APCS holds. In Section 2.1.3, we introduce a way to "symmetrize" a given diffusion process. We then show that by using the symmetrized process which satisfies APCS, the pricing of a barrier option is reduced to that of two plain options. In section 2.2, we give numerical examples under our symmetrized approximation method. The results of the path-wise EM scheme (in section 2.2.1) and our new scheme are given when the underlying asset price process follows CEV with the volatility elasticity $\beta = 1$ which is nothing but a BS model and other elasticities. From Section 2.3, we discuss applications to stochastic volatility models. The symmetrized method is also applicable to the stochastic volatility models where the underlying price process and its volatility follows a (degenerate) 2-dimensional diffusion process. In section 2.3.1, we give numerical results under Heston model and λ-SABR model. In section 2.3.2, we show that the symmetrization scheme also works for the pricing of double barrier option. One will find that our scheme overwhelms the path-wise EM in all numerical results.

Chapter 3 is devoted to a research regarding to a numerical scheme to calculate expectations with first hitting time to a given smooth boundary, in view of the application to the pricing of options with non-linear barriers. We propose a new method to deal with curved boundary cases, which is totally different from the method suggested in [1], but still based on Akahori-Imamura's multi-dimensional symmetrization of the diffusion process. Very roughly speaking, we "lift" a given diffusion to the line bundle of the invariant manifold of the process. Then the first hitting time of the original diffusion process to a domain is equivalently formulated as the first hitting time of the lifted diffusion to a hyperplane. The lifted problem can be handled by Akahori-Imamura's symmetrization. The method is introduced in section 3.1. We first recall Akahori-Imamura's multi-dimensional symmetrization scheme in section 3.1.1, which we will use in section 3.1.2, where we introduce our new framework. In subsection 3.2, we present some results of several numerical experiments that support effectiveness of our scheme. In subsection 3.2.2, we compare a numerical experiment of 1-dimensional symmetrization and multi-dimensional one. Moreover, we give numerical results for hyperbolic reflection principle in subsection 3.2.3. Section 3.3 concludes the chapter.

Chapter 4 is devoted to a related study on stochastic differential equations. In Chapter 4, we proposes concrete and direct constructions of strong solutions for stochastic differential equations (SDEs) with non-Lipschitz coefficients. It is well known that in Ito's classical theory on SDEs (see Ito [19]) under the global Lipschitz condition for coefficients, the existence and the uniqueness hold for their strong solutions. The theory is based on the Picard's iteration method and then the existence and the uniqueness follow naturally by Picard's successive approximation procedure of strong solutions. Although Ito theory is beautifully established, the global Lipschitz condition imposed on coefficients is too strict and too restrictive for the purpose of discussing various SDEs raised both in the theory of stochastic calculus and in its applications. Consider for examples, the SDE associated to square Bessel processes, to Wright-Fischer model in population genetics, to Cox-Ingersoll-Ross (CIR) model in mathematical finance, and also to skewed symmetric Brownian motions. The classical Ito theory covers none of these examples. As is well known that the frame of the weak existence theory [29] [30] is wide enough to cover the all examples cited in the above. We know also that the pathwise uniqueness holds for solutions of SDEs in the above by Yamada-Watanabe condition [32], or by that of Nakao-Le Gall [23][25]. Then the existence of strong solutions for them follows immediately by Yamada-Watanabe Theorem [32].

However, we would like to point out that the proof of the existence by Yamada-Watanabe involves no construction procedure of strong solutions. In this situation, the investigation on concrete construction of strong solutions under non-Lipschitz conditions appears to be interesting. Chapter 4 is motivated by a construction of strong solutions given by Stefan Ebenfeld [5]. His method covers CIR model in mathematical finance. The method is based on a priori estimates and also on Ito classical theory. The benefit of this approach is that the argument only requires some fundamental knowledge about stochastic and functional analysis. The first part of Chapter 4 is devoted to the improvement of the result by Stefan Ebenfeld [5]. We show a concrete construction procedure of strong solutions under Yamada-Watanabe condition. Although our construction method based on a priori estimates is new, the Euler-Maruyama approximation method gives an another construction of strong solutions under the same condition (see Yamada [31], Kaneko-Nakao [21]).

In the final part of Chapter 4, we discuss the existence of strong solutions under Nakao-Le Gall condition. Since coefficients are allowed to be discontinuous, the Euler-Maruyama method based on the continuity of coefficients does not cover this case. Discontinuous points of coefficient raise various difficulties to be discussed carefully in the proof. The stochastic calculus based on local times and their occupation formulas plays important roles to overcome these difficulties.

Chapter 2

A Numerical Scheme Based on Semi-Static Hedging Strategy

In this chapter, we introduce a numerical scheme for the price of a barrier option when the price of the underlying follows a diffusion process. The numerical scheme is based on an extension of a static hedging formula of barrier options. For getting the static hedging formula, the underlying process needs to have a symmetry. We introduce a way to "symmetrize" a given diffusion process. Then the pricing of a barrier option is reduced to that of plain options under the symmetrized process. To show how our symmetrization scheme works, we will present some numerical results applying (path-independent) Euler-Maruyama approximation to our scheme, comparing them with the path-dependent Euler-Maruyama scheme when the model is of the Black-Scholes, CEV, Heston, and (λ)-SABR, respectively. The results show the effectiveness of our scheme.

2.1 The Put-Call Symmetry Method for One Dimensional Diffusions

2.1.1 Arithmetic Put-Call Symmetry

Let X be a real valued diffusion process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$ which satisfies the usual conditions. For fixed $K > 0$, we say that *arithmetic put-call symmetry* (APCS) at K holds for X if the following equation is satisfied ;

$$\mathbf{E}[G(X_t - K) \mid X_0 = K] = \mathbf{E}[G(K - X_t) \mid X_0 = K],$$

for any bounded measurable function G and $t \geq 0$. The APCS at K is alternatively defined to be the equivalence in law between $X_t - X_{t\wedge\tau}$ and $X_{t\wedge\tau} - X_t$ for any $t \geq 0$ and stopping time τ with $X_\tau = K$. Intuitively, the APCS means the following. For every path of X which crosses the level K and is found at time t at a point below K, there is a "shadow path" obtained from the reflection with respect to the level K which exceeds this level at time t, and these two paths have the same probability. For one-dimensional Brownian motion, APCS holds for any $K > 0$ since the reflection principle holds. In [3], the APCS, or more precisely, *PCS*[1], is applied to the pricing and *semi-static* hedging of a barrier options. Semi-static hedging means replication of the barrier contract by trading European-style claims at no more than two times after inception. In more detail, we have the following; if X satisfies APCS at K, then for any bounded measurable f and $T > 0$,

$$\begin{aligned}\mathbf{E}[f(X_T)I_{\{\tau>T\}}] &= \mathbf{E}[f(X_T)I_{\{X_T>K,\ \tau>T\}}] \\ &= \mathbf{E}[f(X_T)I_{\{X_T>K\}}] - \mathbf{E}[f(X_T)I_{\{X_T>K,\ \tau\leq T\}}],\end{aligned}$$

where

$$\tau = \inf\{t \geq 0 : X_T \leq K\}. \tag{2.1}$$

By APCS of X, we see that

$$\begin{aligned}\mathbf{E}[f(X_T)I_{\{X_T>K,\ \tau\leq T\}}] &= \mathbf{E}[\mathbf{E}[f(X_T)I_{\{X_T>K\}}|\mathcal{F}_\tau]I_{\{\tau\leq T\}}] \\ &= \mathbf{E}[\mathbf{E}[f(2K - X_T)I_{\{X_T<K\}}|\mathcal{F}_\tau]I_{\{\tau\leq T\}}].\end{aligned}$$

Hence we obtain the following equation;

$$\begin{aligned}\mathbf{E}[f(X_T)I_{\{\tau>T\}}] = \mathbf{E}[f(X_T)I_{\{X_T>K\}}] \\ - \mathbf{E}[f(2K - X_T)I_{\{X_T<K\}}].\end{aligned} \tag{2.2}$$

Of the equation (2.2), the left-hand-side reads the price of a barrier option written on X, whose pay-off is f, knocked out at K, and the right-hand-side is the price of a combination of two plain-vanilla options. Here is a description of the hedging strategy of a barrier option implied from the right-hand-side of (2.2);

1. Hold a plain-vanilla options whose pay-off is $f(X_T)$ if the price at the maturity is less than K, and is nothing if the price at the maturity is greater than K. Moreover short-sell a plain-vanilla options whose pay-off is $f(2K - X_T)$ if the price at the maturity is greater than K, and is nothing if the price at the maturity is less than K.

2. Keep the above position until the price hits the barrier K. If the price never hits K until the maturity, the pay-off is $f(X_T)$.

3. If the price hits K, clear both plain-vanilla positions at the hitting time. Indeed, the value of two options are exactly the same at τ.

[1]They defined PCS as the equality of the distribution between $\frac{X_T}{X_0}$ under $\mathbf{P}$ and $\frac{X_0}{X_T}$ under $\mathbf{Q}$, where $\frac{d\mathbf{Q}}{d\mathbf{P}} = \frac{X_T}{X_0}$.

2.1.2 APCS of diffusion process

Let X be a solution to the following one-dimensional stochastic differential equation (SDE) driven by a Brownian motion W in $\mathbb{R}$,

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt. \tag{2.3}$$

Here we assume the following hypotheses;

(H1) $\sigma : \mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$ are locally bounded measurable functions such that the linear growth condition is satisfied, ie, for a constant C, $|\sigma(x)| + |\mu(x)| \leq C(1 + |x|)$ for any $x \in \mathbb{R}$.

(H2) The following condition is satisfied;

$$\sigma(y) \neq 0 \Longleftrightarrow \sigma^{-2} \text{ is integrable in a neighborhood of } y.$$

Then we have the following result on the uniqueness of the solution to (2.3);

Theorem 2.1.1 (Theorem 4, [6]). *Under* **(H2)***, there exists a unique (in law) solution satisfying SDE (2.3).*

Moreover, by the linear growth condition **(H1)**, the unique (in law) solution will not explode in finite time. [3] gave a sufficient condition for a solution to (2.3) to satisfy PCS at $0 \in \mathbb{R}$. The following Proposition is essentially a corollary to Theorem 3.1 in [3].

Proposition 2.1.2. *If the coefficients further satisfy the following conditions;*

$$\sigma(x) = \varepsilon(x)\sigma(2K - x) \ (x \in \mathbb{R} \setminus \{K\}), \tag{2.4}$$

for a measurable $\varepsilon : \mathbb{R} \to \{-1, 1\}$ *and*

$$\mu(x) = -\mu(2K - x) \ (x \in \mathbb{R} \setminus \{K\}), \tag{2.5}$$

then APCS at K holds for X.

Proof. By the uniqueness in law, it is sufficient to show that $(X_{t\wedge\tau} - (X_t - X_{t\wedge\tau}))_{t\geq 0}$ solves the SDE (2.3). By the assumptions (2.4) and (2.5), we obtain that

$$\begin{aligned}
X_{t\wedge\tau} - (X_t - X_{t\wedge\tau}) &= X_{t\wedge\tau} - \int_{t\wedge\tau}^t \sigma(X_s)dW_s - \int_{t\wedge\tau}^t \mu(X_s)ds \\
&= X_{t\wedge\tau} - \int_{t\wedge\tau}^t \varepsilon(X_s)\sigma(2K - X_s)dW_s \\
&\quad + \int_{t\wedge\tau}^t \mu(2K - X_s)ds \\
&= X_{t\wedge\tau} - \int_{t\wedge\tau}^t \varepsilon(X_s)\sigma(X_\tau - (X_s - X_\tau))dW_s \\
&\quad + \int_{t\wedge\tau}^t \mu(X_\tau - (X_s - X_\tau))ds.
\end{aligned}$$

We set $W'_t = W_{t\wedge\tau} - \int_{t\wedge\tau}^{t} \varepsilon(X_s) dW_s$. Sce we obtain that

$$\begin{aligned}\langle W' \rangle(t) &= \langle W \rangle(t) \\ &= t,\end{aligned}$$

W' is a Brownian motion (cf. [15] Chapter II, Theorem 6.1.). Therefore we see that

$$\begin{aligned}X_{t\wedge\tau} - (X_t - X_{t\wedge\tau}) = X_0 &+ \int_0^t \sigma(X_{s\wedge\tau} - (X_s - X_{s\wedge\tau}))dW'_s \\ &+ \int_0^t \mu(X_{s\wedge\tau} - (X_s - X_{s\wedge\tau}))ds.\end{aligned}$$

Hence APCS at K holds. □

2.1.3 Symmetrization of Diffusion Processes

We introduce a way to "symmetrize" a given diffusion to satisfy APCS. By using this symmetrized process satisfying APCS, the pricing of a barrier option is reduced to that of plain options. We start with a diffusion process X given as a unique solution to SDE (2.3). We do not assume that the coefficients have the symmetric conditions (2.4) and (2.5). We then construct another diffusion $\tilde{X}$ that satisfies APCS at K in the following way. Put

$$\tilde{\sigma}(x) := \begin{cases} \sigma(x) & x > K \\ \sigma(2K - x) & x \leq K, \end{cases} \tag{2.6}$$

$$\tilde{\mu}(x) := \begin{cases} \mu(x) & x > K \\ -\mu(2K - x) & x \leq K, \end{cases} \tag{2.7}$$

and consider the following SDE;

$$d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t)dW_t + \tilde{\mu}(\tilde{X}_t)dt. \tag{2.8}$$

Again by Theorem 2.1.1, there is a unique (in law) solution $\tilde{X}_t$. Then we obtain the following result;

Theorem 2.1.3. *It holds that*

$$\begin{aligned}\mathbf{E}[f(X_T)I_{\{\tau>T\}}] = &\mathbf{E}[f(\tilde{X}_T)I_{\{\tilde{X}_T>K\}}] \\ &- \mathbf{E}[f(2K - \tilde{X}_T)I_{\{\tilde{X}_T<K\}}].\end{aligned} \tag{2.9}$$

Proof. Since $\tilde{\sigma}$ and $\tilde{\mu}$ satisfy the condition (2.4) and (2.5), APCS at K holds for $\tilde{X}$ by Proposition 2.1.2. Then the equation (2.2) is valid for $\tilde{X}$. Moreover, by the definition of $\tilde{\sigma}$ and $\tilde{\mu}$, we have $\sigma(x) = \tilde{\sigma}(x)$ and $\mu(x) = \tilde{\mu}(x)$ for $x > K$. Therefore by the uniqueness in law of the SDE, we have that $\{X_t\}_{t\leq\tau} = \{\tilde{X}_t\}_{t\leq\tau}$ pathwisely. Then we see that $\tau = \tilde{\tau}$ where $\tilde{\tau} = \inf\{t > 0 : \tilde{X}_t \leq K\}$. Hence we have

$$\mathbf{E}[f(\tilde{X}_T)I_{\{\tilde{\tau}>T\}}] = \mathbf{E}[f(X_T)I_{\{\tau>T\}}].$$

□

2.1.4 Important Remark

We do not anymore regard (2.9) as an equation for semi-static hedging but a relation to give a numerical scheme to calculate the expectation of the diffusion with a Dirichlet boundary condition in terms of those without boundary conditions. The former is very difficult while the latter is rather easier using rapidly developing technique from numerical finance. In the following sections, we will present some results of numerical examples to show the effectivity of the new scheme.

2.2 Numerical Experiments for One Dimensional Models

2.2.1 The Euler-Maruyama Scheme

Here we briefly recall the Euler-Maruyama scheme for a diffusion process given as a solution to SDE (2.3). Fix $T > 0$. For $n \geq 1$, we set a subdivision of the interval $[0, T]$;

$$0 = t_0 < t_1 < \cdots < t_n = T,$$

where $t_k := \frac{kT}{n}$ for $0 \leq k \leq n$, and we denote this net by $\triangle_n$. The Euler-Maruyama scheme is a general method for numerically solving (2.3) by a discretized stochastic process which is given by

$$\begin{aligned} X^n_{t_0} &= X_0, \\ X^n_{t_{k+1}} &= X^n_{t_k} + \sigma(X^n_{t_k})(t_{k+1} - t_k) + \mu(X^n_{t_k})(W_{t_{k+1}} - W_{t_k}), \end{aligned} \tag{2.10}$$

$k = 0, 1, 2, \cdots, n-1$, and for $t_k < t < t_{k+1}$, X_t is given by an interpolation. The approximating process (X^n_T) is simulated by using independent Gaussian random variables for the increments $(W_{t_{k+1}} - W_{t_k})_{0 \leq k \leq n-1}$.

We rely on the following result;

Theorem 2.2.1 (Theorem 3.1, [33]). *If the set of discontinuous points of σ and μ is countable, then the Euler scheme (2.10) converges weakly to the unique weak solution to SDE (2.3) as $n \to \infty$.*

From now on, in addition to **(H1)** and **(H2)**, we assume that σ and μ are piece-wise continuous.

Path-wise Method

Since the convergence is in the space of probability measures on continuous functions, we see that this algorithm can also be used to simulate a path-dependent functional of the process; in particular, $f(X_T)I_{\{\tau > T\}}$, where f is a (bounded) continuous function and τ is the first hitting time defined by (2.1). The functional is approximated by $f(X^n_T)I_{\{\tau^n > T\}}$, where $\tau^n := \inf\{t_k : X^n_{t_k} \leq K\}$ is the discretized first hitting time to K. Then the expectation $\mathbf{E}[f(X_T)I_{\{\tau > T\}}]$ is approximated with a Monte-Carlo algorithm by

Method 1. (Path-wise EM scheme)

$$\frac{1}{M}\sum_{i=1}^{M} f(X_T^n(\omega_i))I_{\{\tau^n(\omega_i)>T\}}. \tag{2.11}$$

By the strong law of large numbers, (2.11) converges to $\mathbf{E}[f(X_T^n)I_{\{\tau^n>T\}}]$ as M goes to infinity. Moreover, as the index n of the net $\triangle_n$ goes to infinity, $\mathbf{E}[f(X_t^n)I_{\{\tau^n>T\}}]$ converges to $\mathbf{E}[f(X_T)I_{\{\tau>T\}}]$. According to [7], the following convergence rate was given;

Theorem 2.2.2 (Theorem 2.3, [7]). *Assume that σ and μ are in C_b^∞, σ is bounded below from zero and a solution is non-explosion. Then for a bounded measurable function f such that $d(\mathrm{supp} f, K) > 0$, there is a constant C such that*

$$|\mathbf{E}[f(X_t^n)I_{\{\tau^n>T\}}] - \mathbf{E}[f(X_T)I_{\{\tau>T\}}]| < C\frac{1}{\sqrt{n}}.$$

Put-Call Symmetry Method

Let $\tilde{X}$ be a solution with coefficients $\tilde{\sigma}$ and $\tilde{\mu}$ given by (2.6) and (2.7), and $(\tilde{X}_t^n)$ be the discretized Euler-Maruyama process with respect to the net $\triangle_n$. Namely,

$$\begin{aligned}\tilde{X}_{t_{k+1}}^n &= \tilde{X}_{t_k}^n + (\sigma(\tilde{X}_{t_k}^n)(t_{k+1}-t_k) + \mu(\tilde{X}_{t_k}^n)(W_{t_{k+1}} - W_{t_k}))I_{\{\tilde{X}_{t_k}^n>K\}} \\ &\quad + (\sigma(2K - \tilde{X}_{t_k}^n)(t-t_k) - \mu(2K-\tilde{X}_{t_k}^n)(W_t - W_{t_k}))I_{\{\tilde{X}_{t_k}^n\le K\}}\end{aligned}$$

for $k = 0, 1, 2, \cdots, n-1$. With an interpolation, $\tilde{X}_t^n$ for $t_k \le t \le t_{k+1}$ is obtained as well. Since the set of the discontinuous points in the coefficients has a null Lebesgue measure, $\tilde{X}^n$ also converges weakly to $\tilde{X}$ by Theorem 2.2.1.

Combining Theorem 2.1.3 and 2.2.1, we may rely on the following algorithm; the expectation $\mathbf{E}[f(X_T)I_{\{\tau>T\}}]$ is approximated with a Monte-Carlo algorithm by

Method 2. (Put-Call symmetry method)

$$\frac{1}{M}\sum_{i=1}^{M}\left\{ f(\tilde{X}_T^n(\omega_i))I_{\{\tilde{X}_T^n(\omega_i)>K\}} - f(2K - \tilde{X}_T^n(\omega_i))I_{\{\tilde{X}_T^n(\omega_i)<K\}}\right\}. \tag{2.12}$$

As M goes to ∞, (2.12) converges to

$$\mathbf{E}[f(\tilde{X}_T^n)I_{\{\tilde{X}_T^n>K\}}] - \mathbf{E}[f(2K-\tilde{X}_T^n)I_{\{\tilde{X}_T^n\le K\}}]. \tag{2.13}$$

By the weak convergence of X^n, (2.13) converges to

$$\mathbf{E}[f(\tilde{X}_T)I_{\{\tilde{X}_T>K\}}] - \mathbf{E}[f(2K-\tilde{X}_T)I_{\{\tilde{X}_T\le K\}}],$$

as $n \to \infty$. However, we don't know the exact rate of convergence in this algorithm since the coefficients are inevitably non-smooth at K [2]

The numerical results in the next section, however, may imply that the convergence rate of Put-Call symmetry method is better than that of the path-wise EM scheme. To prove this *conjecture* would be a very interesting mathematical challenge.

[2]There are many results on the rate of convergence when σ and μ are smooth. For example,

2.2.2 Numerical Results

In this section, we give numerical examples using method 1 (path-wise EM method) and method 2 (Put-Call symmetry method) under Black-Scholes model and other CEV models. Let us consider the value of a barrier call option with strike price S and knockout barrier K.

Black-Scholes Model

The underlying price process of Black-Scholes model is the unique solution to the following SDE;

$$dX_t = rX_t dt + \sigma X_t dW_t, \tag{2.14}$$

for $r,\ \sigma \geq 0$. Then the value of a barrier option is accurately-calculable since the joint distribution of Brownian motion and the hitting time of Brownian motion to a point is computable by using the reflection principle. The exact option price is given by the following;

$$e^{-rT}\mathbf{E}[(X_T - S)^+ I_{\{\tau > T\}}] = V_{call}(X_0) - \left(\frac{K}{X_0}\right)^{\frac{2r}{\sigma^2}-1} V_{call}\left(\frac{K^2}{X_0}\right),$$

where

$$V_{call}(x) = x\left(1 - \Phi(d_+(x))\right) - Se^{-rT}\left(1 - \Phi(d_-(x))\right),$$
$$d_\pm(x) = \frac{\log(\frac{S}{x}) - \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

and Φ is the distribution function of the standard normal distribution.

Fix a maturity time $T > 0$. Tables 2.1 - 2.4 give simulation results for the value of down-and-out call option $\mathbf{E}[(X_T - S)^+ I_{\{\tau > T\}}]$ under the path-wise Euler-Maruyama method (EM) and the Put-Call symmetry method (PCM). We take $[X_0 = 100,\ S = 95,\ K = 90,\ T = 1]$, and

Table 2.1: $\sigma = 0.2,\ r = 0,$

Table 2.2: $\sigma = 0.2,\ r = 0.02,$

Table 2.3: $\sigma = 0.5,\ r = 0,$

Table 2.4: $\sigma = 0.5,\ r = 0.02.$

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when σ and μ are in $\mathbf{C}_p^4$, the space of functions such that 4-th derivative exists and have a polynomial growth, we have

$$|\mathbf{E}[g(X_T^n)] - \mathbf{E}[g(X_T)]| = O(n),$$

for any $g \in \mathbf{C}_p^4$ (See [22], pp. 476).

In the PCM, we symmetrize the functions $\mu(x) = rx$ and $\sigma(x) = \sigma x$ at K. The errors in the last two columns are calculated as

$$\frac{\text{EM(PCM)} - \text{true option price}}{\text{true option price}}.$$

One sees that, in the experiments, the Put-Call symmetry method always beats the path-wise EM method.

Table 2.1: Black-Scholes model; $X_0 = 100$, $S = 95$, $K = 90$, $\sigma = 0.2$, $r = 0$, $T = 1$, true option price $= 8.17140$

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	8.881	7.816	8.7	4.4
20	8000	9.183	8.172	12.4	0.0
30	27000	8.992	8.250	10.0	1.0
40	64000	8.880	8.175	8.7	0.0
50	125000	8.804	8.190	7.7	0.2
60	216000	8.692	8.137	6.4	0.4
70	343000	8.697	8.127	6.4	0.0
80	512000	8.671	8.171	6.1	0.0
90	729000	8.672	8.207	6.1	0.4
100	1000000	8.597	8.135	5.2	0.4

Table 2.2: Black-Scholes model; $X_0 = 100$, $S = 95$, $K = 90$, $\sigma = 0.2$, $r = 0.02$, $T = 1$, true option price$= 9.31138$

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	10.953	9.821	17.6	5.5
20	8000	10.050	9.165	7.9	1.6
30	27000	10.090	9.226	8.4	0.9
40	64000	9.952	9.258	6.9	0.6
50	125000	9.974	9.302	7.1	0.1
60	216000	10.033	9.389	7.7	0.8
70	343000	9.911	9.298	6.4	0.1
80	512000	9.885	9.353	6.2	0.4
90	729000	9.839	9.306	5.7	0.1
100	1000000	9.811	9.309	5.4	0.0

Table 2.3: Black-Scholes model; $X_0 = 100$, $S = 95$, $K = 90$, $\sigma = 0.5$, $r = 0$, $T = 1$, true option price = 9.37170

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	15.981	9.521	70.5	1.6
20	8000	14.455	9.742	54.2	4.0
30	27000	13.074	9.126	39.5	2.6
40	64000	12.837	9.479	37.0	1.1
50	125000	12.281	9.251	31.0	1.3
60	216000	11.942	9.231	27.4	1.5
70	343000	11.838	9.307	26.3	0.7
80	512000	11.750	9.450	25.4	0.8
90	729000	11.549	9.392	23.2	0.2
100	1000000	11.443	9.319	22.1	0.6

Table 2.4: Black-Scholes model; $X_0 = 100$, $S = 95$, $K = 90$, $\sigma = 0.5$, $r = 0.02$, $T = 1$, true option price = 10.02470

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	15.488	9.688	54.5	3.4
20	8000	14.687	9.540	46.5	4.8
30	27000	14.065	10.341	40.3	3.2
40	64000	13.472	10.191	34.4	1.7
50	125000	13.012	9.779	29.8	2.4
60	216000	12.981	10.257	29.5	2.3
70	343000	12.707	9.991	26.8	0.3
80	512000	12.391	9.916	23.6	1.1
90	729000	12.418	10.098	23.9	0.7
100	1000000	12.235	10.025	22.0	0.0

CEV Model

Here the underlying price process is a solution to the following SDE;

$$dX_t = rX_t dt + \sigma X_t^{\beta} dW_t, \tag{2.15}$$

for $r,\ \sigma \geq 0$ and $\beta \geq \frac{1}{2}$. Tables 2.5 - 2.6 are simulation results for down-and-out call options with EM and PCM. We set parameters to $[X_0 = 100,\ S = 95,\ K = 90,\ \beta = 0.75,\ \sigma = 0.45,\ T = 1]$, and

Table 2.5: $r = 0$,

Table 2.6: $r = 0.02$,

in the experiments. For CEV model, we do not have an analytic formula. So, as a benchmark, we used numerical results by the path-wise Euler-Maruyama scheme where the number of time steps for the Euler discretization is $5,000$ and that of a Monte-Carlo simulation is $50,000,000$. Note that since we are calculating the prices of down-and-out call options, we do not need to care about the singularity at $x = 0$ in the SDE.

Table 2.5: CEV model; $X_0 = 100$, $S = 95$, $K = 90$, $\beta = 0.75$, $\sigma = 0.45$, $r = 0$, $T = 1$, benchmark = 7.50095

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	7.781	7.068	3.8	5.7
20	8000	7.997	7.504	6.6	0.1
30	27000	7.805	7.397	4.1	1.4
40	64000	7.758	7.379	3.5	1.6
50	125000	7.730	7.412	3.1	1.2
60	216000	7.733	7.407	3.1	1.2
70	343000	7.714	7.422	2.9	1.0
80	512000	7.691	7.423	2.6	1.0
90	729000	7.680	7.414	2.4	1.1
100	1000000	7.654	7.414	2.1	1.1

Table 2.6: CEV model; $X_0 = 100$, $S = 95$, $K = 90$, $\beta = 0.75$, $\sigma = 0.45$, $r = 0.02$, $T = 1$, benchmark = 8.82718

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	9.418	8.918	6.7	1.0
20	8000	9.349	8.986	5.9	1.8
30	27000	9.242	8.791	4.7	0.4
40	64000	9.193	8.772	4.1	0.6
50	125000	9.109	8.760	3.2	0.8
60	216000	9.089	8.751	3.0	0.9
70	343000	9.063	8.742	2.7	1.0
80	512000	9.009	8.722	2.1	1.2
90	729000	9.027	8.745	2.3	0.9
100	1000000	8.995	8.722	1.9	1.2

2.3 Put-Call Symmetry Method Applied to Stochastic Volatility Models

In this section, we slightly extend the put-call symmetry method to apply it to stochastic volatility models which are described by two-dimensional SDE. Theoretical backgrounds of the extension is given in [1].

A generic stochastic volatility model is given as follows;

$$\begin{aligned} dX_t &= \sigma_{11}(X_t, V_t)dW_t + \mu_1(X_t, V_t)\,dt \\ dV_t &= \sigma_{21}(V_t)dW_t + \sigma_{22}(V_t)dB_t + \mu_2(V_t)\,dt, \end{aligned} \tag{2.16}$$

where W and B are mutually independent (1-dim) Wiener processes,

$$\sigma(x,v) = \begin{pmatrix} \sigma_{11}(x,v) & 0 \\ \sigma_{21}(v) & \sigma_{22}(v) \end{pmatrix}$$

and $\mu(x,v) = (\mu_1(x,v), \mu_2(v))$ are continuous functions on $\mathbb{R}^2$. Here we simply assume that σ and μ are sufficiently regular (not so irregular) to allow a unique weak solution to (2.16). The independence of V against X plays an important role in applying our scheme. In fact, thanks to the property, we may simply work on the symmetrization with respect to the reflection $(x,y) \mapsto (2K - x, y)$. Let us be more precise. Let (X,V) be a 2-dimensional diffusion process given as a (weak) unique solution to SDE (2.16), and τ be the first passage time of X to K. We say that *arithmetic put-call symmetry* at K holds for (X,V) if

$$(X_t, V_t)1_{\{\tau \le t\}} \stackrel{d}{=} (2K - X_t, V_t)1_{\{\tau \le t\}}$$

for any $t > 0$.

Mathematically, we rely on the following result from [1].

Proposition 2.3.1 ([1]). *If the coefficients satisfy the following conditions;*

$$\sigma_{11}(x, v) = -\sigma_{11}(2K - x, v), \tag{2.17}$$

$$\mu_1(x, v) = -\mu_1(2K - x, v), \tag{2.18}$$

for $(x, v) \in (\mathbb{R} \setminus \{K\}) \times \mathbb{R}$*, then APCS at* K *holds for* (X, V)*.*

On the basis of Proposition 2.3.1, we construct another diffusion $(\tilde{X}, V)$ that satisfies APCS at K in a totally similar way as the one dimensional case, and we obtain a static hedging formula corresponding to Theorem 2.1.3.

Proposition 2.3.2. *Let* $K > 0$ *and put*

$$\tilde{\sigma}_{11}(x, v) = \begin{cases} \sigma_{11}(x, v) & x \geq K \\ -\sigma_{11}(2K - x, v) & x < K \end{cases},$$

$$\tilde{\mu}_1(x, v) = \begin{cases} \mu_1(x, v) & x \geq K \\ -\mu_1(2K - x, v) & x < K \end{cases},$$

and let $\tilde{X}$ *be the unique (weak) solution to*

$$d\tilde{X}_t = \tilde{\sigma}_{11}(\tilde{X}_t, V_t) dW_t + \tilde{\mu}_1(\tilde{X}_t, V_t)\, dt,$$

where V *is the solution to (2.16). Then, it holds for any bounded Borel function* f *and* $t > 0$ *that*

$$\begin{aligned} &E[f(X_t) 1_{\{X_t > K\}} 1_{\{\tau_K > t\}}] \\ &\quad = E[f(\tilde{X}_t) 1_{\{\tilde{X}_t > K\}}] - E[f(2K - \tilde{X}_t) 1_{\{\tilde{X}_t < K\}}], \end{aligned} \tag{2.19}$$

where X *is the solution to (2.16) with* $X_0 > K$*.*

Proof. Omitted. □

2.3.1 Numerical Results on Single Barrier Options under Stochastic Volatility Models

In this section we give numerical examples of the price of a single barrier option under Heston's and SABR type stochastic volatility models, using numerical method based on (2.19).

The Euler-Maruyama scheme of the solution to SDE (2.16) with respect to the net $\triangle_n = \{t_0, t_1, \cdots, t_n\}$ is given by the following;

$$\begin{aligned} X^n_{t_0} &= X_0, \\ V^n_{t_0} &= V_0 \\ X^n_{t_{k+1}} &= X^n_{t_k} + \sigma_{11}(X^n_{t_k}, V^n_{t_k})(W_{t_{k+1}} - W_{t_k}) + \mu_1(X^n_{t_k}, V^n_{t_k})(t_{k+1} - t_k), \\ V^n_{t_{k+1}} &= V^n_{t_k} + \sigma_{21}(V^n_{t_k})(W_{t_{k+1}} - W_{t_k}) + \sigma_{22}(V^n_{t_k})(B_{t_{k+1}} - B_{t_k}) \\ &\quad + \mu_2(V^n_{t_k})(t_{k+1} - t_k), \end{aligned}$$

for $k = 0, 1, 2, \cdots, n-1$. With an interpolation, X_t for $t_k < t \le t_{k+1}$ is obtained as well. Here W and B denotes two independent 1-dimensional Brownian motions. The increments $W_{t_{k+1}} - W_{t_k}$ and $B_{t_{k+1}} - B_{t_k}$ are simulated by independent Gaussian random variables. The underlying price process of Heston model is given as follows;

$$\begin{cases} dX_t = rX_t dt + \sqrt{V_t} X_t dW_t, \\ dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t}(\rho dW_t + \sqrt{1-\rho^2} dB_t) \end{cases} \tag{2.20}$$

for r, κ, θ, $\nu > 0$ and $-1 \le \rho \le 1$. Then the symmetrized path $\tilde{X}$ is constructed as a solution to the following SDE;

$$\begin{aligned} d\tilde{X}_t = &\left(r\tilde{X}_t I_{\{\tilde{X}_t > K\}} - r(2K - \tilde{X}_t) I_{\{\tilde{X}_t < K\}} \right) dt \\ &+ \left(\sqrt{V_t}\tilde{X}_t I_{\{\tilde{X}_t > K\}} - \sqrt{V_t}(2K - \tilde{X}_t) I_{\{\tilde{X}_t < K\}} \right) dW_t, \end{aligned}$$

where V is the solution to SDE (2.20).

The underlying asset price of λ-SABR model is described as

$$\begin{cases} dX_t = rX_t dt + V_t X_t^\beta dW_t, \\ dV_t = \lambda(\theta - V_t)dt + \nu V_t(\rho dW_t + \sqrt{1-\rho^2} dB_t) \end{cases} \tag{2.21}$$

for r, λ, θ, $\nu > 0$, $\beta \ge \frac{1}{2}$ and $-1 \le \rho \le 1$. Then the symmetrized process $\tilde{X}$ is given by the following SDE;

$$\begin{aligned} d\tilde{X}_t = &\left(r\tilde{X}_t I_{\{\tilde{X}_t > K\}} - r(2K - \tilde{X}_t) I_{\{\tilde{X}_t < K\}} \right) dt \\ &+ \left(V_t \tilde{X}_t^{\beta} I_{\{\tilde{X}_t > K\}} - V_t(2K - \tilde{X}_t)^\beta I_{\{\tilde{X}_t < K\}} \right) dW_{1,t}, \end{aligned}$$

where V is the solution to SDE (2.21).

Tables 2.7 - 2.10 below are simulation results of the price of a single barrier call option under Heston's and λ-SABR model, respectively. We set the parameters as $[X_0 = 100,\ V_0 = 0.03,\ K = 95,\ H = 90,\ \theta = 0.03,\ r = 0,\ T = 1,\ \kappa = 1,\ \rho = -0.7,\ \nu = 0.03]$ in Heston model (Table 2.7 and Table 2.8), and $[X_0 = 100,\ V_0 = 0.5,\ S = 95,\ K = 90,\ \theta = 0.03,\ r = 0,\ T = 1,\ \beta = 0.75,\ \lambda = 1.0,\ \rho = -0.7,\ \nu = 0.3]$ in λ-SABR model (Table 2.9 and Table 2.10), and

Table 2.7 and 2.9: $r = 0$,

Table 2.8 and 2.10: $r = 0.02$,

in the experiments. Benchmark is given in the same setting of Section 2.2.2. We again observe the superiority of our scheme.

Table 2.7: Heston model; $X_0 = 100$, $V_0 = 0.03$, $K = 95$, $H = 90$, $\theta = 0.03$, $r = 0$, $T = 1$, $\kappa = 1$, $\rho = -0.7$, $\nu = 0.03$, benchmark = 7.92706

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	8.638	7.953	9.0	0.3
20	8000	8.761	8.167	10.5	3.0
30	27000	8.466	7.932	6.8	0.1
40	64000	8.477	8.017	6.9	1.1
50	125000	8.366	7.892	5.5	0.4
60	216000	8.301	7.877	4.7	0.6
70	343000	8.246	7.875	4.0	0.7
80	512000	8.273	7.902	4.4	0.3
90	729000	8.221	7.875	3.7	0.7
100	1000000	8.212	7.871	3.6	0.7

Table 2.8: Heston model; $X_0 = 100$, $V_0 = 0.03$, $K = 95$, $H = 90$, $\theta = 0.03$, $r = 0.02$, $T = 1$, $\kappa = 1$, $\rho = -0.7$, $\nu = 0.03$, benchmark = 9.15602

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	10.308	9.192	12.6	0.4
20	8000	9.828	9.197	7.3	0.5
30	27000	9.572	8.953	4.5	2.2
40	64000	9.674	9.133	5.7	0.3
50	125000	9.632	9.134	5.2	0.2
60	216000	9.552	9.093	4.3	0.7
70	343000	9.525	9.096	4.0	0.7
80	512000	9.524	9.135	4.0	0.2
90	729000	9.498	9.116	3.7	0.4
100	1000000	9.454	9.106	3.2	0.5

Table 2.9: λ-SABR model; $X_0 = 100$, $V_0 = 0.5$, $S = 95$, $K = 90$, $\theta = 0.03$, $r = 0$, $T = 1$, $\beta = 0.75$, $\lambda = 1.0$, $\rho = -0.7$, $\nu = 0.3$, benchmark = 6.59534

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	6.643	6.478	0.7	1.8
20	8000	6.708	6.591	1.7	0.1
30	27000	6.701	6.584	1.6	0.2
40	64000	6.671	6.565	1.1	0.5
50	125000	6.668	6.568	1.1	0.4
60	216000	6.672	6.581	1.2	0.2
70	343000	6.669	6.585	1.1	0.2
80	512000	6.671	6.597	1.1	0.0
90	729000	6.655	6.579	0.9	0.2
100	1000000	6.646	6.576	0.8	0.3

Table 2.10: λ-SABR model; $X_0 = 100$, $V_0 = 0.5$, $S = 95$, $K = 90$, $\theta = 0.03$, $r = 0.02$, $T = 1$, $\beta = 0.75$, $\lambda = 1.0$, $\rho = -0.7$, $\nu = 0.3$, benchmark = 8.71005

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	9.493	8.779	9.0	0.8
20	8000	9.081	8.582	4.3	1.5
30	27000	9.106	8.723	4.5	0.2
40	64000	9.029	8.656	3.7	0.6
50	125000	9.007	8.683	3.4	0.3
60	216000	9.008	8.710	3.4	0.0
70	343000	8.988	8.707	3.2	0.0
80	512000	8.940	8.670	2.6	0.5
90	729000	8.923	8.671	2.4	0.4
100	1000000	8.929	8.680	2.5	0.3

2.3.2 Application to Pricing Double Barrier Options under the Stochastic Volatility Models

Fix $K, K' > 0$. Let us consider a double barrier option knocked out if price process X hit either the boundary K or $K+K'$. The price of a double barrier option with payoff function f and barriers K and $K+K'$ is given by $\mathbf{E}[f(X_T)I_{\{\tau_{(K,K+K')}>T\}}]$, where $\tau_{(K,K+K')}$ is the first exit time of X from $(K, K+K')$. In a similar way as the static hedging formula of a single barrier option, we obtain a static hedging formula if the price process satisfies APCS both at K and $K+K'$.

Proposition 2.3.3 ([1])**.** *If X satisfies APCS at $K+nK'$ ($n \in \mathbf{Z}$), then for any bounded Borel function f and $T>0$, we have*

$$\begin{aligned}
&\mathbf{E}[f(X_T)I_{\{\tau_{(K,K+K')}>T\}}] \\
&= \sum_{n\in\mathbf{Z}} \mathbf{E}[f(X_T - 2nK')I_{[K+2nK', K+(2n+1)K')}(X_T)] \\
&\quad - \sum_{n\in\mathbf{Z}} \mathbf{E}f(2K - (X_T - 2nK'))I_{[K+(2n-1)K', K+2nK')}(X_T)],
\end{aligned} \tag{2.22}$$

In formula (2.22), the left-hand-side is the price of a barrier option, and the right-hand-side is an infinite series of the prices of plain-vanilla options. It means that a double barrier option can be hedged by infinite plain-vanilla options. Practically, the series should be approximated by finite terms. In our numerical scheme, however, finite sum approximation is not necessary as we will explain later in Remark 2.3.5.

We give a numerical scheme of a double barrier option under stochastic volatility model by using the symmetrized process which satisfies APCS at $K+nK'$ ($n \in \mathbf{Z}$). The scheme is summarized as

Proposition 2.3.4. *Set*

$$\begin{aligned}
&\hat{\sigma}_{11}(x,v) \\
&= \sum_{n\in\mathbf{Z}} \sigma_{11}(x-2nK', v)I_{[K+2nK', K+(2n+1)K')}(x) \\
&\quad - \sum_{n\in\mathbf{Z}} \sigma_{11}(2K-(x-2nK'), v)I_{[K+(2n-1)K', K+2nK')}(x),
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&\hat{\mu}_1(x,v) \\
&= \sum_{n\in\mathbf{Z}} \mu_1(x-2nK', v)I_{[K+2nK', K+(2n+1)K')}(x) \\
&\quad - \sum_{n\in\mathbf{Z}} \mu_1(2K-(x-2nK'), v)I_{[K+(2n-1)K', K+2nK')}(x),
\end{aligned} \tag{2.24}$$

and let $\hat{X}$ be the unique (weak) solution to

$$d\hat{X}_t = \hat{\sigma}_{11}(\hat{X}_t, V_t)dW_t + \hat{\mu}_1(\hat{X}_t, V_t)\,dt,$$

where V is the solution to SDE (2.16). Then, it holds for any bounded Borel function f and $t > 0$ that

$$\begin{aligned} &E[f(X_t)1_{\{\tau_{(K,K+K')}>t\}}] \\ &= \sum_{n\in\mathbf{Z}} E[f(\hat{X}_t - 2nK')I_{[K+2nK',K+(2n+1)K')}(\hat{X}_t)] \\ &\quad - \sum_{n\in\mathbf{Z}} E[f(2K-(\hat{X}_t - 2nK'))I_{[K+(2n-1)K',K+2nK')}(\hat{X}_t)]. \end{aligned} \tag{2.25}$$

Proof. This is an easy consequence of Proposition 2.3.3. □

Remark 2.3.5. *The infinite series of the right hand side in (2.23) and (2.24) is expressed by the following;*

$$\begin{aligned} &(\textit{the right hand side of } (2.23)) \\ &= \begin{cases} \sigma_{11}(x - [\frac{x-K}{K'}]K', v) & \textit{if } [\frac{x-K}{K'}] \equiv 0 \mod 2, \\ -\sigma_{11}(2K-(x-([\frac{x-K}{K'}]-1)K'), v) & \textit{if } [\frac{x-K}{K'}] \equiv 1 \mod 2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &(\textit{the right hand side of } (2.24)) \\ &= \begin{cases} \mu_1(x - [\frac{x-K}{K'}]K', v) & \textit{if } [\frac{x-K}{K'}] \equiv 0 \mod 2, \\ -\mu_1(2K-(x-([\frac{x-K}{K'}]-1)K'), v) & \textit{if } [\frac{x-K}{K'}] \equiv 1 \mod 2, \end{cases} \end{aligned}$$

where $[\cdot]$ is floor function, i.e. $[x]$ is the largest integer not greater than x. Therefore the discretized process of $(\hat{X}, V)$ by Euler-Maruyama scheme can be simulated without approximating the infinite series by finite sums. Similarly, we have

$$\begin{aligned} &(\textit{the right hand side of } (2.25)) \\ &= \mathbf{E}\Bigg[f(\hat{X}_t - [\frac{\hat{X}_t - K}{K'}]K', V_t) I_{\{[\frac{\hat{X}_t-K}{K'}]\equiv 0 \mod 2\}} \\ &\qquad - f(2K - (\hat{X}_t - ([\frac{\hat{X}_t - K}{K'}] - 1)K'), V_t) I_{\{[\frac{\hat{X}_t-K}{K'}]\equiv 1 \mod 2\}} \Bigg]. \end{aligned}$$

Therefore Put-Call symmetry method is available for the pricing of a barrier option.

Table 2.11 and Table 2.12 below are numerical results of the price of a double barrier call option under Heston model and λ-SABR model, respectively. We take

Table 2.11: $X_0 = 100,\ V_0 = 0.03,\ S = 95,\ K + K' = 115,\ K = 85,\ \theta = 0.03,\ r = 0.02,\ T = 1,\ \kappa = 1,\ \rho = -0.7,\ \nu = 0.03,$

Table 2.12: $X_0 = 100,\ V_0 = 0.3,\ S = 95,\ K + K' = 110,\ K = 90,\ \theta = 0.3,\ r = 0.02,\ T = 1,\ \beta = 0.75, \lambda = 1,\ \rho = -0.7,\ \nu = 0.3,$

in the experiments. Benchmark is given by the same setting of Section 2.2.2.

We still see that the put-call symmetry method beats the path-wise EM.

Table 2.11: Heston model; $X_0 = 100$, $V_0 = 0.03$, $S = 95$, $K + K = 115$, $K = 85$, $\theta = 0.03$, $r = 0.02$, $T = 1$, $\kappa = 1$, $\rho = -0.7$, $\nu = 0.03$, benchmark = 1.40319930

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	2.987	1.671	112.9	19.1
20	8000	2.368	1.498	68.8	6.7
30	27000	2.144	1.588	52.8	13.2
40	64000	2.045	1.475	45.8	5.1
50	125000	1.921	1.402	36.9	0.1
60	216000	1.876	1.453	33.7	3.6
70	343000	1.820	1.411	29.7	0.6
80	512000	1.792	1.438	27.7	2.5
90	729000	1.765	1.411	25.8	0.6
100	1000000	1.744	1.416	24.3	0.9

Table 2.12: λ-SABR model; $X_0 = 100$, $V_0 = 0.3$, $S = 95$, $K + K = 110$, $K = 90$, $\theta = 0.3$, $r = 0.02$, $T = 1$, $\beta = 0.75$, $\lambda = 1$, $\rho = -0.7$, $\nu = 0.3$, benchmark = 2.46950606

No. of time steps	No. of simulation trials	EM	PCM	EM error (%)	PCM error (%)
10	1000	3.779	2.451	53.0	0.8
20	8000	3.427	2.566	38.8	3.9
30	27000	3.164	2.442	28.1	1.1
40	64000	3.037	2.489	23.0	0.8
50	125000	2.955	2.514	19.6	1.8
60	216000	2.915	2.480	18.0	0.4
70	343000	2.875	2.481	16.4	0.5
80	512000	2.838	2.478	14.9	0.4
90	729000	2.806	2.464	13.6	0.2
100	1000000	2.779	2.465	12.5	0.2

2.4 Concluding Remark

The new scheme, which is based on the symmetrization of diffusion process, is, though not theoretically, experimentally proven to be more effective than the path-wise Euler-Maruyama approximation scheme. The scheme is also applicable to stochastic volatility models including Heston's and SABR type.

Chapter 3

A Numerical Scheme for Expectations with First Hitting Time to Smooth Boundary

In this chapter, we propose a numerical scheme to calculate expectations with first hitting time to a given smooth boundary, in view of the application to the pricing of options with non-linear barriers. To attack the problem, we rely on the symmetrization technique in [1] and [18], with some modifications. To see the effectiveness, we perform some numerical experiments.

3.1 The Framework

3.1.1 Symmetrization in $\mathbb{R}^d$

We begin with a quick review of (a reduced version of) Akahori-Imamura's multi-dimensional symmetrization scheme [1].

We suppose that the following d dimensional stochastic differential equation (often abbreviated as SDE)

$$X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t \mu(X_s) ds \tag{3.1}$$

has a law-unique solution for each initial point X_0 in a domain in $\mathbb{R}^d$. Here, for $d' \leq d$, $W \equiv (W_t)_{t \geq 0}$ is a d'-dimensional Wiener process, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^{d'}$, $\mu : \mathbb{R}^d \to \mathbb{R}^d$, are piecewise smooth and at most linear growth. For $\alpha \in \mathbb{R}^d \backslash \{0\}$ and $h \in \mathbb{R}$, let

$$H_{\alpha,h} := \{x \in \mathbb{R}^d : \langle \alpha, x \rangle = h\},$$

and $s_{\alpha,h} : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection associated with $H_{\alpha,h}$;

$$s_{\alpha,h}(x) := x - (\langle x, \alpha \rangle - h) \frac{2\alpha}{|\alpha|^2} = \left(I - \frac{2}{|\alpha|^2} \alpha \otimes \alpha \right) x + \frac{2h}{|\alpha|^2} \alpha.$$

Denote

$$T_\alpha := I - \frac{2}{|\alpha|^2}\alpha \otimes \alpha.$$

Then, T_α is a $d \times d$ orthogonal matrix such that ${T_\alpha}^2 = I$ and $T_\alpha \alpha = -\alpha$, and therefore $s^2_{\alpha,h} : \mathbb{R}^d \to \mathbb{R}^d$ is an identity map. We write

$$H^+_{\alpha,h} := \{x \in \mathbb{R}^d : \langle \alpha, x\rangle > h \ \},$$

then

$$\mathbb{R}^d = \overline{H^+_{\alpha,h}} \uplus s_{\alpha,h}(H^+_{\alpha,h}).$$

For a piecewise smooth map $U_\bullet : \mathbb{R}^d \to O(d')$, define

$$\begin{aligned} \tilde{\sigma}(x) &:= \sigma(x)U_x 1_{\overline{H^+_{\alpha,h}}}(x) + T_\alpha \sigma(s_{\alpha,h}(x))U_x 1_{s_{\alpha,h}(H^+_{\alpha,h})}(x) \\ \tilde{\mu}(x) &:= \mu(x)1_{\overline{H^+_{\alpha,h}}}(x) + T_\alpha \mu(s_{\alpha,h}(x))1_{s_{\alpha,h}(H^+_{\alpha,h})}(x), \end{aligned} \tag{3.2}$$

and consider the following stochastic differential equation with starting point $X_0 \in \mathbb{R}^d$:

$$\tilde{X}_t = X_0 + \int_0^t \tilde{\sigma}(\tilde{X}_s)dW_s + \int_0^t \tilde{\mu}(\tilde{X}_s)ds, \tag{3.3}$$

which we call *symmetrization* of X with respect to $H_{\alpha,h}$.

Let

$$\tau_{\alpha,h} := \inf\left\{t > 0 : X_t \notin H^+_{\alpha,h}\right\}.$$

Theorem 3.1.1 (Akahori and Imamura [1]). *Suppose that* (3.3) *has a law-unique solution. Then, for any* $f \in b\mathcal{B}(\mathbb{R}^d)$ *with* $\mathrm{supp}(f) \equiv \overline{\{x \in \mathbb{R}^d; f\ (x) \neq 0\}} \subset H^+_{\alpha,h}$, *it holds that*

$$\mathbf{E}[f(X_t)1_{\{\tau_{\alpha,h}>t\}}] = \mathbf{E}[f(\tilde{X}_t)] - \mathbf{E}[f(s_{\alpha,h}(\tilde{X}_t))]. \tag{3.4}$$

Remark 3.1.2. We can easily extend the validity of the formula (3.4) for any positive f such that both of the expectations in the right-hand-side are finite. Our experiments below are based on this observation.

3.1.2 Symmetrization over a line bundle

Let $d(\geq 2)$, and consider a solution to the following d-dimensional stochastic differential equation:

$$V_t = V_0 + \int_0^t \sigma(V_s)dW_s + \int_0^t \mu(V_s)ds. \tag{3.5}$$

For $g \in C^2(\mathbb{R}^d)$ and $c \in \mathbb{R}$, define

$$\tau = \inf\{t > 0 : g(V_t) < c\}.$$

Our target is to obtain a numerical approximation of

$$\pi = \mathbf{E}\left[F(V_T)1_{\{\tau>T\}}\right]$$

for a bounded measurable function F with

$$\mathrm{supp}F \subset \{v \in \mathbb{R}^d : g(v) \geq c\}.$$

Our scheme consists of lifting it to the following $d+1$ dimensional stochastic differential equation:

$$\begin{cases} V_t = V_0 + \int_0^t \sigma(V_s)dW_s + \int_0^t \mu(V_s)ds \\ Z_t = g(V_0) + \int_0^t \nabla g(V_s)\sigma(V_s)dW_s \\ \quad + \frac{1}{2}\int_0^t (2\nabla g \cdot \mu(V_s) + \nabla \otimes \nabla g \cdot \sigma \otimes \sigma(V_s))\, ds, \end{cases} \tag{3.6}$$

to which we apply Akahori-Imamura's symmetrization. It is possible because the hitting time is now lifted to the one to a hyperplane:

$$H := \{(x_1, \cdots, x_{d+1}) \in \mathbb{R}^{d+1} : x_{d+1} = c\}.$$

The reflection associated with H is now obtained as

$$s_{\alpha,c}\begin{pmatrix} x_1 \\ \vdots \\ x_d \\ x_{d+1} \end{pmatrix} := \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ x_{d+1} \end{pmatrix} - 2(x_{d+1} - c)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 2c - x_{d+1} \end{pmatrix}.$$

Let

$$T_\alpha := I - \frac{2}{|\alpha|^2}\alpha \otimes \alpha = \begin{pmatrix} 1 & & & O \\ & \ddots & & \\ & & 1 & \\ O & & & -1 \end{pmatrix},$$

where

$$\alpha = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}.$$

By putting

$$\begin{aligned} \sigma_2(x_1, \cdots, x_d, x_{d+1}) &:= \begin{pmatrix} I_d \\ \nabla g(x_1, \cdots, x_d) \end{pmatrix} \begin{pmatrix} \sigma(x_1, \cdots, x_d) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma(x_1, \cdots, x_d) & 0 \\ \nabla g \cdot \sigma(x_1, \cdots, x_d) & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\mu_2(x_1, \cdots, x_d, x_{d+1}) \\ &:= \begin{pmatrix} I_d \\ \nabla g(x_1, \cdots, x_d) \end{pmatrix} \mu(x_1, \cdots, x_d) \\ &\quad + \frac{1}{2}\begin{pmatrix} 0 \\ \nabla \otimes \nabla g \cdot \sigma \otimes \sigma(x_1, \cdots, x_d) \end{pmatrix}, \end{aligned}$$

the stochastic differential equation (3.6) is now rewritten as

$$dX_t = \sigma_2(X_t)dW_t + \mu_2(X_t)dt, \quad X_0 = (V_0, g(V_0)) \tag{3.7}$$

with $X = (V, Z)$.

The symmetrization of (3.7) is

$$\tilde{X}_t = X_0 + \int_0^t \tilde{\sigma_2}(\tilde{X}_s)dW_s + \int_0^t \tilde{\mu_2}(\tilde{X}_s)ds, \tag{3.8}$$

where

$$\begin{aligned}\tilde{\sigma_2}(x) &= \begin{cases} \sigma_2(x) & x_{d+1} > c \\ T_\alpha \sigma_2(s_{\alpha,c}(x)) & x_{d+1} \le c \end{cases} \\ &= \begin{cases} \sigma_2(x) & x_{d+1} > c \\ \begin{pmatrix} \sigma(x) & 0 \\ -\nabla g(x) \cdot \sigma(x) & 0 \end{pmatrix} & x_{d+1} \le c \end{cases}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mu_2}(x) &= \begin{cases} \mu_2(x) & x_{d+1} > c \\ T_\alpha \mu(s_{\alpha,c}(x)) & x_{d+1} \le c \end{cases} \\ &= \begin{cases} \mu_2(x) & x_{d+1} > c \\ \begin{pmatrix} \mu(x) \\ -\nabla g(x) \cdot \mu_2(x) - \frac{1}{2}\nabla \otimes \nabla g(x) \cdot \sigma \otimes \sigma(x) \end{pmatrix} & x_{d+1} \le c. \end{cases}\end{aligned}$$

Theorem 3.1.1 could imply

$$\pi = \mathbf{E}\left[F(\tilde{V_T})1_{\{\tilde{Z_T} \ge c\}}\right] - \mathbf{E}\left[F(\tilde{V_T})1_{\{c \ge \tilde{Z_T}\}}\right], \tag{3.9}$$

where $\tilde{X} = (\tilde{V}, \tilde{Z})$, provided that the SDE of (3.8) has a law-unique solution.

Remark 3.1.3. The above procedure, that is, symmetrization of (3.5) by way of (3.6), may well be called symmetrization over a *line bundle*, since the process defined by (3.6) can be regarded as one in the line bundle of the invariant manifold of the process V.

3.2 Numerical Experiments

3.2.1 Description of Experiments

As mentioned, the mathematical validity of (3.9) is dependent of the law-uniqueness of (3.8). Instead of proving it mathematically, in this section we give some results of numerical experiments that convince us the validity. We work on two examples where the expectation with hitting time can be calculated in a direct way. Then we compare it with the one by Euler-Maruyama approximation of the symmetrized process, to see if the formula (3.9) is valid or not.

The results show that the "symmetrization over a line bundle" works properly, and even as efficient as the standard symmetrization whose efficiency is confirmed in [18]. in the sense that it is much more efficient than the order-1/2 scheme of the path-wise Euler-Maruyama approximation of the original process.

Our experiments are done with Euler-Maruyama discretization scheme with Monte-Carlo simulation, described below.

Let n be the number of discretization and M be the number of simulations. For a generic SDE (3.1) and a function f and a hitting time τ to a domain D, we obtain PW-EM(n, M) and EM(n, M) as an approximation of $E[f(X_1)1_{\{\tau>1\}}]$ and $\mathbf{E}[f(X_1)]$ respectively, in the following way. We set $X_0^n = X_0$ and for $t_k = k/n$, $k = 1, \cdots, n$,

$$X_{t_k}^n = X_{t_{k-1}}^n + \sigma(X_{t_{k-1}}^n)\Delta W_{t_k}^n + \mu(X_{t_{k-1}}^n)n^{-1},$$

where $\{\Delta W_{t_k}^n : k = 1, 2, \cdots, n\}$ simulates, by pseudo random numbers, independent copies of d-dimensional centered Gaussian random variables with variance $n^{-1}\times$(identity matrix). The approximation values are given by

$$\text{EM}(n, M) := \frac{1}{M}\sum_{m=1}^{M} f(X^{n,m})$$

and

$$\text{PW-EM}(n, M) := \frac{1}{M}\sum_{m=1}^{M} f(X_1^{n,m})1_{\{\tau^{n,m}>1\}},$$

where $X^{n,m}$ stands for the m-th simulation of X^n, and

$$\tau^{n,m} = \min\left\{t_k : X_{t_k}^{n,m} \in D\right\}.$$

In each experiment, a benchmark value is calculated by a "direct" method, described in each subsection. The errors are calculated accordingly as

$$\text{EM Error}(n, M) = \frac{\text{EM}(n, M) - \text{Benchmark Value}}{\text{Benchmark Value}}, \tag{3.10}$$

and

$$\text{PW-EM Error}(n, M) = \frac{\text{PW-EM}(n, M) - \text{Benchmark Value}}{\text{Benchmark Value}}, \tag{3.11}$$

respectively.

The number of simulation trials M is set to be $10,000$ when the number of time steps for the Euler discretization n is less than 30, and otherwise we set $M = n^3$.

3.2.2 Comparison with 1-dimensional symmetrization

Let

$$X_t = |W_t^x|^2,$$

where W^x is a 2-dimensional Wiener process starting from $x \in \mathbb{R}^2$. We assume $|x| > 1$. Let

$$\tau := \inf\{s > 0 : X_s < 1\}. \tag{3.12}$$

There are two ways to symmetrize X to obtain numerical value of $\mathbf{E}[f(X_t)1_{\{\tau>t\}}]$(say).

The first one is based on the framework in section 3.1.2. We consider 3-dimensional diffusion process (Y^1, Y^2, Y^3) given by

$$\begin{aligned} dY_t^1 &= dW_t^1 \\ dY_t^2 &= dW_t^2 \\ dY_t^3 &= 2Y_t^1 dW_t^1 + 2Y_t^2 dW_t^2 + 2dt. \end{aligned} \tag{3.13}$$

The associated symmetrization is then, by letting

$$\tilde{\sigma}(y_1, y_2, y_3) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2y_1 & 2y_2 & 0 \end{pmatrix} & (y_3 \geq 1) \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2y_1 & -2y_2 & 0 \end{pmatrix} & (y_3 < 1) \end{cases}$$

and

$$\tilde{\mu}(y_1, y_2, y_3) = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} & (y_3 \geq 1) \\ \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} & (y_3 < 1), \end{cases}$$

given by

$$dY_t = \tilde{\sigma}(Y_t)\, dW_t + \tilde{\mu}(Y_t)\, dt. \tag{3.14}$$

Then, by (3.9), we have that

$$\begin{aligned} &\mathbf{E}[f(X_t)1_{\{\tau>t\}}] \\ &= \mathbf{E}[f((Y_t^1)^2 + (Y_t^2)^2)1_{\{Y_t^3>1\}}] - \mathbf{E}[f((Y_t^1)^2 + (Y_t^2)^2)1_{\{Y_t^3\leq 1\}}] \end{aligned} \tag{3.15}$$

for any bounded continuous function f which is zero if $|x| \leq 1$.

The second one is based on the following reduction to one dimensional case. As is well-known (see e.g.[27]), X_t can be realized as a strong solution to

$$dX_t = 2\sqrt{X_t} dB_t + 2dt. \tag{3.16}$$

Therefore, to calculate $\mathbf{E}[f(X_t)1_{\{\tau>t\}}]$, we can rely on the original one dimensional symmetrization [18]:

$$d\tilde{X}_t = 2\sqrt{\tilde{X}_t}dB_t + \tilde{\nu}(\tilde{X}_t)dt, \tag{3.17}$$

where

$$\nu(x) = \begin{cases} 2 & x > 1 \\ -2 & x \leq 1. \end{cases}$$

By the conversion formula (see [18]), we have that

$$\mathbf{E}[f(X_t)1_{\{\tau>t\}}] = \mathbf{E}[f(X_t)] - \mathbf{E}[f(1 - X_t)]. \tag{3.18}$$

The benchmark value is set to be EM$(500, 500^3)$ for (3.17) with (3.18). Table 3.1 shows the results for the experiments with $f(x) = \max(x - 1.1, 0)$ and $X_0 = 2$. There "1-dim PW-EM" refers to the results from PW-EM(n, M) for the discretization of (3.16), "3-dim PW-EM" are PW-EM(n, M) for (3.13), and "3-dim Sym" stands for EM(n, M) for the discretization (3.14) with (3.15). Errors are calculated accordingly by (3.10) or (3.11). The results convince us that *the symmetrization over a line bundle* works well, and much more efficient than the path-wise method.

Table 3.1: Simulation results for the setting described in Section 3.2.2.

No. of time steps	No. of simulation trials	1-dim PW-EM	3-dim PW-EM	3-dim Sym	1-dim PW-EM error (%)	3-dim PW-EM error (%)	3-dim Sym error (%)
10	10000	2.608	2.655	2.191	23.0	25.3	3.3
20	10000	2.467	2.496	2.108	16.4	17.7	-0.5
30	27000	2.408	2.414	2.130	13.6	13.9	0.5
40	64000	2.367	2.375	2.152	11.7	12.0	1.5
50	125000	2.347	2.346	2.149	10.7	10.7	1.4
60	216000	2.330	2.330	2.148	9.9	9.9	1.3
70	343000	2.314	2.316	2.146	9.2	9.2	1.2
80	512000	2.304	2.306	2.151	8.7	8.8	1.5
90	729000	2.292	2.292	2.153	8.1	8.1	1.6
100	1000000	2.284	2.289	2.157	7.7	8.0	1.7

3.2.3 Comparison with hyperbolic reflection principle

Let $X = (X^1, X^2)$ be the unique solution to

$$\begin{aligned} dX_t^1 &= X_t^2 dW_t^1 \\ dX_t^2 &= X_t^2 dW_t^2, \\ X_0 &= x = (x_1, x_2), \end{aligned} \tag{3.19}$$

which is so-called Hyperbolic Brownian motion. We again assume $|x| > 1$ and let τ be as (3.12). Like above, there are two ways to symmetrize X to obtain numerical value of $\mathbf{E}[f(X_t)1_{\{\tau>t\}}]$(say). The first one is, as above, based on the framework in section 3.1.2. We consider 3-dimensional diffusion process (Y^1, Y^2, Y^3) given by

$$\begin{aligned} dY_t^1 &= Y_t^2 dW_t^1 \\ dY_t^2 &= Y_t^2 dW_t^2 \\ dY_t^3 &= 2Y_t^1 dY_t^1 + 2Y_t^2 dY_t^2 + 2(Y_t^2)^2 dt \\ &= 2Y_t^1 Y_t^2 dW_t^1 + 2(Y_t^2)^2 dW_t^2 + 2(Y_t^2)^2 dt. \end{aligned} \tag{3.20}$$

The associated symmetrization is, by letting

$$\tilde{\sigma}(y_1, y_2, y_3) = \begin{cases} \begin{pmatrix} y_2 & 0 & 0 \\ 0 & y_2 & 0 \\ 2y_1 y_2 & 2(y_2)^2 & 0 \end{pmatrix} & (y_3 \geq 1) \\ \begin{pmatrix} y_2 & 0 & 0 \\ 0 & y_2 & 0 \\ -2y_1 y_2 & -2(y_2)^2 & 0 \end{pmatrix} & (y_3 < 1) \end{cases}$$

and

$$\tilde{\mu}(y_1, y_2, y_3) = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 2(y_2)^2 \end{pmatrix} & (y_3 \geq 1) \\ \begin{pmatrix} 0 \\ 0 \\ -2(y_2)^2 \end{pmatrix} & (y_3 < 1), \end{cases}$$

given by

$$dY_t = \tilde{\sigma}(Y_t)\, dW_t + \tilde{\mu}(Y_t)\, dt. \tag{3.21}$$

Then, as above, we obtain the same equation (3.15).

The second one is hyperbolic reflection principle (see Appendix 3.4 for detail) diffusion (3.19). In this case, actually we do not need to symmetrize, but we have readily

$$(X_t^1, X_t^2) \overset{\text{law}}{=} \left(\frac{X_t^1}{(X_t^1)^2 + (X_t^2)^2}, \frac{X_t^2}{(X_t^1)^2 + (X_t^2)^2} \right) \tag{3.22}$$

as a stochastic process provided that $(X_0^1)^2 + (X_0^2)^2 = 1$. Here

$$(x_1, x_2) \mapsto \left(\frac{x_1}{(x_1)^2 + (x_2)^2}, \frac{x_2}{(x_1)^2 + (x_2)^2} \right)$$

is the reflection with respect to the (semi) circle $|x| = 1$. Thanks to the reflection principle (3.22), we have that

$$\mathbf{E}[f(X_t)1_{\{\tau>t\}}] = \mathbf{E}[f(X_t)] - \mathbf{E}\left[f\left(\frac{X_t^1}{(X_t^1)^2 + (X_t^2)^2}, \frac{X_t^2}{(X_t^1)^2 + (X_t^2)^2} \right) \right], \tag{3.23}$$

for any bounded continuous function f which is zero if $|x| \leq 1$.

The benchmark value is set to be $\mathrm{EM}(500, 500^3)$ for (3.19) with (3.23). Table 3.2 shows the results for the experiments with $f(x) = \max(|x| - 1.1, 0)$ and $(X_0, Y_0) = (1, 1)$. There "2-dim PW-EM" refers to the results from PW-EM(n, M) for the discretization of (3.19), "3-dim PW-EM" are PW-EM(n, M) for (3.20), and "3-dim Sym" stands for EM(n, M) for the discretization (3.21) with (3.15). Errors are calculated accordingly by (3.10) or (3.11). The results again convince us that our new scheme works well, and much more efficient than the path-wise method.

Table 3.2: Simulation result for the setting described in Section 3.2.3.

No. of time steps	No. of simulation trials	2-dim PW-EM	3-dim PW-EM	3-dim Sym	2-dim PW-EM error (%)	3-dim PW-EM error (%)	3-dim Sym error (%)
10	10000	3.619	3.945	3.554	-2.6	6.2	-4.3
20	10000	3.555	3.965	3.230	-4.3	6.7	-13.1
30	27000	4.007	3.985	3.599	7.9	7.2	-3.1
40	64000	3.644	3.934	3.598	-1.9	5.9	-3.2
50	125000	3.939	3.954	3.665	6.0	6.4	-1.4
60	216000	3.969	3.882	3.550	6.8	4.5	-4.5
70	343000	3.838	3.910	3.577	3.3	5.2	-3.7
80	512000	3.868	3.834	3.691	4.1	3.2	-0.6
90	729000	3.848	3.828	3.719	3.6	3.0	0.1
100	1000000	3.819	3.833	3.696	2.8	3.2	-0.5

3.3 Conclusion

This paper proposed a numerical scheme to calculate the expectation with hitting time to a domain with smooth boundary. The new scheme is not theoretically valid, but experimentally shown to be robust and more efficient than the path-wise Euler-Maruyama approximation scheme.

3.4 Appendix to Chapter 3: Hyperbolic Reflection Principle

In the complex coordinate $Z = X + iY$, its law is invariant under the action of $SL(2, \mathbb{R})$; for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A \in SL(2, \mathbb{R}),$$

$$(Z_t) \stackrel{\text{law}}{=} \left(\frac{aZ_t + b}{cZ_t + d} \right) =: \Phi_A(Z_t)$$

provided $Z_0 = (aZ_0 + b)/(cZ_0 + d)$. More generally, it is invariant under isometry (a map preserving distance of any two given points) of the hyperbolic space $\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, which is equipped with the distance given by

$$d(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}\right).$$

Thus the Hyperbolic Brownian motion to the hyperbolic space $\mathbb{H}^2$ is the standard Brownian motion to the Euclidean space. We can then define "reflection" in $\mathbb{H}^2$ and then associated reflection principle holds. Associated symmetrization has been introduced by Y. Ida and her collaborators in [13], see also [14].

Chapter 4

On a construction of strong solutions for stochastic differential equations with non-Lipschitz coefficients; a priori estimates approach

Given a stochastic differential equation of which coefficients satisfy Yamada-Watanabe condition or Nakao-Le Gall condition. We prove that its strong solution can be constructed on any probability space using a priori estimates and also using Ito theory based on Picard's approximation scheme.

4.1 The Main Result under Yamada-Watanabe condition

4.1.1 Assumptions

We discuss under the following assumptions.

1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space where the filtration satisfies the usual conditions.

2. Let W be a Brownian motion with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$.

3. Let $T > 0$, and let $X_0 \in \mathbb{R}$.

4. Let $b \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and let $\sigma \in C([0,T] \times \mathbb{R}, \mathbb{R})$.

5. Let b and σ satisfy the following linear growth condition

$$\exists C > 0 \quad \forall t \in [0,T] \quad \forall x \in \mathbb{R},$$
$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|).$$

6. Let b satisfy the following Lipschitz continuity condition

$$\exists C > 0 \quad \forall t \in [0, T] \quad \forall x, y \in \mathbb{R},$$
$$|b(t, x) - b(t, y)| \leq C|x - y|.$$

7. Let σ satisfy the following continuity condition

$$\forall \epsilon > 0 \quad \forall t \in [0, T] \quad \forall x, y \in \mathbb{R},$$
$$|\sigma(t, x) - \sigma(t, y)| \leq h(|x - y|),$$

where h is a strictly increasing continuous function defined on $[0, \infty)$ with $h(0) = 0$ such that

(a)
$$h(x) \leq C(1 + x); \quad \exists C > 0 \quad \forall x \in [0, \infty),$$

(b)
$$\int_{(0,\epsilon)} h^{-2}(u)du = \infty; \quad \forall \epsilon > 0.$$

We consider the following SDE

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{4.1}$$

The following result shows that the assumptions are sufficient to guarantee pathwise uniqueness of solutions.

Proposition 4.1.1 (Yamada-Watanabe(1971)[32])**.** *Under the assumptions (1-7), pathwise uniqueness holds for SDE (4.1).*

A detailed proof of Proposition 4.1.1 can be found for example in the book of Karatzas and Shreve [20], Proposition 5.3.20.

4.1.2 Main Theorem

Here, we present the main theorem. The proof of the main theorem is discussed along the lines of Stefan Ebenfeld [5].

Theorem 4.1.2 (Strong Existence)**.** *Under the assumptions (1-7), the SDE (4.1) has a strong solution.*

The proof of the theorem is based on a particular approximation of the SDE (4.1). Let $\chi_\epsilon(x)$ be a function in $C^\infty(\mathbb{R})$, such that

$$\chi_\epsilon(x) := \begin{cases} 0 & ; x \leq -\epsilon, \\ \text{positive} & ; -\epsilon < x < \epsilon, \\ 0 & ; x \geq \epsilon, \end{cases}$$

and

$$\int_{-\infty}^{\infty} \chi_\epsilon(x)dx = 1.$$

Moreover, we define that

$$\sigma_\epsilon(t,x) := \int_{-\infty}^{\infty} \sigma(t, x-y)\chi_\epsilon(y)dy = (\sigma * \chi_\epsilon)(x),$$

where the symbol $*$ stands for the convolution operator. Consider the following approximated SDE

$$X_t^{(\epsilon)} = X_0 + \int_0^t b(s, X_s^{(\epsilon)})ds + \int_0^t \sigma_\epsilon(s, X_s^{(\epsilon)})dW_s. \tag{4.2}$$

First, we mention that σ_ϵ belongs to $C^\infty(\mathbb{R})$ and satisfies following properties.

Lemma 4.1.3. *Under the assumption 7 for σ,*

(i) for any $0 < \delta < 1$, there exists a constant $\epsilon(\delta) > 0$ such that

$$|\sigma_\epsilon(t,x) - \sigma(t,x)| \le \delta \quad \textit{holds}; \quad \forall \epsilon \in (0, \epsilon(\delta)] \quad \forall t \in [0,T] \quad \forall x \in \mathbb{R},$$

(ii) for any $\epsilon > 0$,

$$|\sigma_\epsilon(t,x) - \sigma(t,x)| \le h(\epsilon) \quad \textit{holds}; \quad \forall t \in [0,T] \quad \forall x \in \mathbb{R},$$

(iii) for σ_ϵ,

$$|\sigma_\epsilon(t,x) - \sigma_\epsilon(t,y)| \le h(|x-y|) \quad \textit{holds}; \quad \forall t \in [0,T] \quad \forall x, y \in \mathbb{R}.$$

Proof of (i). Since $h(t)$ is continuous and $h(0) = 0$, there exists $\epsilon(\delta) > 0$ such that

$$h(|u|) \le \delta; \quad 0 \le |u| \le \epsilon(\delta).$$

Therefore, we have for $0 \le \epsilon \le \epsilon(\delta)$,

$$\begin{aligned}
|\sigma_\epsilon(t,x) - \sigma(t,x)| &= |\int_{-\infty}^{\infty} \sigma(t,x-y)\chi_\epsilon(y)dy - \int_{-\infty}^{\infty} \sigma(t,x)\chi_\epsilon(y)dy| \\
&\le \int_{-\infty}^{\infty} h(|y|)\chi_\epsilon(y)dy \\
&= \int_{-\epsilon}^{\epsilon} h(|y|)\chi_\epsilon(y)dy \\
&\le \delta \int_{-\epsilon}^{\epsilon} \chi_\epsilon(y)dy \\
&= \delta.
\end{aligned}$$

□

Proof of (ii). (ii) can be proved in the same way as (i). □

Proof of (iii). From the assumption 7 and the definition of χ_ϵ, we have

$$\begin{aligned}
|\sigma_\epsilon(t,x) - \sigma_\epsilon(t,y)| &= |\int_{-\infty}^{\infty} \sigma(t,x-u)\chi_\epsilon(u)du - \int_{-\infty}^{\infty} \sigma(t,y-u)\chi_\epsilon(u)du| \\
&= |\int_{-\infty}^{\infty} \chi_\epsilon(u)[\sigma(t,x-u) - \sigma(t,y-u)]du| \\
&\leq \int_{-\infty}^{\infty} \chi_\epsilon(u)|\sigma(t,x-u) - \sigma(t,y-u)|du \\
&\leq \int_{-\infty}^{\infty} \chi_\epsilon(u)h(|(x-u)-(y-u)|)du \\
&= \int_{-\infty}^{\infty} \chi_\epsilon(u)h(|x-y|)du \\
&= h(|x-y|)\int_{-\infty}^{\infty} \chi_\epsilon(u)du \\
&= h(|x-y|).
\end{aligned}$$

□

Remark 4.1.4. *From Lemma 4.1.3, there exists $C > 0$ such that*

$$|b(t,x)| + |\sigma_\epsilon(t,x)| \leq C(1+|x|)$$

holds for any $t \in [0,T]$, $x \in \mathbb{R}$, $\epsilon > 0$.

Since σ_ϵ belongs to $C^\infty(\mathbb{R})$ and satisfies obviously local Lipschitz condition, it is shown that the approximated SDE (4.2) has a unique strong solution. See, for example, Theorem 5.12.1 in the book of Rogers and Williams [28].

4.2 A priori estimates

4.2.1 The High Norm

We will mention the well known result on the boundedness of solutions for approximated SDE (4.2) in the sense of the high norm.

Lemma 4.2.1 (A priori estimate in The High Norm)**.** *Solutions of the approximated SDE (4.2) satisfy the following estimate*

$$\begin{aligned}
&\exists C > 0 \quad \forall \epsilon \leq 1, \\
&\sup_{t\in[0,T]} \mathbf{E}[|X_t^{(\epsilon)}|^4] \leq C.
\end{aligned} \tag{4.3}$$

A detailed proof of the lemma can be found, for example, in the book of Karatzas and Shreve [20], Problem 5.3.15.

4.2.2 The Low Norm

The next lemma on a priori estimate in the low norm for the approximated SDE (4.2) will play essential roles in the proof of our main theorem. The lemma requires a smooth approximation of the function $|x|$. Therefore, we introduce a sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers by

$$a_0 := 1; \quad \int_{a_n}^{a_{n-1}} \frac{1}{nh^2(x)} dx = 1.$$

Next, we choose a sequence $(\rho_n)_{n\in\mathbb{N}}$ of smooth mollifiers with the following properties

$$supp(\rho_n) \subset [a_n, a_{n-1}]; \quad 0 \le \rho_n(x) \le \frac{2}{nh^2(x)}; \quad \int_{a_n}^{a_{n-1}} \rho_n(x)dx = 1.$$

Finally, we define a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of smooth functions by

$$\varphi_n(x) := \int_0^{|x|} \int_0^y \rho_n(z)dzdy + a_{n-1}.$$

Then, $(\varphi_n)_{n\in\mathbb{N}}$ has the following properties

$$\varphi_n(x) \ge |x|; \quad |\varphi_n'(x)| \le 1; \quad \varphi_n''(x) = \rho_n(|x|).$$

In other words, φ_n is a smooth approximation of the function $|x|$ from above with a bounded first-order derivative and a second-order derivative having support in the interval $[a_n, a_{n-1}]$.

Lemma 4.2.2 (A priori estimate in The Low Norm)**.** *Solutions of the approximated SDE (4.2) satisfy the following a priori estimate*

$$\begin{aligned} &\forall \alpha > 0 \quad \exists 0 < \beta \le 1 \quad \forall 0 < \epsilon_1, \epsilon_2 \le \beta, \\ &\sup_{t\in[0,T]} \mathbf{E}[|X_t^{(\epsilon_1)} - X_t^{(\epsilon_2)}|] \le \alpha. \end{aligned} \tag{4.4}$$

Proof. Put

$$\begin{aligned} \Delta_t^{(\epsilon_1,\epsilon_2)} &:= X_t^{(\epsilon_1)} - X_t^{(\epsilon_2)} \\ &= \int_0^t [b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]ds + \int_0^t [\sigma_{\epsilon_1}(s, X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s, X_s^{(\epsilon_2)})]dW_s. \end{aligned}$$

Applying Ito formula to the approximated SDE (4.2), we obtain the following representation

$$\begin{aligned} \varphi_n(\Delta_t^{(\epsilon_1,\epsilon_2)}) = \varphi_n(0) &+ \int_0^t \varphi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]ds \\ &+ \frac{1}{2}\int_0^t \varphi_n''(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(s, X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s, X_s^{(\epsilon_2)})]^2 ds \\ &+ \int_0^t \varphi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(s, X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s, X_s^{(\epsilon_2)})]dW_s. \end{aligned}$$

We note that due to the uniform boundedness of φ_n' and the linear growth condition (see Remark 4.1.4), the Ito integral in the above is a martingale with mean 0. Let $0 < \beta \leq 1$, $0 < \epsilon_1, \epsilon_2 \leq \beta$. By Ito formula, we have

$$\begin{aligned}
\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|] &\leq \mathbf{E}[\varphi_n(\Delta_t^{(\epsilon_1,\epsilon_2)})] \\
&= \mathbf{E}[\varphi_n(0)] + \mathbf{E}[\int_0^t \varphi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[b(s,X_s^{(\epsilon_1)}) - b(s,X_s^{(\epsilon_2)})]ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \varphi_n''(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(s,X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s,X_s^{(\epsilon_2)})]^2 ds] \\
&\quad + \mathbf{E}[\int_0^t \varphi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(s,X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s,X_s^{(\epsilon_2)})]dW_s] \\
&= a_{n-1} + \mathbf{E}[\int_0^t \varphi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[b(s,X_s^{(\epsilon_1)}) - b(s,X_s^{(\epsilon_2)})]ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \varphi_n''(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(s,X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s,X_s^{(\epsilon_2)})]^2 ds],
\end{aligned}$$

using $\varphi_n''(x) = \rho_n(|x|)$;

$$\begin{aligned}
&\leq a_{n-1} + \mathbf{E}[\int_0^t |\varphi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[b(s,X_s^{(\epsilon_1)}) - b(s,X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)[\sigma_{\epsilon_1}(s,X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s,X_s^{(\epsilon_2)})]^2 ds],
\end{aligned}$$

using $|\varphi_n'(x)| \leq 1$;

$$\begin{aligned}
&\leq a_{n-1} + \mathbf{E}[\int_0^t |[b(s,X_s^{(\epsilon_1)}) - b(s,X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)|\sigma_{\epsilon_1}(s,X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s,X_s^{(\epsilon_2)})|^2 ds] \\
&= a_{n-1} + \mathbf{E}[\int_0^t |[b(s,X_s^{(\epsilon_1)}) - b(s,X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)|\sigma_{\epsilon_1}(s,X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(s,X_s^{(\epsilon_2)}) \\
&\qquad - \sigma(s,X_s^{(\epsilon_1)}) + \sigma(s,X_s^{(\epsilon_1)}) + \sigma(s,X_s^{(\epsilon_2)}) - \sigma(s,X_s^{(\epsilon_2)})|^2 ds] \\
&\leq a_{n-1} + \mathbf{E}[\int_0^t |[b(s,X_s^{(\epsilon_1)}) - b(s,X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)(3|\sigma(s,X_s^{(\epsilon_1)}) - \sigma(s,X_s^{(\epsilon_2)})|^2 \\
&\qquad + 3|\sigma_{\epsilon_1}(s,X_s^{(\epsilon_1)}) - \sigma(s,X_s^{(\epsilon_1)})|^2 + 3|\sigma(s,X_s^{(\epsilon_2)}) - \sigma_{\epsilon_2}(s,X_s^{(\epsilon_2)})|^2)ds].
\end{aligned}$$

Here, we give $0 < \delta < 1$. Then, by the assumption 7 and Lemma 4.1.3, we have for any $0 < \epsilon_1, \epsilon_2 \leq \beta \leq \epsilon(\delta)$,

$$\begin{aligned}
\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|] &\leq a_{n-1} + \mathbf{E}[\int_0^t |[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)(3[h(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)]^2 + 3\delta^2 + 3\delta^2)ds] \\
&\leq a_{n-1} + \mathbf{E}[\int_0^t |[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)(3[h(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)]^2 + 3\delta + 3\delta)ds],
\end{aligned}$$

using $0 \leq \rho_n(x) \leq \frac{2}{nh^2(x)}$;

$$\begin{aligned}
&\leq a_{n-1} + \mathbf{E}[\int_0^t |[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}\mathbf{E}[\int_0^t \frac{2I_{[a_n,a_{n-1}]}(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)}{nh^2(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)}(3[h(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)]^2 + 6\delta)ds] \\
&\leq a_{n-1} + \mathbf{E}[\int_0^t |[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]|ds] \\
&\quad + \frac{1}{2}T\frac{2 \cdot 6\delta}{nh^2(a_n)} + \frac{1}{2}\mathbf{E}[\int_0^t \frac{2}{nh^2(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)} \cdot 3|h(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)|^2 ds] \\
&= a_{n-1} + \mathbf{E}[\int_0^t |[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]|ds] \\
&\quad + T\frac{6\delta}{nh^2(a_n)} + \frac{3}{n}\mathbf{E}[\int_0^t 1ds],
\end{aligned}$$

with $t \in [0, T]$;

$$\begin{aligned}
&\leq a_{n-1} + \mathbf{E}[\int_0^t |[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]|ds] \\
&\quad + T\frac{6\delta}{nh^2(a_n)} + \frac{3}{n}\mathbf{E}[\int_0^T 1ds] \\
&= a_{n-1} + \mathbf{E}[\int_0^t |[b(s, X_s^{(\epsilon_1)}) - b(s, X_s^{(\epsilon_2)})]|ds] \\
&\quad + T\frac{6\delta}{nh^2(a_n)} + \frac{3}{n}T,
\end{aligned}$$

by the assumption 6;

$$\begin{aligned}
&\leq a_{n-1} + \mathbf{E}[\int_0^t C|\Delta_s^{(\epsilon_1,\epsilon_2)}|ds] + T\frac{6\delta}{nh^2(a_n)} + \frac{3}{n}T \\
&= a_{n-1} + \frac{T}{n}(\frac{6\delta}{h^2(a_n)} + 3) + C\int_0^t \mathbf{E}[|\Delta_s^{(\epsilon_1,\epsilon_2)}|]ds.
\end{aligned}$$

Combining all the estimates stated earlier we obtain our final estimate

$$\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|] \leq \gamma(n,\delta) + C\int_0^t \mathbf{E}[|\Delta_s^{(\epsilon_1,\epsilon_2)}|]ds, \tag{4.5}$$

$$\gamma(n,\delta) := a_{n-1} + \frac{T}{n}(\frac{6\delta}{h^2(a_n)} + 3).$$

Choosing n sufficiently large first and then choosing δ sufficiently small, we can take $\gamma(n,\delta)$ arbitrarily small. By Gronwall's inequality, the equation (4.5) implies

$$\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|] \leq \gamma(n,\delta)e^{Ct} \leq \gamma(n,\delta)e^{CT}. \tag{4.6}$$

Thus, for any $\alpha > 0$, choosing n and δ such that $\gamma(n,\delta)e^{CT} < \alpha$, we have for any $0 < \epsilon_1, \epsilon_2 \leq \beta \leq \epsilon(\delta)$,

$$\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|] \leq \alpha. \tag{4.7}$$

□

4.3 Uniform Integrability

Consider the following Banach spaces ($1 \leq p < \infty$)

$$H^p := C([0,T], L^p(\Omega, \mathcal{F}, \mathbf{P})),$$

$$||X||_p := \sup_{t\in[0,T]} (\mathbf{E}[|X_t|^p])^{\frac{1}{p}}.$$

Moreover, we consider the following subsets ($1 \leq p < \infty$)

$$N^p := \{X \in H^p | X \text{ is adapted with respect to the filtration} \{\mathcal{F}_t\}\}.$$

Since N^p is a closed subspace of H^p, it is also a Banach space with respect to the norm $||\cdot||_p$. We define $\epsilon_n := \frac{1}{n}$ and write $X^{(n)}$ instead of $X^{(\epsilon_n)}$. By Lemma 4.2.1, the sequence $(X^{(n)})_{n\in\mathbb{N}}$ is bounded in N^4. Moreover, according to Lemma 4.2.2, the sequence $(X^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in N^1. We will show that the sequence $(X^{(n)})_{n\in\mathbb{N}}$ is also a Cauchy sequence in N^2.

Lemma 4.3.1. *$(X^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in N^2.*

Proof. Lemma 4.2.1 implies immediately

$$\sup_{n\in\mathbb{N}} ||X^{(n)}||_2 < \infty. \tag{4.8}$$

If $(X^{(n)})$ is not a Cauchy sequence in N^2, there exist a positive constant C and some subsequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sup_{0\leq s\leq T} \mathbf{E}[|X_s^{(p_n)} - X_s^{(q_n)}|^2] = C > 0. \tag{4.9}$$

Note that

$$\mathbf{E}[\int_0^T |X_s^{(p_n)} - X_s^{(q_n)}|ds] = \int_0^T \mathbf{E}[|X_s^{(p_n)} - X_s^{(q_n)}|]ds$$
$$\leq \int_0^T (\sup_{0\leq s\leq T} \mathbf{E}[|X_s^{(p_n)} - X_s^{(q_n)}|])ds$$
$$= T||X^{(p_n)} - X^{(q_n)}||_1.$$

Since $(X^{(n)})$ is a Cauchy sequence in N^1,

$$\lim_{n\to\infty} \mathbf{E}[\int_0^T |X_s^{(p_n)} - X_s^{(q_n)}|ds] \leq \lim_{n\to\infty} T||X^{(p_n)} - X^{(q_n)}||_1 = 0. \tag{4.10}$$

We can choose subsequences $(p_n^{'})$ and $(q_n^{'})$ such that

$$\lim_{n\to\infty} |X_s^{(p_n^{'})} - X_s^{(q_n^{'})}| = 0 \tag{4.11}$$

almost everywhere on $[0, T]\times\Omega$ w.r.t. $dt\times d\mathbf{P}$. Assume that for some subsequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \sup_{0\leq s\leq T} \mathbf{E}[|X_s^{(p_n)} - X_s^{(q_n)}|^2] = C > 0. \tag{4.12}$$

Since

$$\lim_{n\to\infty} ||X^{(p_n)} - X^{(q_n)}||_1 = 0,$$

we can choose subsequences $(p_n')_{n\in\mathbb{N}}$, $(q_n')_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} |X_s^{(p_n')} - X_s^{(q_n')}| = 0 \tag{4.13}$$

almost surely on $[0, T] \times \Omega$ w.r.t. $dt \times d\mathbf{P}$. Let $\epsilon_n' = \frac{1}{p_n'}$ and $\epsilon_n'' = \frac{1}{q_n'}$. We have

$$||X^{(p_n')} - X^{(q_n')}||_2^2 = \sup_{0\leq s\leq T} \mathbf{E}[|X_s^{(p_n')} - X_s^{(q_n')}|^2]$$
$$\leq 2\mathbf{E}[\int_0^T (\sigma_{\epsilon_n'}(s, X_s^{(p_n')}) - \sigma_{\epsilon_n''}(s, X_s^{(q_n')}))^2 ds]$$
$$+ 2\mathbf{E}[\int_0^T (b(s, X_s^{(p_n')}) - b(s, X_s^{(q_n')}))^2 ds]$$

Now, we define as follows;

$$L_1^{(n)}(T) := \mathbf{E}[\int_0^T (\sigma_{\epsilon_n'}(s, X_s^{(p_n')}) - \sigma_{\epsilon_n''}(s, X_s^{(q_n')}))^2 ds], \tag{4.14}$$

$$L_2^{(n)}(T) := \mathbf{E}[\int_0^T (b(s, X_s^{(p_n')}) - b(s, X_s^{(q_n')}))^2 ds]. \tag{4.15}$$

For $L_1^{(n)}(T)$, we observe that

$$L_1^{(n)}(T) \leq 3\mathbf{E}[\int_0^T (\sigma_{\epsilon'_n}(s, X_s^{(p'_n)}) - \sigma(s, X_s^{(p'_n)}))^2 ds]$$
$$+ 3\mathbf{E}[\int_0^T (\sigma(s, X_s^{(p'_n)}) - \sigma(s, X_s^{(q'_n)}))^2 ds]$$
$$+ 3\mathbf{E}[\int_0^T (\sigma_{\epsilon''_n}(s, X_s^{(q'_n)}) - \sigma(s, X_s^{(q'_n)}))^2 ds].$$

Using Lemma 4.1.3 (ii) and the assumption 7, we have

$$L_1^{(n)}(T) \leq 3T(h^2(\epsilon'_n) + h^2(\epsilon''_n))$$
$$+ 3\mathbf{E}[\int_0^T h^2(|X_s^{(p'_n)} - X_s^{(q'_n)}|) ds].$$

Note that

$$h^2(|X_s^{(p'_n)} - X_s^{(q'_n)}|) \leq 2C^2 + 2C^2|X_s^{(p'_n)} - X_s^{(q'_n)}|^2.$$

Since

$$\sup_n ||X^{(p'_n)} - X^{(q'_n)}||_4 < \infty$$

holds, the family of processes

$$h^2(|X_s^{(p'_n)} - X_s^{(q'_n)}|)$$

is uniformly integrable on $[0, T] \times \Omega$ w.r.t. $dt \times d\mathbf{P}$. Since $h(\epsilon)$ tends to 0 $\quad (\epsilon \downarrow 0)$, we have by the equation (4.13),

$$\lim_{n\to\infty} L_1^{(n)}(T) = 0.$$

For $L_2^{(n)}(T)$, we have by the assumption 6 that

$$L_2^{(n)}(T) \leq \mathbf{E}[\int_0^T C^2 |X_s^{(p'_n)} - X_s^{(q'_n)}|^2 ds].$$

Since the family of processes $|X_s^{(p'_n)} - X_s^{(q'_n)}|^2$ is uniformly integrable, the equation (4.13) implies

$$\lim_{n\to\infty} L_2^{(n)}(T) = 0.$$

Thus we observe

$$\lim_{n\to\infty} ||X_s^{(p'_n)} - X_s^{(q'_n)}||_2^2 = 0.$$

This fact contradicts the equation (4.12). $\square$

Therefore, the sequence $(X^{(n)})_{n\in\mathbb{N}}$ converges to some $\widetilde{X} \in N^2$. Since the convergence in N^2 implies in N^1, we have

$$\lim_{n\to\infty} (||X^{(n)} - \widetilde{X}||_1 + ||X^{(n)} - \widetilde{X}||_2) = 0. \quad (4.16)$$

4.4 Proof of The Main Result

Now, we are in a position to prove our main theorem.

Proof of Theorem 4.1.2. We use the following notation for the right-hand sides of the SDEs under consideration

$$RHS_t^{(n)} := X_0 + \int_0^t b(s, X_s^{(n)})ds + \int_0^t \sigma_{\epsilon_n}(s, X_s^{(n)})dW_s,$$

$$\widetilde{RHS}_t := X_0 + \int_0^t b(s, \widetilde{X}_s)ds + \int_0^t \sigma(s, \widetilde{X}_s)dW_s.$$

Fix $N > 0$. Since σ_ϵ is C^∞-function, it satisfies Lipschitz condition on $(t, x) \in [0, T] \times [-N, N]$. Let

$$\tau_N := \inf\{s : X_s^{(n)} \notin [-N, N]\}. \tag{4.17}$$

Note that

$$|\sigma_\epsilon(t, x)| + |b(t, x)| \leq C(1 + |x|), \tag{4.18}$$

it is well known that

$$\lim_{N\to\infty} T \wedge \tau_N = T \quad (a.s.), \tag{4.19}$$

see, for example, Theorem 5.12.1 in Rogers and Williams [28]. We know also by [28] that for the strong solutions $X^{(n)}$ satisfy their respective SDEs in the following sense

$$\mathbf{E}[\sup_{0\leq t\leq T\wedge\tau_N} |X_t^{(n)} - RHS_t^{(n)}|^2] \leq 4\mathbf{E}[|X_{T\wedge\tau_N}^{(n)} - RHS_{T\wedge\tau_N}^{(n)}|^2] = 0. \tag{4.20}$$

By Lemma 4.2.1, we know that the family of variables

$$|X_t^{(n)} - RHS_t^{(n)}|^2, \quad t \in [0, T]$$

is uniformly integrable. Thus, letting $N \to \infty$, we have

$$\mathbf{E}[\sup_{0\leq t\leq T} |X_t^{(n)} - RHS_t^{(n)}|^2] = 0. \tag{4.21}$$

This implies the following weaker condition

$$||X^{(n)} - RHS^{(n)}||_2 = 0. \tag{4.22}$$

In the following, $C_1, C_2 > 0$ denotes some generic constants independent of n. With the help of the linear growth condition (see 5 in 4.1.1 Assumptions), we obtain the following estimate

$$\mathbf{E}[\int_0^T (|b(s, \widetilde{X}_s)|^2 + |\sigma(s, \widetilde{X}_s)|^2)ds] \leq C^2T(1 + ||\widetilde{X}||_2^2) < \infty.$$

This implies that $\widetilde{RHS}$ has continuous paths $\mathbf{P}$-a.s. and satisfies the following regularity condition

$$\mathbf{E}[\sup_{t\in[0,T]} |\widetilde{RHS}_t|^2] < \infty.$$

With the help of the Lipschitz continuity condition for b and the modulus of continuity condition for σ and σ_ϵ (see Lemma 4.1.3), we obtain the following statement of convergence

$$\begin{aligned}
&\lim_{n\to\infty} ||RHS^{(n)} - \widetilde{RHS}||_2^2 \\
&\leq 2 \lim_{n\to\infty} (\sup_{[0,T]} \mathbf{E}[| \int_0^t (b(s, X_s^{(n)}) - b(s, \widetilde{X}_s))ds|^2] \\
&\qquad + \sup_{[0,T]} \mathbf{E}[| \int_0^t (\sigma_{\epsilon_n}(s, X_s^{(n)}) - \sigma(s, \widetilde{X}_s))dW_s|^2]) \\
&\leq C_1 (\lim_{n\to\infty} \mathbf{E}[\int_0^T |b(s, X_s^{(n)}) - b(s, \widetilde{X}_s)|^2 ds] \\
&\qquad + \lim_{n\to\infty} \mathbf{E}[\int_0^T |\sigma_{\epsilon_n}(s, X_s^{(n)}) - \sigma(s, \widetilde{X}_s)|^2 ds]) \\
&= C_1 (\lim_{n\to\infty} \mathbf{E}[\int_0^T |b(s, X_s^{(n)}) - b(s, \widetilde{X}_s)|^2 ds] \\
&\qquad + \lim_{n\to\infty} \mathbf{E}[\int_0^T |\sigma_{\epsilon_n}(s, X_s^{(n)}) - \sigma(s, X_s^{(n)}) + \sigma(s, X_s^{(n)}) - \sigma(s, \widetilde{X}_s)|^2 ds]) \\
&\leq C_1 (\lim_{n\to\infty} \mathbf{E}[\int_0^T |b(s, X_s^{(n)}) - b(s, \widetilde{X}_s)|^2 ds] \\
&\qquad + \lim_{n\to\infty} \mathbf{E}[\int_0^T |\sigma_{\epsilon_n}(s, X_s^{(n)}) - \sigma(s, X_s^{(n)})|^2 + |\sigma(s, X_s^{(n)}) - \sigma(s, \widetilde{X}_s)|^2 ds]),
\end{aligned}$$

using Lemma 4.1.3;

$$\begin{aligned}
&\leq C_2 (\lim_{n\to\infty} \mathbf{E}[\int_0^T (|X_s^{(n)} - \widetilde{X}_s|)^2 ds] \\
&\qquad + \lim_{n\to\infty} \mathbf{E}[\int_0^T |\delta|^2 + |h^2(|X_s^{(n)} - \widetilde{X}_s|)| ds]),
\end{aligned}$$

using Lemma 4.3.1 and noticing that $\delta > 0$ is arbitrary;

$$= 0.$$

Combining the estimates stated earlier, we see that $\widetilde{X}$ satisfies the SDE (4.1) in the following sense

$$||\widetilde{X} - \widetilde{RHS}||_2 = 0. \tag{4.23}$$

Although $\widetilde{RHS}$ is a modification of $\widetilde{X}$ having continuous path $\mathbf{P}$-a.s., the same is generally not true for $\widetilde{X}$. Therefore, we consider $\widetilde{RHS}$ instead of $\widetilde{X}$ using the following notation

$$X := \widetilde{RHS},$$

$$RHS_t := X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

Since X and $\widetilde{X}$ coincide as elements of N^2, the linear growth condition (see 5 in 4.1.1 Assumptions) yields the following estimate

$$\mathbf{E}[\int_0^T (|b(s, X_s)|^2 + |\sigma(s, |X_s|)|^2)ds] \leq C^2(1 + ||X||_2^2) < \infty. \tag{4.24}$$

This implies that RHS has continuous paths $\mathbf{P}$-a.s. and satisfies the following regularity condition

$$\mathbf{E}[\sup_{t\in[0,T]} |RHS_t|^2] < \infty. \tag{4.25}$$

Finally, with the help of the Lipschitz continuity condition for b (see 6 in 4.1.1 Assumptions), the modulus of continuity condition for σ (see 7 in 4.1.1 Assumptions), and Doob's maximal inequality, we see that X satisfies the SDE (4.1) in the sense of Ito theory

$$\begin{aligned}
\mathbf{E}[\sup_{t\in[0,T]} |X_t - RHS_t|^2] &= \mathbf{E}[\sup_{t\in[0,T]} |\widetilde{RHS}_t - RHS_t|^2] \\
&\leq 2(\mathbf{E}[\sup_{t\in[0,T]} |\int_0^t (b(s, \widetilde{X}_s) - b(s, X_s))ds|^2] \\
&\quad + \mathbf{E}[\sup_{t\in[0,T]} |\int_0^t (\sigma(s, \widetilde{X}_s) - \sigma(s, X_s))dW_s|^2]) \\
&\leq C_1(\mathbf{E}[\int_0^T |b(s, \widetilde{X}_s) - b(s, X_s)|^2 ds] \\
&\quad + \mathbf{E}[\int_0^T |\sigma(s, \widetilde{X}_s) - \sigma(s, X_s)|^2 ds]) \\
&\leq C_1(||\widetilde{X} - X||_2 + \int_0^T \mathbf{E}[h^2(|\widetilde{X}_s - X_s|)]ds) \\
&= 0.
\end{aligned}$$

Thus, X is the desired strong solution. This concludes the proof. □

4.5 The Main Result under Nakao-Le Gall condition

In the present section, we construct concretely a strong solution of SDE under Nakao-Le Gall condition. We consider the following SDE

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s. \tag{4.26}$$

We assume that σ satisfies Nakao-Le Gall condition.

Definition 4.5.1 (Nakao-Le Gall condition)**.** *σ be $\mathbb{R} \to \mathbb{R}$, Borel measurable. There exist two positive constants $0 < k < K < \infty$ such that*

$$0 < k \leq \sigma(x) \leq K < \infty \quad \forall x \in \mathbb{R}.$$

And, there exists bounded increasing function f such that

$$|\sigma(x) - \sigma(y)|^2 \leq |f(x) - f(y)| \quad \forall x, y \in \mathbb{R}, \tag{4.27}$$

where f is not necessarily continuous.

The main result in this section is the following theorem. Although the result of the theorem is known, our proof of the theorem proposes a concrete construction of strong solution.

Theorem 4.5.2. *Under Nakao-Le Gall condition, the SDE (4.26) has a strong solution.*

To prove the theorem, we prepare some approximation techniques. Here, let $f(-\infty)$ and $f(\infty)$ be

$$f(-\infty) := \lim_{x \to -\infty} f(x), \tag{4.28}$$

$$f(\infty) := \lim_{x \to \infty} f(x), \tag{4.29}$$

then we obtain $-\infty < f(-\infty) < f(\infty) < \infty$. Let $v(f)$ be

$$v(f) := f(\infty) - f(-\infty), \tag{4.30}$$

$v(f)$ is called the total variation of f.

Remark 4.5.3. *Let D be a set of the discontinuous points of f. Since f is a bounded increasing function, it is well known that D is a countable set. Let $(f_l)_{l \in \mathbb{N}}$ be a sequence of C^∞-functions such that*

$$f_l \leq f, \tag{4.31}$$

$$\lim_{l \to \infty} f_l(x) = f(x) \quad for\ x \notin D, \tag{4.32}$$

and

$$v(f_l) \leq v(f). \tag{4.33}$$

We will construct an example of such sequence (f_l). Let $(g_l)_{l \in \mathbb{N}}$ be a sequence of C^∞-functions such that

$$g_l(u) = \begin{cases} 0 & ; u \leq 0, \\ g(u) > 0 & ; 0 < u < \frac{1}{l}, \\ 0 & ; u \geq \frac{1}{l}, \end{cases}$$

and

$$\int_{-\infty}^{\infty} g_l(u)du = 1.$$

Put

$$f_l(x) := \int_{-\infty}^{\infty} f(x-u)g_l(u)du. \tag{4.34}$$

Note that

$$\begin{aligned} f_l(x) &= \int_{-\infty}^{\infty} f(x-u)g_l(u)du \\ &\leq \int_{-\infty}^{\infty} f(x)g_l(u)du \\ &= f(x). \end{aligned}$$

This implies (4.31). Let $x \notin D$*. For any* $\epsilon > 0$*, there exists* l *such that*

$$f(x) - \epsilon \leq f(x-u) \leq f(x) \tag{4.35}$$

holds for $0 \leq u \leq \frac{1}{l}$*. We have*

$$\begin{aligned} f(x) - \epsilon &= \int_{-\infty}^{\infty} (f(x) - \epsilon)g_l(u)du \\ &\leq \int_{-\infty}^{\infty} f(x-u)g_l(u)du \\ &= f_l(x) \\ &\leq f(x). \end{aligned}$$

Thus,

$$\lim_{l\to\infty} f_l(x) = f(x) \tag{4.36}$$

holds for $x \notin D$*. By the definition of* $f_l(x)$*, we observe that*

$$f(x - \frac{1}{l}) \leq f_l(x) \leq f(x). \tag{4.37}$$

This implies $f_l(-\infty) = f(-\infty)$*, and also* $f_l(\infty) \leq f(\infty)$*. Therefore, we have*

$$v(f_l) = f_l(\infty) - f_l(-\infty) \leq v(f). \tag{4.38}$$

Let $\sigma_\epsilon(x)$ be

$$\sigma_\epsilon(x) := \int_{-\infty}^{\infty} \sigma(x-y)\chi_\epsilon(y)dy = (\sigma * \chi_\epsilon)(x), \tag{4.39}$$

where the function $\chi_\epsilon(x)$ is given in section 4.1.2. Then, $\sigma_\epsilon(x)$ is $C^\infty(\mathbb{R})$ function. Consider the following approximated SDE

$$X_t^{(\epsilon)} = X_0 + \int_0^t \sigma_\epsilon(X_s^{(\epsilon)})dW_s. \tag{4.40}$$

Lemma 4.5.4.

(i) σ_ϵ *is a function in* C^∞ *and* $0 < k \le \sigma_\epsilon(x) \le K < \infty$*,*

(ii) let x *be continuous point of* σ*, then* $\lim_{\epsilon\downarrow 0} \sigma_\epsilon(x) = \sigma$*,*

(iii) $|\sigma_\epsilon(x) - \sigma_\epsilon(y)|^2 \le |f(x \vee y + \epsilon) - f(x \wedge y - \epsilon)|$*, where* $x \vee y := max(x, y)$ *and* $x \wedge y := min(x, y)$*.*

Proof of (i).

$$\sigma_\epsilon(x) = \int_{-\infty}^{\infty} \sigma(x-y)\chi_\epsilon(y)dy \ge \int_{-\infty}^{\infty} k\chi_\epsilon(y)dy = k, \tag{4.41}$$

and

$$\sigma_\epsilon(x) = \int_{-\infty}^{\infty} \sigma(x-y)\chi_\epsilon(y)dy \le \int_{-\infty}^{\infty} K\chi_\epsilon(y)dy = K. \tag{4.42}$$

□

Proof of (ii). Let x be a continuous point of σ. $\forall \eta > 0, \exists \delta > 0$ such that

$$\sigma(x) - \eta \le \sigma(y) \le \sigma(x) + \eta, \tag{4.43}$$

for any y such that $|x - y| < \delta$. For $0 < \epsilon < \delta$,

$$\begin{aligned} |\sigma(x) - \sigma_\epsilon(x)| &= |\int_{-\infty}^{\infty} [\sigma(x) - \sigma(x-y)]\chi_\epsilon(y)dy| \\ &\le \int_{-\infty}^{\infty} |[\sigma(x) - \sigma(x-y)]|\chi_\epsilon(y)dy \\ &\le \eta. \end{aligned}$$

□

Proof of (iii). Assume that $x > y$. By Schwarz inequality, we have

$$\begin{aligned} |\sigma_\epsilon(x) - \sigma_\epsilon(y)|^2 &= |\int_{-\infty}^{\infty} \sigma(x-u)\chi_\epsilon(u)du - \int_{-\infty}^{\infty} \sigma(y-u)\chi_\epsilon(u)du|^2 \\ &\le \int_{-\infty}^{\infty} |\sigma(x-u) - \sigma(y-u)|^2\chi_\epsilon(u)du \\ &\le \int_{-\infty}^{\infty} f(x-u)\chi_\epsilon(u)du - \int_{-\infty}^{\infty} f(y-u)\chi_\epsilon(u)du \\ &\le f(x+\epsilon) - f(y-\epsilon). \end{aligned}$$

By similar arguments for $y \ge x$, we have (iii). □

Here we introduce some local times which play important roles in the proof of Lemma 4.5.4. Let $L_t^a(X_\bullet^{(\epsilon)})$ be the local time at a of the process $X_\bullet^{(\epsilon)}$ such that

$$L_t^a(X_\bullet^{(\epsilon)}) := |X_t^{(\epsilon)} - a| - |X_0^{(\epsilon)} - a| - \int_0^t \mathrm{sgn}(X_s^{(\epsilon)} - a)dX_s^{(\epsilon)}, \tag{4.44}$$

(see Revuz-Yor [27] Chapter 6).
Let $Z_t^{(\epsilon_1,\epsilon_2,\theta)}$ be

$$Z_t^{(\epsilon_1,\epsilon_2,\theta)} := X_t^{(\epsilon_1)} + \theta(X_t^{(\epsilon_2)} - X_t^{(\epsilon_1)}), \quad 0 < \epsilon_1, \epsilon_2 \le 1, \quad 0 \le \theta \le 1. \tag{4.45}$$

Let $L_t^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)})$ be the local time at a of the process $Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)}$ such that

$$L_t^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)}) := |Z_t^{(\epsilon_1,\epsilon_2,\theta)} - a| - |Z_0^{(\epsilon_1,\epsilon_2,\theta)} - a| - \int_0^t \operatorname{sgn}(Z_s^{(\epsilon_1,\epsilon_2,\theta)} - a) dZ_s^{(\epsilon_1,\epsilon_2,\theta)}. \tag{4.46}$$

We have the next lemma.

Lemma 4.5.5.
(i) Let C_L be the constant such that

$$C_L := \sup_{\epsilon\in(0,1]} \sup_{a\in\mathbb{R}} \mathbf{E}[L_T^a(X_\bullet^{(\epsilon)})]. \tag{4.47}$$

Then, $C_L < \infty$ holds and it is independent of ϵ.
(ii) Let $\widetilde{C}_L$ be the constant such that

$$\widetilde{C}_L := \sup_{(\epsilon_1,\epsilon_2,\theta)\in(0,1]\times(0,1]\times[0,1]} \sup_{a\in\mathbb{R}} \mathbf{E}[L_T^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)})]. \tag{4.48}$$

Then, $\widetilde{C}_L < \infty$ holds and it is independent of $(\epsilon_1, \epsilon_2, \theta)$.

Proof of (ii). By the definition of $L_t^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)})$, we have

$$\begin{aligned} 0 &\le L_t^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)}) \\ &\le L_T^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)}) \\ &\le |Z_T^{(\epsilon_1,\epsilon_2,\theta)} - Z_0^{(\epsilon_1,\epsilon_2,\theta)}| - \int_0^T \operatorname{sgn}(Z_s^{(\epsilon_1,\epsilon_2,\theta)}) dZ_s^{(\epsilon_1,\epsilon_2,\theta)}. \end{aligned}$$

Then,

$$\begin{aligned} (L_T^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)}))^2 \le 2(&\int_0^T (\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) + \theta(\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}) - \sigma_{\epsilon_1}(X_s^{(\epsilon_1)}))dW_s)^2 \\ &+ 2(\int_0^T \operatorname{sgn}(Z_s^{(\epsilon_1,\epsilon_2,\theta)} - a)(\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) \\ &\qquad + \theta(\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}) - \sigma_{\epsilon_1}(X_s^{(\epsilon_1)}))dW_s)^2. \end{aligned} \tag{4.49}$$

Therefore, we have

$$\begin{aligned} \mathbf{E}[(L_T^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)}))^2] \le 2\mathbf{E}[&\int_0^T (\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) + \theta(\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}) - \sigma_{\epsilon_1}(X_s^{(\epsilon_1)})))^2 ds] \\ &+ 2\mathbf{E}[\int_0^T (\operatorname{sgn}(Z_s^{(\epsilon_1,\epsilon_2,\theta)}))^2 (\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) \\ &\qquad + \theta(\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}) - \sigma_{\epsilon_1}(X_s^{(\epsilon_1)})))^2 ds]. \end{aligned} \tag{4.50}$$

Using the assumption on σ, we have

$$\mathbf{E}[(L^a_T(Z^{(\epsilon_1,\epsilon_2,\theta)}_\bullet))^2] \leq 36K^2T < \infty, \tag{4.51}$$

where K is independent of $(a, \epsilon_1, \epsilon_2, \theta)$. This implies immediately (ii). $\square$

Now, the proof of (i) is similar. It is well known that we have following occupation formulas. Let g be a non-negative Borel function. We have

$$\int_0^t g(X^{(\epsilon)}_s) d < X^{(\epsilon)}_\bullet >_s = \int_{-\infty}^{\infty} g(a) L^a_t(X^{(\epsilon)}_\bullet) da, \tag{4.52}$$

and also

$$\int_0^t g(Z^{(\epsilon_1,\epsilon_2,\theta)}_s) d < Z^{(\epsilon_1,\epsilon_2,\theta)}_\bullet >_s = \int_{-\infty}^{\infty} g(a) L^a_t(Z^{(\epsilon_1,\epsilon_2,\theta)}_\bullet) da, \tag{4.53}$$

where $< X^{(\epsilon)}_\bullet >$ is the quadratic variation of the process $X^{(\epsilon)}_\bullet$ such that

$$< X^{(\epsilon)}_\bullet >_t := \int_0^t (\sigma_\epsilon(X^{(\epsilon)}_s))^2 ds. \tag{4.54}$$

And also, $< Z^{(\epsilon_1,\epsilon_2,\theta)}_\bullet >_t$ is the quadratic variation of the process $Z^{(\epsilon_1,\epsilon_2,\theta)}_\bullet$ such that

$$< Z^{(\epsilon_1,\epsilon_2,\theta)}_\bullet >_t := \int_0^t (\sigma_{\epsilon_1}(X^{(\epsilon_1)}_s) + \theta(\sigma_{\epsilon_2}(X^{(\epsilon_2)}_s) - \sigma_{\epsilon_1}(X^{(\epsilon_1)}_s)))^2 ds, \tag{4.55}$$

(see Revuz-Yor [27] Chapter 6).
Here, we state a lemma which is very useful in the rest of the paper. Let $B \subset [0, T]$ be a Borel set. Leb.B means the Lebesgue measure of the set B.

Lemma 4.5.6. *We have*
(i) *Leb.*$\{s\,;\, 0 \leq s \leq T,\, X^{(\epsilon)}_s \in D\} = 0$ *(a.s.),*
(ii) *Leb.*$\{s\,;\, 0 \leq s \leq T,\, Z^{(\epsilon_1,\epsilon_2,\theta)}_s \in D\} = 0$ *(a.s.).*

Proof of (i). Note that by Lemma 4.5.4,

$$< X^{(\epsilon)}_\bullet >_t = \int_0^t \sigma_{\epsilon_1}(X^{(\epsilon_1)}_s)^2 ds \geq k^2 t, \quad k > 0. \tag{4.56}$$

Since D is a countable set and $a \to L^a_T$ is non-negative continuous,

$$\begin{aligned} k^2 \int_0^T I_D(X^{(\epsilon)}_s) ds &\leq \int_0^T I_D(X^{(\epsilon)}_s) d < X^{(\epsilon)} >_s \\ &= \int_{-\infty}^{\infty} I_D(a) L^a_T(X^{(\epsilon)}_\bullet) da \\ &= 0 \quad (a.s.). \end{aligned}$$

This implies (i). $\square$

Proof of (ii). Note that for $k > 0$,

$$< Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)} >_t = \int_0^t ((1-\theta)\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) + \theta\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}))^2 ds \geq k^2 t. \tag{4.57}$$

The similar argument as in the proof of (i) implies (ii), see, for example, Exercise 1.32 p.237 in Revuz-Yor [27]. Closely related technique to Lemma 4.5.6 is employed in Hashimoto-Tsuchiya [9]. □

The next lemma is crucial in the proof of Theorem 4.5.2.

Lemma 4.5.7 (A priori estimates)**.** *For any α, there exits $0 < \beta \leq 1$ such that $\forall 0 < \epsilon_1, \epsilon_2 \leq \beta$,*

$$\sup_{t\in[0,T]} \mathbf{E}[|X_t^{(\epsilon_1)} - X_t^{(\epsilon_2)}|] \leq \alpha. \tag{4.58}$$

Proof. Put

$$\Delta_t^{(\epsilon_1,\epsilon_2)} := X_t^{(\epsilon_1)} - X_t^{(\epsilon_2)}.$$

Let $a_0 = 1 > a_1 > \cdots > a_{n-1} > a_n \cdots$, such that

$$\int_{a_n}^{a_{n-1}} \frac{dx}{x} = n.$$

We choose a sequence $(\rho_n)_{n\in\mathbb{N}}$ of smooth functions such that

$$supp(\rho_n) \subset [a_n, a_{n-1}]; \quad 0 \leq \rho_n(x) \leq \frac{2}{nx}; \quad \int_{a_n}^{a_{n-1}} \rho_n(x)dx = 1.$$

We define a sequence $(\psi_n)_{n\in\mathbb{N}}$ of smooth functions by

$$\psi_n(x) := \int_0^{|x|} \int_0^y \rho_n(u)dudy + a_{n-1}.$$

Then, $(\psi_n)_{n\in\mathbb{N}}$ has the following properties

$$\psi_n(x) \geq |x|; \quad |\psi_n'(x)| \leq 1; \quad \psi_n''(x) = \rho_n(|x|).$$

Moreover, we have

$$\begin{aligned} |\Delta_t^{(\epsilon_1,\epsilon_2)}| &\leq \psi_n(\Delta_t^{(\epsilon_1,\epsilon_2)}) \\ &= a_{n-1} + \int_0^t \psi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(X_s^{(\epsilon_2)})]dW_s \\ &+ \frac{1}{2}\int_0^t \psi_n''(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(X_s^{(\epsilon_2)})]^2 ds. \end{aligned}$$

Since $|\psi_n'(x)| \leq 1$, σ_{ϵ_1} and σ_{ϵ_2} are bounded, then

$$\int_0^t \psi_n'(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(X_s^{(\epsilon_2)})]dW_s \tag{4.59}$$

is a martingale with mean 0. Therefore, we have

$$\begin{aligned}\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|] &\le a_{n-1} + \frac{1}{2}\mathbf{E}[\psi_n''(\Delta_s^{(\epsilon_1,\epsilon_2)})[\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) - \sigma_{\epsilon_2}(X_s^{(\epsilon_2)})]^2 ds] \\ &\le a_{n-1} + \frac{3}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)][\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) - \sigma(X_s^{(\epsilon_1)})]^2 ds] \\ &+ \frac{3}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)[\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}) - \sigma(X_s^{(\epsilon_2)})]^2 ds] \\ &+ \frac{3}{2}\mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)[\sigma(X_s^{(\epsilon_1)}) - \sigma(X_s^{(\epsilon_2)})]^2 ds].\end{aligned}$$

Here, we define

$$J_t^{(\epsilon_1,\epsilon_2)}(1) := \mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)][\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) - \sigma(X_s^{(\epsilon_1)})]^2 ds], \tag{4.60}$$

$$J_t^{(\epsilon_1,\epsilon_2)}(2) := \mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)[\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}) - \sigma(X_s^{(\epsilon_2)})]^2 ds], \tag{4.61}$$

and

$$J_t^{(\epsilon_1,\epsilon_2)}(3) := \mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)[\sigma(X_s^{(\epsilon_1)}) - \sigma(X_s^{(\epsilon_2)})]^2 ds]. \tag{4.62}$$

Now, we remember that

$$\rho_n(|x|) \le I_{[a_n,a_{n-1}]}(|x|)\frac{2}{n|x|},$$

and

$$\int_{a_n}^{a_{n-1}} \rho_n(u)du = 1.$$

Consider $J_t^{(\epsilon_1,\epsilon_2)}(1)$ and $J_t^{(\epsilon_1,\epsilon_2)}(2)$, we have

$$J_t^{(\epsilon_1,\epsilon_2)}(1) \le \frac{2}{na_n}\mathbf{E}[\int_0^t [\sigma_{\epsilon_1}(X_s^{(\epsilon_1)}) - \sigma(X_s^{(\epsilon_1)})]^2 ds], \tag{4.63}$$

and also

$$J_t^{(\epsilon_1,\epsilon_2)}(2) \le \frac{2}{na_n}\mathbf{E}[\int_0^t [\sigma_{\epsilon_2}(X_s^{(\epsilon_2)}) - \sigma(X_s^{(\epsilon_2)})]^2 ds]. \tag{4.64}$$

Now, we consider the term $J_\bullet^{(\epsilon_1,\epsilon_2)}(3)$. By the equation (4.27), we note that

$$J_t^{(\epsilon_1,\epsilon_2)}(3) \le \mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)|f(X_s^{(\epsilon_1)}) - f(X_s^{(\epsilon_2)})|ds]. \tag{4.65}$$

Let $\widetilde{J}_t^l = J_t^{(\epsilon_1,\epsilon_2,l)}(3)$ be

$$\widetilde{J}_t^l = J_t^{(\epsilon_1,\epsilon_2,l)}(3) := \mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)|f_l(X_s^{(\epsilon_1)}) - f_l(X_s^{(\epsilon_2)})|ds].$$

By Hadamard formula;

$$f_l(x) - f_l(y) = (x-y)\int_0^1 f_l^{'}(x+\theta(y-x))d\theta, \tag{4.66}$$

we have

$$\begin{aligned}\widetilde{J}_t^l &= \mathbf{E}[\int_0^t \rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)|f_l(X_s^{(\epsilon_1)}) - f_l(X_s^{(\epsilon_2)})|ds] \\ &\leq 2\mathbf{E}[\int_0^t I_{[a_n,a_{n-1}]}(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)\frac{|f_l(X_s^{(\epsilon_1)}) - f_l(X_s^{(\epsilon_2)})|}{n|\Delta_s^{(\epsilon_1,\epsilon_2)}|}ds] \\ &= \frac{2}{n}\mathbf{E}[\int_0^t\int_0^1 f_l^{'}(X_s^{(\epsilon_1)} + \theta(X_s^{(\epsilon_2)} - X_s^{(\epsilon_1)}))d\theta ds].\end{aligned}$$

Let g be a non-negative Borel function. We have the occupation formula

$$\int_0^t g(Z_s^{(\epsilon_1,\epsilon_2,\theta)})d < Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)} >_s = \int_{-\infty}^{\infty} g(a)L_t^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)})da. \tag{4.67}$$

Since we know that

$$< Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)} >_t \geq k^2 t,$$

we obtain

$$\begin{aligned}\widetilde{J}_t^l &\leq \frac{2}{n}\mathbf{E}[\int_0^1 d\theta\int_0^t f_l^{'}(Z_s^{(\epsilon_1,\epsilon_2,\theta)})ds] \\ &\leq \frac{2}{nk^2}\mathbf{E}[\int_0^1 d\theta\int_{-\infty}^{\infty} L_t^a(Z_\bullet^{(\epsilon_1,\epsilon_2,\theta)})f_l^{'}(a)da].\end{aligned}$$

For $\theta \in [0,1]$, $0 < \epsilon_1, \epsilon_2 \leq 1$, we have

$$\begin{aligned}\widetilde{J}_t^l &\leq \frac{2}{n}\cdot\frac{\widetilde{C}_L}{k^2}\int_{-\infty}^{\infty} f_l^{'}(a)da \\ &\leq \frac{2}{n}\cdot\frac{\widetilde{C}_L}{k^2}v(f_l) \\ &\leq \frac{2}{n}\cdot\frac{\widetilde{C}_L}{k^2}v(f).\end{aligned} \tag{4.68}$$

Since

$$\lim_{l\to\infty} f_l = f(x), \quad x \notin D \tag{4.69}$$

and

$$Leb.\{s\,;\, 0\leq s\leq T,\, X_s^{(\epsilon_1)} \in D \text{ } or \text{ } X_s^{(\epsilon_2)} \in D\} = 0 \quad (a.s.), \tag{4.70}$$

we have

$$\lim_{l\to\infty}|f_l(X_s^{(\epsilon_1)}) - f_l(X_s^{(\epsilon_2)})| = |f(X_s^{(\epsilon_1)}) - f(X_s^{(\epsilon_2)})| \tag{4.71}$$

almost surely on $[0,T]\times\Omega$, w.r.t. $dt\times d\mathbf{P}$. Note that f and f_l are uniformly bounded. We have

$$\lim_{l\to\infty}\widetilde{J}^{(\epsilon_1,\epsilon_2,l)}=\mathbf{E}[\int_0^t\rho_n(|\Delta_s^{(\epsilon_1,\epsilon_2)}|)|f(X_s^{(\epsilon_1)})-f(X_s^{(\epsilon_2)})|ds]. \tag{4.72}$$

By the inequalities (4.68), we obtain

$$J_t^{(\epsilon_1,\epsilon_2)}(3)\le\frac{2}{n}\cdot\frac{\widetilde{C}_L}{k^2}v(f). \tag{4.73}$$

Finally, we will come back to estimate $\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|]$. We know that

$$\mathbf{E}[|\Delta_t^{(\epsilon_1,\epsilon_2)}|]\le a_{n-1}+\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(1)+\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(2)+\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(3).$$

By (4.63) and also by (4.64) we obtain that

$$\begin{aligned}\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(1)+\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(2)\le&\frac{3}{na_n}\mathbf{E}[\int_0^t[\sigma_{\epsilon_1}(X_s^{(\epsilon_1)})-\sigma(X_s^{(\epsilon_1)})]^2ds]\\&+\frac{3}{na_n}\mathbf{E}[\int_0^t[\sigma_{\epsilon_2}(X_s^{(\epsilon_2)})-\sigma(X_s^{(\epsilon_2)})]^2ds].\end{aligned} \tag{4.74}$$

Let $\alpha>0$ be given, choose n such that

$$a_{n-1}<\frac{\alpha}{3},$$

and also

$$\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(3)\le\frac{3}{n}\cdot\frac{\widetilde{C}_L}{k^2}v(f)<\frac{\alpha}{3}. \tag{4.75}$$

For this n, we have

$$\begin{aligned}\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(1)+\frac{3}{2}J_t^{(\epsilon_1,\epsilon_2)}(2)\le&\frac{3}{na_n}(\mathbf{E}[\int_0^t(\sigma_{\epsilon_1}(X_s^{(\epsilon_1)})-\sigma(X_s^{(\epsilon_1)}))^2ds]\\&+\mathbf{E}[\int_0^t(\sigma_{\epsilon_2}(X_s^{(\epsilon_2)})-\sigma(X_s^{(\epsilon_2)}))^2ds])\\\le&\frac{3}{na_nk^2}(\mathbf{E}[\int_{-\infty}^{\infty}(\sigma_{\epsilon_1}(a)-\sigma(a))^2L_t^a(X_\cdot^{(\epsilon_1)})da]\\&+\mathbf{E}[\int_{-\infty}^{\infty}(\sigma_{\epsilon_2}(a)-\sigma(a))^2L_t^a(X_\cdot^{(\epsilon_2)})da]).\end{aligned}$$

As is well known, the local time $L_t^a(X_\cdot^{(\epsilon_1)})$ can be written as $L_t^a(X_\cdot^{(\epsilon_1)})=L^a_{<X^{(\epsilon_1)}>_t}(B_\cdot)$, where B is called the Dambis, Dubins-Schwarz Brownian motion. See, Chapter 5 and Chapter 6 in Revuz-Yor [27]. Note that

$$<X^{(\epsilon_1)}>_t\le k^2t$$

and

$$0 \le L^a_{<X^{(\epsilon_1)}>_t}(B_\cdot) \le L^a_{k^2t}(B_\cdot),$$

hold. Then we obtain

$$\begin{aligned}\mathbf{E}[\int_{-\infty}^{\infty}(\sigma_{\epsilon_1}(a)-\sigma(a))^2 L^a_t(X^{(\epsilon_1)}_\cdot)da] &\le \mathbf{E}[\int_{-\infty}^{\infty}(\sigma_{\epsilon_1}(a)-\sigma(a))^2 L^a_{k^2t}(B_\cdot)da] \\ &\le 4k^2\cdot k^2 t.\end{aligned}$$

Since $a \mapsto L^a_{k^2t}$ is a continuous function with a compact support a.s.,

$$\lim_{\epsilon_1\to 0}\int_{-\infty}^{\infty}(\sigma_{\epsilon_1}(a)-\sigma(a))^2 L^a_{k^2t}(B_\cdot)da = 0, \tag{4.76}$$

holds a.s.. By Lebesgue convergence theorem, we obtain

$$\begin{aligned}\lim_{\epsilon_1\to 0}\mathbf{E}[\int_{-\infty}^{\infty}(\sigma_{\epsilon_1}(a)-\sigma(a))^2 L^a_t(X^{(\epsilon_1)}_\cdot)da] &\le \lim_{\epsilon_1\to 0}\mathbf{E}[\int_{-\infty}^{\infty}(\sigma_{\epsilon_1}(a)-\sigma(a))^2 L^a_{k^2t}(B_\cdot)da] \\ &= 0.\end{aligned}$$

Thus we have proved Lemma 4.5.7. □

Proof of Theorem 4.5.2. In this part, we use the Notation and some basic arguments on functional analysis employed in the section 4. We define $\epsilon_n := 1/n$ and write $X^{(n)}$ instead of $X^{(\epsilon_n)}$. Since $0 < k \le \sigma_{\epsilon_n} \le K < \infty$, there exists $C > 0$ such that for $n \in \mathbb{N}$

$$\sup_{t\in[0,T]} \mathbf{E}[|X^{(n)}_t|^4] \le C. \tag{4.77}$$

This result is called a priori estimate in the High Norm.

Lemma 4.5.8. $(X^{(n)})_{n\in\mathbb{N}}$ *is a Cauchy sequence in* N^2*.*

Proof. Let for some subsequences $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty}\sup_{0\le t\le T}\mathbf{E}[|X^{(p_n)}_t - X^{(q_n)}_t|^2] = C > 0 \tag{4.78}$$

holds. Since

$$\lim_{n\to\infty}||X^{(p_n)} - X^{(q_n)}||_1 = 0,$$

we can choose subsequences $(p'_n)_{n\in\mathbb{N}}$, $(q'_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}|X^{(p'_n)}_T - X^{(q'_n)}_T| = 0 \quad (a.s.).$$

Using

$$\sup_n \mathbf{E}[|X^{(p'_n)}_T - X^{(q'_n)}_T|^4] < \infty,$$

the family of variables $|X_T^{(p'_n)} - X_T^{(q'_n)}|^2$ is uniformly integrable. Note that $X_t^{(p'_n)} - X_t^{(q'_n)}$ $(0 \le t \le T)$ is a martingale, we have by Doob's maximal inequality

$$\begin{aligned} ||X^{(p'_n)} - X^{(q'_n)}||_2^2 &\le \mathbf{E}[\sup_{0\le t\le T} |X_t^{(p'_n)} - X_t^{(q'_n)}|^2] \\ &\le 4\mathbf{E}[|X_T^{(p'_n)} - X_T^{(q'_n)}|^2]. \end{aligned} \tag{4.79}$$

Thus, we observe that

$$\lim_{n\to\infty} ||X^{(p'_n)} - X^{(q'_n)}||_2 = 0. \tag{4.80}$$

This fact contradicts the equation (4.78). □

Therefore, the sequence $(X^{(n)})_{n\in\mathbb{N}}$ converges to some $\widetilde{X} \in N^2$. Moreover, we observe that

$$\lim_{n\to\infty} (\| X^{(n)} - \widetilde{X} \|_1 + \| X^{(n)} - \widetilde{X} \|_2) = 0. \tag{4.81}$$

Let

$$RHS_t^{(n)} := X_0 + \int_0^t \sigma_{\epsilon_n}(X_s^{(n)}) dW_s. \tag{4.82}$$

From Ito theory, we know that the strong solutions $X^{(n)}$ satisfy their respective SDEs in the following sense

$$\mathbf{E}[\sup_{0\le t\le T} |X_t^{(n)} - RHS_t^{(n)}|^2] = 0. \tag{4.83}$$

This implies

$$\| X_t^{(n)} - RHS^{(n)} \|_2^2 = 0. \tag{4.84}$$

Since $X^{(n)} \in N^2$ $(n = 1, 2, \cdots)$ is a sequence of martingales, there exists a martingale version of the process $\widetilde{X}$. Let X be a martingale version of $\widetilde{X}$. Here, let

$$RHS_t := X_0 + \int_0^t \sigma(X_s) dW_s. \tag{4.85}$$

Lemma 4.5.9.

$$Leb.\{s\,;\, 0 \le s \le T,\, X_s \in D\} = 0 \quad (a.s.) \tag{4.86}$$

holds.

Proof. Since the sequence of $< X^{(n)} >_t$

$$k^2 t \le < X^{(n)} >_t, \quad n = 1, 2, \cdots, \quad (a.s.)$$

converges to

$$< X >_t, \quad 0 \le t \le T, \quad (a.s.),$$

we have

$$k^2 t \le < X >_t, \quad 0 \le t \le T, \quad (a.s.). \tag{4.87}$$

Let $L_t^a(X)$ be the local time at a of X. We have

$$\begin{aligned} k^2 \int_0^T I_D(X_s)ds &\le \int_0^T I_D(X_s)d < X >_s \\ &= \int_{-\infty}^{\infty} L_T^a(X) I_D(a)da. \end{aligned} \tag{4.88}$$

Let

$$\widehat{C}_L := \sup_{a\in\mathbb{R}} \mathbf{E}[L_T^a(X)]. \tag{4.89}$$

By the argument employed in the proof of Lemma 4.5.5, we can prove that $\widehat{C}_L < \infty$ holds. Thus, we have

$$\begin{aligned} \mathbf{E}[\int_0^T I_D(X_s)ds] &\le \frac{1}{k^2}\mathbf{E}[\int_{-\infty}^{\infty} L_T^a(X)I_D(a)da] \\ &\le \frac{\widehat{C}_L}{k^2}\int_{-\infty}^{\infty} I_D(a)da \\ &= 0. \end{aligned}$$

This implies

$$Leb.\{s\,;\, 0 \le s \le T,\, X_s \in D\} = 0 \quad (a.s.). \tag{4.90}$$

□

Now, we will show that $RHS^{(n)}$ converges RHS in N^2. Observe using Lemma 4.5.4 (iii) that

$$\begin{aligned} &\lim_{n\to\infty} \| RHS^{(n)} - RHS \|_2^2 \\ &= \lim_{n\to\infty} \sup_{t\in[0,T]} \mathbf{E}[|\int_0^t \sigma_{\epsilon_n}(X_s^{(n)}) - \sigma(X_s)dW_s|^2] \\ &= \lim_{n\to\infty} \mathbf{E}[\int_0^T \left\{\sigma_{\epsilon_n}(X_s^{(n)}) - \sigma(X_s)\right\}^2 ds] \\ &= \lim_{n\to\infty} \mathbf{E}[\int_0^T \left\{\sigma_{\epsilon_n}(X_s^{(n)}) - \sigma_{\epsilon_n}(X_s) + \sigma_{\epsilon_n}(X_s) - \sigma(X_s)\right\}^2 ds] \\ &\le 2\lim_{n\to\infty} \mathbf{E}[\int_0^T |\sigma_{\epsilon_n}(X_s^{(n)}) - \sigma_{\epsilon_n}(X_s)|^2 ds] \\ &\qquad\qquad + 2\lim_{n\to\infty} \mathbf{E}[\int_0^T |\sigma_{\epsilon_n}(X_s) - \sigma(X_s)|^2 ds] \\ &\le 2\lim_{n\to\infty} \mathbf{E}[\int_0^T |f(X_s^{(n)} \vee X_s + \epsilon_n) - f(X_s^{(n)} \wedge X_s - \epsilon_n)|ds] \\ &\qquad\qquad + 2\lim_{n\to\infty} \mathbf{E}[\int_0^T (\sigma_{\epsilon_n}(X_s) - \sigma(X_s))^2 ds]. \end{aligned}$$

Let

$$S_1^{(n)}(T) := \mathbf{E}[\int_0^T |f(X_s^{(n)} \vee X_s + \epsilon_n) - f(X_s^{(n)} \wedge X_s - \epsilon_n)|ds], \tag{4.91}$$

and also

$$S_2^{(n)}(T) := \mathbf{E}[\int_0^T (\sigma_{\epsilon_n}(X_s) - \sigma(X_s))^2 ds]. \tag{4.92}$$

By Doob's maximal inequality, we have

$$\begin{aligned} \mathbf{E}[\sup_{0\le s\le T} |X_s^{(n)} - X_s|^2] &\le \mathbf{E}[|X_T^{(n)} - X_T|^2] \\ &\le 4||X^{(n)} - X||_2^2. \end{aligned} \tag{4.93}$$

This implies

$$\lim_{n\to\infty} \sup_{0\le s\le T} |X_s^{(n)} - X_s| = 0 \quad (a.s.). \tag{4.94}$$

By Lemma 4.5.9, we have

$$Leb.\{s\,;\, 0 \le s \le T,\, X_s \in D\} = 0 \quad (a.s.). \tag{4.95}$$

Note that $X_s^{(n)} \vee X_s + \epsilon_n$ and $X_s^{(n)} \wedge X_s - \epsilon_n$ converge to X_s. We observe that

$$\begin{aligned} Leb.\{s\,;\, 0 \le s \le T, \lim_{n\to\infty} |f(X_s^{(n)} \vee X_s + \epsilon_n) - f(X_s^{(n)} \wedge X_s - \epsilon_n)| \\ \textit{does not converge to } 0\} = 0 \quad (a.s.). \end{aligned}$$

Since f is a bounded function, we have

$$\lim_{n\to\infty} S_1^{(n)}(T) = 0. \tag{4.96}$$

For $S_2^{(n)}(T)$, we have

$$\begin{aligned} S_2^{(n)}(T) &= \mathbf{E}[\int_0^T (\sigma_{\epsilon_n}(X_s) - \sigma(X_s))^2 ds] \\ &\le \frac{1}{k^2}\mathbf{E}[\int_0^T (\sigma_{\epsilon_n}(X_s) - \sigma(X_s))^2 d < X >_s] \\ &\le \frac{1}{k^2}\mathbf{E}[\int_{-\infty}^{\infty} (\sigma_{\epsilon_n}(a) - \sigma(a))^2 L_T^a(X.) da]. \end{aligned}$$

Since $(\sigma_{\epsilon_n}(a) - \sigma(a))^2$ is uniformly bounded by $4k^2$, and $a \mapsto L_T^a(X.)$ is a continuous function with a compact support a.s., we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (\sigma_{\epsilon_n}(a) - \sigma(a))^2 L_T^a(X.) da = 0 \quad (a.s.).$$

By Lebesgue convergence theorem, we can conclude

$$\lim_{n\to\infty} S_2^{(n)}(T) = 0. \tag{4.97}$$

Therefore, we have

$$\lim_{n\to\infty} ||RHS^{(n)} - RHS||_2^2 = 0. \tag{4.98}$$

Since $X^{(n)} = RHS^{(n)}$ converges to $\widetilde{X}$ in N^2, $X^{(n)} = RHS^{(n)}$ converges to X in N^2. We have

$$||X - RHS||_2^2 = 0. \tag{4.99}$$

Note that X is a martingale having continuous paths,

$$\begin{aligned} \mathbf{E}[\sup_{0\le s\le T} |X_t - RHS_t|^2] &\le 4\mathbf{E}[|X_T - RHS_T|^2] \\ &\le 4||X - RHS||_2^2 \\ &= 0. \end{aligned} \tag{4.100}$$

Thus, X is the desired strong solution. This concludes the proof of Theorem 4.5.2. □

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