Doctoral Thesis

Asymptotic distribution of eigenvalues of the Zakharov-Shabat operators in the semiclassical limit

March 2019

Doctoral Program in Advanced Mathematics and Physics
Graduate School of Science and Engineering
Ritsumeikan University

HIROTA Koki

Doctoral Thesis Reviewed
by Ritsumeikan University

Asymptotic distribution of eigenvalues of the Zakharov-Shabat operators in the semiclassical limit

(Zakharov-Shabat 作用素の固有値の準古典極限における漸近分布)

March 2019
2019 年 3 月

Doctoral Program in Advanced Mathematics and Physics
Graduate School of Science and Engineering
Ritsumeikan University
立命館大学大学院理工学研究科
基礎理工学専攻博士課程後期課程

HIROTA Koki
広田　高輝

Supervisor: Professor FUJIIÉ Setsuro
研究指導教員: 藤家　雪朗　教授

Acknowledgments

I would like to express my deepest gratitude to my Ph.D. supervisor Professor Setsuro Fujiié for his guidance. I have been supported since 2013 when I was a fourth-year undergraduate student, and received a lot of appropriate and valuable suggestions. His advice helped me in all the time of making my study and writing of this thesis.

I am also grateful to Professor Takuya Watanabe for his kind support and useful comments to my study, as well as Professor Hiroyuki Osaka for giving me insightful comments and suggestions.

My sincere thanks also go to Professor Spyros Kamvissis of University of Crete. I was inspired to study by his innovative ideas and suggestions.

I would like to give my thanks to Professor Jens Wittsten of University of Borås for giving me a lot of advices. His constructive ideas and thoughtful comments contribute to a part of this thesis.

I am also very grateful to Professor Masafumi Yoshino of Hiroshima University for his kindness and giving me a great opportunity of a discussion about my study.

I am thankful to Professor Yuuki Ida, and our laboratory members in Ritsumeikan University for giving me a lot of support.

Finally, I would like to express my thanks to my parents in Shizuoka for their support and encouragement even though I have been far away from them for a long time.

Abstract

In this doctoral thesis, we present the results related to the eigenvalues of the Zakharov-Shabat operator in the semiclassical limit. The Zakharov-Shabat operator is a kind of the one-dimensional Dirac operator and appears as one of the Lax pair for the defocusing or focusing nonlinear Schrödinger equation. This operator is self-adjoint in the defocusing case and non-self-adjoint in the focusing case provided the potential is real-valued.

First, we consider the self-adjoint Zakharov-Shabat operator. Real eigenvalues exist when the square of the potential has a simple well. We derive the distribution for the eigenvalues by using the exact WKB method. Moreover we show that the eigenvalues stay real for a sufficiently small non-self-adjoint perturbation when the potential has some $\mathcal{PT}$-like symmetry.

Second, we consider the distribution of eigenvalues of the non-self-adjoint Zakharov-Shabat operator with a simple well type potential and a double well type potential. In the simple well case, we show that all of the eigenvalues are purely imaginary for sufficiently small semiclassical parameter. In the symmetric double well case, we observe the eigenvalue splitting with an exponential estimate described by the action between the two wells.

Contents

1 Introduction **9**

2 Exact WKB method **13**

2.1 General theory 13

2.2 Exact WKB solution to the Zakharov-Shabat equation 15

3 Eigenvalues of the self-adjoint Zakharov-Shabat operator and its $\mathcal{PT}$ symmetric perturbation **19**

3.2 Quantization condition for the eigenvalues of L_+ 22

3.3 Real eigenvalues of the $\mathcal{PT}$ symmetric perturbation 28

4 Eigenvalues of the non-self-adjoint Zakharov-Shabat operator and complex splitting **37**

4.1 Eigenvalues for a simple well potential 38

4.2 Eigenvalues for a double well potential 42

Chapter 1

Introduction

Our aim is the study of the eigenvalues of non-self-adjoint differential operators. The spectral theory for the self-adjoint differential operators has been established since the beginning of 20th century (e.g., by von Neumann in [24]) with the development of the quantum mechanics. On the contrary, the spectral theory for the non-self-adjoint operators is still under developing in spite of its importance. It is now one of the central problems in mathematics and physics. For example the pseudo-spectrum of non-self-adjoint operators is studied in connection with the numerical analysis (see for example Davies [5, 6], Trefethen [21], Dencker-Sjöstrand-Zworski [8] etc.), and the real-valuedness of eigenvalues of $\mathcal{PT}$-symmetric operators is studied for non-self adjoint Schrödinger operators (for example Bender and Boettcher [2], Boussekkine, Mecherout, Ramond and Sjöstrand [3], [23] etc.).

We are interested in the eigenvalues of the non-self-adjoint first order systems. Dencker studied the pseudospectrum for such operators in 2008, and established some basic theory in [7]. Under this background, we study the eigenvalues of the the so-called Zakharov-Shabat operators defined by

$$L_\pm = \begin{pmatrix} ih\dfrac{d}{dx} & -iA(x) \\ \pm iA(x) & -ih\dfrac{d}{dx} \end{pmatrix},$$

where h is a positive small parameter (called semiclassical parameter) and $A(x)$ is a potential. Notice that L_+ is self-adjoint and L_- is non-self-adjoint if the potential is real-valued.

These operators appear in the study of the soliton theory for the nonlinear Schrödinger equations

$$ih\frac{\partial\psi}{\partial t} + \frac{h^2}{2}\frac{\partial^2\psi}{\partial x^2} \mp |\psi|^2\psi = 0, \quad \psi = \psi(t,x)$$

with the initial condition $\psi(0,x) = A(x)$. The equation is called *defocusing* (resp. *focusing*) for the minus (resp. plus) sign of the nonlinear term. It is also known that these nonlinear equations have soliton solutions, and can be solved by using the inverse scattering method, which reduces the nonlinear equations to the direct and inverse eigenvalue (and scattering) problems of linear operators, called Lax pair. The Zakharov-Shabat operator is one of this Lax pair, and each eigenvalue corresponds to a soliton solution for the nonlinear equation.

This doctoral thesis consists of two subjects.

The first one is concerned with the self-adjoint Zakharov-Shabat operator L_+. Assume that the potential $A(x)$ has a simple well at a fixed real energy level λ_0. Adding a small complex perturbation $i\varepsilon B(x)$ to the potential and assuming $\mathcal{PT}$-symmetry, which means that A is even and B is odd, or A is odd and B is even, we prove that the eigenvalues stay real for suficiently small ε and h. This is an analogy of the result by Boussekkine, Mecherout ([3]) about the Schrödinger operator.

The second one is concerned with the non-self-adjoint Zakharov-Shabat operator L_-. The potential is supposed to decay at infinity. Then the spectrum of L_- consists of essential spectrum on $\mathbb{R}$ and eigenvalues near the imaginary axis. We give the quantization conditions of the eigenvalues near $i\lambda_0$ for a fixed $\lambda_0 > 0$ in case where the potential has a simple well at the energy level λ_0 and in case where it has double wells. In the simple well case, we deduce that the eigenvalues near $i\lambda_0$ are all purely imaginary. This is a recovery of a result by Klaus and Shaw ([16]) in a different semiclassical setting. In the double well case, we observe an interesting splitting phenomenon. If the potential A is an even function, each pair of eigenvalues near $i\lambda_0$ corresponding to the two symmetric wells split into two purely imaginary eigenvalues, and if A is an odd function, they split into two non-purely imaginary eigenvalues. In both cases, the difference of the two eigenvalues is exponentially small with respect to h and the exponential rate is given by the action of the barrier between two wells. These results also give an answer to the question by Klaus and Shaw ([17]) who conjectured the existence of non-purely imaginary eigenvalues for double lobe potentials (a stronger condition than double well potential), although h is assumed to be small enough in our results.

The main tool of our study is the exact WKB method. WKB solution is an asymptotic solution in powers of the small parameter h. The name comes from the three physicists Wentzel, Kramers and Brillouin. It is used since the beginning of 20th century for the study of quantum mechanics, or even in the 19th century by mathematicians such as Liouville or Green for the study of Sturm-Liouville problems. The essential difficulty lies in the divergence of the power series in h. In 1970's, Ecalle and Voros successed to overcome this difficulty by using a Borel resummation of this series. Since that time, WKB method (now called exact WKB method) has developed in a rigorous way. The method we use here was first discovered for Schrödinger operators by Gérard and Grigis in [12] and later extended to 2×2 systems by

Fujiié, Lasser and Nédélec in [9]. It uses another resummation to construct an exact solution. This method is reviewed in Chapter 2.

Chapter 2

Exact WKB method

2.1 General theory

We review the construction of the exact WKB solutions in [9] and apply to the 2×2 system of the first order differential equations

$$\frac{h}{i}\frac{d}{dx}\boldsymbol{u}(x) = \begin{pmatrix} 0 & F(x) \\ -G(x) & 0 \end{pmatrix} \boldsymbol{u}(x), \tag{2.1.1}$$

where $F(x)$ and $G(x)$ are holomorphic function in a simply connected complex domain $D \subset \mathbb{C}$. Let us take a point $\gamma \in D$, and define the phase function by

$$z(x;\gamma) = \int_\gamma^x \sqrt{F(t)G(t)}dt$$

Note that the zeros of $F(x)$ and $G(x)$ are referred to as *turning points* of the system (2.1.1). We write the solutions of the equation (2.1.1) in the form

$$\boldsymbol{u}^\pm(x,h;\gamma) = e^{\pm z(x;\gamma)/h} Q(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{(1\pm 1)/2} \boldsymbol{w}^\pm(x,h),$$

where $Q(x)$ is 2×2 matrix-valued functions defined by

$$Q(x) = \begin{pmatrix} H(x)^{-1} & H(x)^{-1} \\ iH(x) & -iH(x) \end{pmatrix}, \quad H(x) = \left(\frac{F(x)}{G(x)}\right)^{\frac{1}{4}}.$$

Then, $\boldsymbol{w}^\pm(x,h)$ satisfy the equation

$$\frac{d}{dx}\boldsymbol{w}^\pm + \begin{pmatrix} 0 & 0 \\ 0 & \pm 2z'/h \end{pmatrix} \boldsymbol{w}^\pm = \begin{pmatrix} 0 & c(x) \\ c(x) & 0 \end{pmatrix} \boldsymbol{w}^\pm, \quad \text{where } c(x) = \frac{H'(x)}{H(x)}. \tag{2.1.2}$$

Let $\{w_n^\pm(x)\}_{n=-1}^\infty$ be a sequence of functions satisfying the recurrence differential equations

$$\begin{cases} w_{-1}^\pm = 0, \quad w_0^\pm = 1, \\ \dfrac{d}{dx} w_{2n}^\pm = c(x) w_{2n-1}^\pm, \\ \left(\dfrac{d}{dx} \pm \dfrac{2z'}{h}\right) w_{2n-1}^\pm = c(x) w_{2n-2}^\pm, \end{cases} \tag{2.1.3}$$

the vector-valued infinite series

$$\boldsymbol{w}^\pm(x,h) = \sum_{n=0}^\infty \begin{pmatrix} w_{2n}^\pm(x,h) \\ w_{2n-1}^\pm(x,h) \end{pmatrix}$$

formally satisfies the equation (2.1.2). Each $w_n^\pm$ is uniquely determined when an initial condition is given at some point x_0 which is not a turning point,

$$w_n^\pm|_{x=x_0} = 0, \quad n \geq 1, \tag{2.1.4}$$

We denote these formal solutions constructed above by $\boldsymbol{u}(x,h;\gamma,x_0)$. The points γ and x_0 are called base points for the phase function $z(x)$ and the amplitude $\boldsymbol{w}(x,h)$. We also write the series $\sum w_{2n}^\pm$ as $w_{even}^\pm(x,h;x_0)$, and $\sum w_{2n-1}^\pm$ as $w_{odd}^\pm(x,h;x_0)$.

We recall here the following three propositions. The proofs are found in [9] or [12]. The first is about the convergence of the series.

Proposition 2.1 ([9, Lemma 3.2])**.** *Two series $w_{even}^\pm(x,h;x_0)$ and $w_{odd}^\pm(x,h;x_0)$ are absolutely convergent in a neighborhood of x_0. These solutions extend analytically to any turning point free subdomain Ω of D.*

The second property is about the Wronskian between two exact WKB solutions.

Proposition 2.2 ([9, Lemma 3.4])**.** *Let $\gamma \in D$ and $x_0, x_1 \in \Omega$. Then, the Wronskian of the exact WKB solutions $\boldsymbol{u}^\pm(x,h;\gamma,x_0)$ and $\boldsymbol{u}^\mp(x,h;\gamma,x_1)$ is given by*

$$\mathcal{W}\left(\boldsymbol{u}^\pm(x,h;\gamma,x_0), \boldsymbol{u}^\mp(x,h;\gamma,x_1)\right) = \pm 2i w_{even}^\pm(x_1,h;x_0),$$

where $\mathcal{W}(\boldsymbol{f},\boldsymbol{g}) := \det(\boldsymbol{f},\boldsymbol{g})$.

The final proposition is about the asymptotic property of the exact WKB solution. Let $x_0 \in \Omega$ be fixed.

Definition 2.3. We denote by Ω_+ (resp. Ω_-) the subset of all $x \in \Omega$ such that there exists a path in Ω from x_0 to x along which $\mathrm{Re}\, z(x;x_0)$ is strictly increasing (resp. decreasing).

Proposition 2.4 ([9, Proposition 3.3], [12, Proposition 1.2])**.** *The functions $w^+_{even}(x,h;x_0)$ and $w^+_{odd}(x,h;x_0)$ have the asymptotic expansions as $h \to 0$ in all compact subsets of Ω_+:*

$$w^+_{even}(x,h;x_0) - \sum_{n=0}^{N} w^+_{2n}(x,h;x_0) = \mathcal{O}(h^{N+1}),$$
$$w^+_{odd}(x,h;x_0) - \sum_{n=0}^{N} w^+_{2n-1}(x,h;x_0) = \mathcal{O}(h^{N+1}).$$

Similarly, The functions $w^-_{even}(x,h;x_0)$ and $w^-_{odd}(x,h;x_0)$ have the asymptotic expansions as $h \to 0$ in all compact subsets of Ω_-:

$$w^-_{even}(x,h;x_0) - \sum_{n=0}^{N} w^-_{2n}(x,h;x_0) = \mathcal{O}(h^{N+1}),$$
$$w^-_{odd}(x,h;x_0) - \sum_{n=0}^{N} w^-_{2n-1}(x,h;x_0) = \mathcal{O}(h^{N+1}).$$

It is sometimes convenient to consider the Stokes lines in order to apply these asymptotic properties.

Definition 2.5 (Stokes line)**.** The Stokes lines are defined as the level curves of the real part of the phase function $z(x;\gamma)$

$$\left\{ x \in D; \operatorname{Re} z(x;\gamma) = \operatorname{Re} \int_\gamma^x \sqrt{F(t)G(t)}dt = \text{const.} \right\}.$$

In particular, the Stokes lines passing through the point $\gamma = \gamma_0$ in D are defined by

$$\left\{ x \in D; \operatorname{Re} z(x;\gamma_0) = \operatorname{Re} \int_{\gamma_0}^x \sqrt{F(t)G(t)}dt = 0 \right\}.$$

Along a path which intersects transversally with the Stokes lines, $\text{Re}z(x)$ or $-\text{Re}z(x)$ is strictly increasing. Such a path is sometimes called *canonical*. From a turning point, which is a singularity around which the phase function is multi-valued, Stokes lines present a particular configuration. In particular, three Stokes lines emanate from a simple turning point with an asymptotic angle $2\pi/3$ between two neighboring Stokes lines.

2.2 Exact WKB solution to the Zakharov-Shabat equation

We first construct the exact WKB solution to the equation

$$L_+ \boldsymbol{u}(x) = \lambda \boldsymbol{u}(x) \tag{2.2.1}$$

for the operator

$$L_+ = \begin{pmatrix} ih\dfrac{d}{dx} & -iA(x) \\ iA(x) & -ih\dfrac{d}{dx} \end{pmatrix}.$$

Here, $A(x)$ is a real-valued function and analytic in the complex domain D. This equation is also written in the form

$$\frac{h}{i}\frac{d}{dx}\boldsymbol{u}(x) = \begin{pmatrix} -\lambda & -iA(x) \\ -iA(x) & \lambda \end{pmatrix}\boldsymbol{u}(x).$$

After a change of unknown function $\boldsymbol{u} \mapsto \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\boldsymbol{u}$, this system can be reduced to the form (2.1.1) with $F(x) = A(x) + \lambda$ and $G(x) = A(x) - \lambda$, i.e.,

$$\frac{h}{i}\frac{d}{dx}\boldsymbol{u}(x) = \begin{pmatrix} 0 & A(x)+\lambda \\ -(A(x)-\lambda) & 0 \end{pmatrix}\boldsymbol{u}(x).$$

Hence, we can construct exact WKB solutions for (2.2.1) as in the previous section

$$\boldsymbol{u}^{\pm}(x,h;\gamma,x_0) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} e^{\pm z(x;\gamma)/h} Q(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\frac{1\pm 1}{2}} \boldsymbol{w}^{\pm}(x,h;x_0), \quad (2.2.2)$$

where the phase function $z(x;\gamma)$ is defined by

$$z(x;\gamma) := \int_{\gamma}^{x} \sqrt{A(t)^2 - \lambda^2}\,dt,$$

and the 2×2 matrix function $Q(x)$ is given by

$$Q(x) = \begin{pmatrix} H(x)^{-1} & H(x)^{-1} \\ iH(x) & -iH(x) \end{pmatrix}, \quad H(x) = \left(\frac{A(x)+\lambda}{A(x)-\lambda}\right)^{\frac{1}{4}}. \quad (2.2.3)$$

The vector-valued function $\boldsymbol{w}^{\pm}(x,h;x_0)$ are the series

$$\boldsymbol{w}^{\pm}(x,h;x_0) = \begin{pmatrix} w^{\pm}_{even}(x,h;x_0) \\ w^{\pm}_{odd}(x,h;x_0) \end{pmatrix} := \sum_{n=0}^{\infty} \begin{pmatrix} w^{\pm}_{2n}(x,h) \\ w^{\pm}_{2n-1}(x,h) \end{pmatrix}$$

constructed inductively by (2.1.3) and (2.1.4).

As in the previous section, this formal solution converges in a neighborhood of x_0, and defines an analytic exact solution in Ω. The asymptotic properties (Propsition 2.4) also hold. The Wronskian formula (Proposition 2.2) slightly changes because of the matrix

$$\begin{aligned}\mathcal{W}\left(\boldsymbol{u}^{+}(x,h;\gamma,x_0), \boldsymbol{u}^{-}(x,h;\gamma,x_1)\right) &= 2i\det\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} w^{+}_{even}(x_1,h;x_0) \\ &= -4w^{+}_{even}(x_1,h;x_0).\end{aligned}$$

If we find a canonical curve along which $\mathrm{Re} z(x) = \mathrm{Re} \int^x \sqrt{A^2(t) - \lambda^2} dt$ strictly increases, we obtain from Proposition 2.4

$$\mathcal{W}\left(\boldsymbol{u}^+(x, h; \gamma, x_0), \boldsymbol{u}^-(x, h; \gamma, x_1)\right) = -4 + \mathcal{O}(h)$$

as $h \to 0$.

In the same way, we construct the exact WKB solutions to the equation

$$L_- \boldsymbol{u} = \lambda u(x) \tag{2.2.4}$$

with the operator

$$L_- = \begin{pmatrix} ih\dfrac{d}{dx} & -iA(x) \\ -iA(x) & -ih\dfrac{d}{dx} \end{pmatrix}.$$

Let us introduce the change of the unknown function $\boldsymbol{u} \to \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \boldsymbol{u}$. Then, this system reduce to

$$\frac{h}{i}\frac{d}{dx}\boldsymbol{u}(x) = \begin{pmatrix} 0 & i(-A(x) + i\lambda) \\ -i(A(x) - i\lambda) & 0 \end{pmatrix} \boldsymbol{u}(x).$$

Therefore, the exact WKB solutions to the (2.2.4) are constructed in the form

$$\boldsymbol{u}^\pm(x, h; \gamma, x_0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{\pm z(x;\gamma)/h} Q(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\frac{1\pm 1}{2}} \boldsymbol{w}^\pm(x, h; x_0), \tag{2.2.5}$$

where the phase functions $z(x; \gamma)$ is in the form

$$z(x; \gamma) = \int_\gamma^x \sqrt{-A(t)^2 - \lambda^2} dt = i \int_\gamma^x \sqrt{A(t)^2 + \lambda^2} dt \tag{2.2.6}$$

for the base point $\gamma \in \Omega$. The matrix $Q(x)$ is the same form as in (2.2.3), where we take the function $H(x)$ as

$$H(x) = \left(\frac{A(x) + i\lambda}{A(x) - i\lambda}\right)^{\frac{1}{4}}.$$

The series $\boldsymbol{w}^\pm(x, h; x_0)$ also are constructed by (2.1.3) and (2.1.4).

In this case, the Wronskian formula for the exact WKB solutions $\boldsymbol{u}^+(x, h; \gamma, x_0)$ and $\boldsymbol{u}^-(x, h; \gamma, x_1)$ with $\gamma \in D$ and $x_0, x_1 \in \Omega$ is given by

$$\mathcal{W}\left(\boldsymbol{u}^+(x, h; \gamma, x_0), \boldsymbol{u}^-(x, h; \gamma, x_1)\right) = 4iw^+_{even}(x_1, h; x_0).$$

We also have asymptotic property of this

$$\mathcal{W}\left(\boldsymbol{u}^+(x, h; \gamma, x_0), \boldsymbol{u}^-(x, h; \gamma, x_1)\right) = 4i + \mathcal{O}(h)$$

when we have a canonical curve along which $+\mathrm{Re} z(x) = +\mathrm{Im} \int^x \sqrt{A^2(t) + \lambda^2} dt$ strictly increases.

Chapter 3

Eigenvalues of the self-adjoint Zakharov-Shabat operator and its $\mathcal{PT}$ symmetric perturbation

In this chapter, we consider the eigenvalue problem

$$L_+\boldsymbol{u}(x) = \lambda\boldsymbol{u}(x) \tag{3.1.1}$$

for the first order 2×2 differential system

$$L_+ = \begin{pmatrix} ih\dfrac{d}{dx} & -iA(x) \\ iA(x) & -ih\dfrac{d}{dx} \end{pmatrix}$$

with a real-valued function $A(x)$ and a small positive parameter h.

Note that this operator is (formally) self-adjoint since $A(x)$ is real-valued. Indeed, we have

$$\langle L_+\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, L_+\boldsymbol{v}\rangle$$

for any pair $\boldsymbol{u}, \boldsymbol{v} \in C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$. Here $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product. We will see below that, anagolouly to the Schrödinger case, L_+ has real eigenvalues when $A(x)^2$ form a well at an energy level. Notice that the potential $V(x)$ in the Schrödinger operator corresponds to $A(x)^2$ in our Dirac case.

We assume the following assumption.

Assumption 1. Let $A(x)$ be a real-valued function analytic in $D := \{z \in \mathbb{C}; |\mathrm{Imz}| < \delta\}$ for some $\delta > 0$, and λ_0 a positive real number satisfying the following conditions:

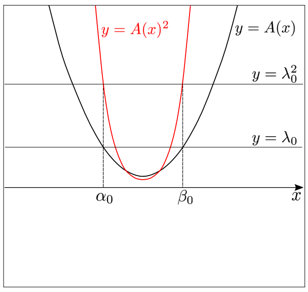
1. There exist two real numbers, α_0 and β_0 ($\alpha_0 < \beta_0$) such that $|A(x)| = \lambda_0, x \in \mathbb{R}$ if and only if $x = \alpha_0,\ \beta_0$.

2. $A'(\alpha_0)A'(\beta_0) \neq 0$.

3. $|A(x)| < \lambda_0$ for $\alpha_0 < x < \beta_0$, and $|A(x)| > \lambda_0$ for $x < \alpha_0$ and $x > \beta_0$.

4. $\liminf_{|x|\to\infty} |A(x)| > \lambda_0$.

This assumption permits two types of potentials. One is a simple well type where $A(\alpha_0) = A(\beta_0)$, and the other is monotonic type where $A(\alpha_0) = -A(\beta_0)$. In both cases, $A(x)^2$ has a simple well, see Figure 3.



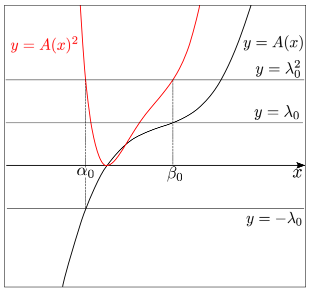


Figure 3.1: Examples of the potential $A(x)$.

Recall that the phase function of the solution is of the form

$$z(x) = \int_\gamma^x \sqrt{A(t)^2 - \lambda^2}dt.$$

Under Assumption 1, the solution is oscillating on the interval $[\alpha(\lambda), \beta(\lambda)]$ for real λ since $\sqrt{A(x)^2 - \lambda^2}$ is purely imaginary for $x \in [\alpha(\lambda), \beta(\lambda)]$, and is increasing or decreasing exponentially outside the interval. In other words, the region $\left\{x \in \mathbb{R};\ A(x)^2 - \lambda^2 < 0\right\}$ corresponds to the *classically allowed region* and $\left\{x \in \mathbb{R};\ A(x)^2 - \lambda^2 > 0\right\}$ corresponds to the *classically forbidden region* as in the case of the Schrödinger operator with a simple well potential.

We then introduce the action integral. For $\lambda \in \mathbb{R}$ close enough to λ_0, the function $\lambda^2 - A(x)^2$ has exactly two real zeros $\alpha(\lambda)$ and $\beta(\lambda)$ close to α_0 and β_0 respectively, and we define the action integral

$$I(\lambda) := \int_{\alpha(\lambda)}^{\beta(\lambda)} \sqrt{\lambda^2 - A(t)^2}dt. \tag{3.1.2}$$

This is a real analytic function of $\lambda \in \mathbb{R}$ in a neighborhood of λ_0. The quantization condition of eigenvalues near λ_0 is given by the following Bohr-Sommerfeld type formula:

Theorem 3.1. *Suppose Assumption 1. In the case* $A(\alpha_0) = A(\beta_0)$*, there exist positive constants* δ *and* h_0*, and a function* $r_+(\lambda, h)$ *bounded on* $[\lambda_0 - \delta, \lambda_0 + \delta] \times (0, h_0]$ *such that* $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ *is an eigenvalue of* L *for* $h \in (0, h_0]$ *if and only if*

$$I(\lambda) = \left(k + \frac{1}{2}\right)\pi h + h^2 r_+(\lambda, h) \tag{3.1.3}$$

holds for some integer k*. In the case* $A(\alpha_0) = -A(\beta_0)$*, there exist positive constants* δ *and* h_0*, and a function* $r_-(\lambda, h)$ *bounded on* $[\lambda_0-\delta, \lambda_0+\delta]\times(0, h_0]$ *such that* $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ *is an eigenvalue of* L *for* $h \in (0, h_0]$ *if and only if*

$$I(\lambda) = k\pi h + h^2 r_-(\lambda, h) \tag{3.1.4}$$

holds for some integer k*.*

We give the proof to this theorem in Section 3.3.

Next, we add a small complex perturbation to the potential $A(x)$:

$$A_\varepsilon(x) = A(x) + i\varepsilon B(x)$$

with a real-valued function $B(x)$ and a positive small parameter ε, and consider the eigenvalues of L_ε

$$L_\varepsilon := \begin{pmatrix} ih\dfrac{d}{dx} & -iA_\varepsilon(x) \\ iA_\varepsilon(x) & -ih\dfrac{d}{dx} \end{pmatrix}.$$

This operator is no longer self-adjoint, and eigenvalues become complex in general.

There are special cases that the complex eigenvalues of the non-self-adjoint operator become real when the operator has some symmetry. In the case of Schrödinger operator, $\mathcal{PT}$-symmetry has been expected to be an alternative to the self-adjointness in order to have real eigenvalues. In recent studies, Boussekkine and Mecherout considered in [3] the Schrödinger operator with $\mathcal{PT}$-symmetry. Let P_ε be the Schrödinger with a complex perturbation

$$P_\varepsilon := -h^2\frac{d^2}{dx^2} + V(x) + i\varepsilon W(x),$$

where $V(x)$ and $W(x)$ are real-valued function. It is said that that P_ε is $\mathcal{PT}$-symmetry if $P_{\_\varepsilon}$ is commutative for the product of two operators $\mathcal{P}$ and $\mathcal{T}$, that is, $\mathcal{PT}P_\varepsilon = P_\varepsilon\mathcal{PT}$. Here, $\mathcal{P}$ and $\mathcal{T}$ are given by

$$\mathcal{P}u_x = u(-x), \quad \mathcal{T}u(x) = \overline{u(x)},$$

(see [3, 23]). Moreover, the condition that P_ε is $\mathcal{PT}$-symmetry is equivalent to that $V(x)$ is an even function and $W(x)$ is an odd function. Under the simple well condition on $V(x)$ and λ_0, Boussekkine and Mecherout showed that the eigenvalues stay real near λ_0 for sufficiently small h and ε. After that, Boussekkine, Mecherout, Ramond and Sjöstrand studied in [23] the double well case with $\mathcal{PT}$-symmetry, and found that the eigenvalues stay real only for exponentially small ε with respect to h.

We continue in this direction and prove that a sufficiently small complex perturbation of the self-adjoint Zakharov-Shabat operator L_ε has real eigenvalues when $A(x)$ and $B(x)$ have some $\mathcal{PT}$-like symmetry in the case where $A(x)^2$ has a simple well, even though the perturbed operator L_ε is non-self-adjoint. Here assume the following symmetry condition on $A(x)$ and $B(x)$.

Assumption 2. The function $B(x)$ is real-valued, analytic and bounded on $\mathbb{R}$. $A(x)$ and $B(x)$ satisfy for $x \in \mathbb{R}$ either

$$A(x) = A(-x), \quad B(x) = -B(-x), \tag{3.1.5}$$

or

$$A(x) = -A(-x), \quad B(x) = B(-x). \tag{3.1.6}$$

The following theorem shows that the eigenvalues of L_ε are real for sufficiently small ε and h.

Theorem 3.2. *Suppose Assumption 1 and 2. Then there exist positive constants ε_0 and h_0 such that $\sigma_p(L_\varepsilon) \cap \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon_0\} \subset \mathbb{R}$ when $0 < \varepsilon \le \varepsilon_0$ and $0 < h \le h_0$.*

3.2 Quantization condition for the eigenvalues of L_+

In this section, we derive the quantization condition for the eigenvalues of L_+. In order to describe the condition, we define the exact WKB solutions to the equation (3.1.1) on D, and consider the connections of the solutions near the turning points $\alpha(\lambda)$ and $\beta(\lambda)$, simple zeros of $A(x)^2 - \lambda^2$.

Now we choose the points $\alpha(\lambda)$ and $\beta(\lambda)$ as the base point of the phase function $z(x)$ of the exact WKB solutions to the equation (3.1.1). We introduce the Stokes lines by the Definition 2.5, that is,

$$\left\{ x \in D; \operatorname{Re} z(x;\gamma) = \operatorname{Re} \int_\gamma^x \sqrt{A(t)^2 - \lambda^2} dt = \text{const.} \right\}.$$

By a simple calculation, we see that the Stokes lines emanating from $\alpha(\lambda)$ (resp. $\beta(\lambda)$) have angles of $0, 2\pi/3$, and $4\pi/3$ (resp. $\pi/3, \pi$, and $5\pi/3$). These Stokes lines separate the complex domain D into four sectors as in Figure 3.2 if δ is chosen sufficiently small.

Here, we put two branch cuts on the Stokes lines emanating from α at an angle $2\pi/3$ and from β at an angle $5\pi/3$ since the functions $\sqrt{A(x)^2 - \lambda^2}$ and $H(x)$ are multi-valued in the complex plane around the turning points α and β. We choose the branches such that $\sqrt{A(x)^2 - \lambda^2}$ and $H(x)$ are both positive when λ is real and $x > \beta(\lambda)$.

We take x_1 and x_4 as in Figure 3.2 for the base point of amplitude, and define the exact WKB solutions u_1 and u_4 so that $u_1 \in L^2(\mathbb{R}^+)$ and $u_4 \in L_2(\mathbb{R}_-)$ respectively, i.e.

$$\boldsymbol{u}_1 = \boldsymbol{u}^+(x, h; \alpha, x_1), \quad \boldsymbol{u}_4 = \boldsymbol{u}^-(x, h; \beta, x_4).$$

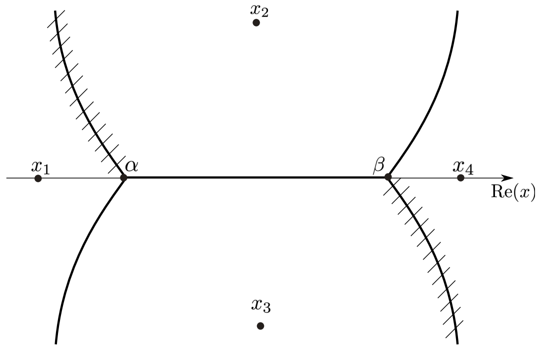


Figure 3.2: The Stokes lines and base points

Then, μ near μ_0 is an eigenvalue of the equation (3.1.1) if and only if two solutions u_1 and u_4 are linearly dependent. This means that the quantization condition is given by

$$\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_4) = 0. \tag{3.2.1}$$

To calculate this Wronskian, we also define four solutions

$$\begin{aligned} \boldsymbol{u}_2 &= \boldsymbol{u}^+(x, h; \alpha, x_2), \quad \boldsymbol{u}_3 = \boldsymbol{u}^-(x, h; \alpha, x_3), \\ \tilde{\boldsymbol{u}}_2 &= \boldsymbol{u}^+(x, h; \beta, x_2), \quad \tilde{\boldsymbol{u}}_3 = \boldsymbol{u}^-(x, h; \beta, x_3) \end{aligned}$$

with base points x_2 and x_3, and write u_1 and u_4 by the linear combination of them:

$$\boldsymbol{u}_1 = c_2 \boldsymbol{u}_2 + c_3 \boldsymbol{u}_3, \quad \boldsymbol{u}_4 = d_2 \tilde{\boldsymbol{u}}_2 + d_3 \tilde{\boldsymbol{u}}_3.$$

Then, the Wronskian of $\boldsymbol{u}_1$ and $\boldsymbol{u}_4$ is represented by

$$\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_4) = e^{2iI(\lambda)/h}(c_2 d_3 - c_3 d_2)\mathcal{W}(\boldsymbol{u}_2, \boldsymbol{u}_3),$$

where $I(\lambda)$ is action integral defined by (3.1.2). We can easily check that the solutions u_2 and u_3 are linearly independent. Therefore, we see that the condition (3.2.1) is equivalent to

$$\exp\left[\frac{2i}{h}\left(I(\lambda) + \frac{h}{2i}\log\left(-\frac{c_2 d_3}{c_3 d_2}\right)\right)\right] = -1,$$

that is,

$$I(\lambda) + \frac{h}{2i}\log\left(-\frac{c_2 d_3}{c_3 d_2}\right) = \left(k + \frac{1}{2}\right)\pi h \tag{3.2.2}$$

for some $k \in \mathbb{Z}$.

To complete the proof of Theorem 3.2.2, we calculate the connection coefficients c_j and d_k ($j, k \in \{2, 3\}$). Note that each coefficient is represented in terms of the Wronskians as

$$c_2 = \frac{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_3)}{\mathcal{W}(\boldsymbol{u}_2, \boldsymbol{u}_3)}, \quad c_3 = \frac{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_2)},$$
$$\widetilde{c}_2 = \frac{\mathcal{W}(\boldsymbol{u}_4, \widetilde{\boldsymbol{u}}_3)}{\mathcal{W}(\widetilde{\boldsymbol{u}}_2, \widetilde{\boldsymbol{u}}_3)}, \quad \widetilde{c}_3 = \frac{\mathcal{W}(\boldsymbol{u}_4, \widetilde{\boldsymbol{u}}_2)}{\mathcal{W}(\widetilde{\boldsymbol{u}}_3, \widetilde{\boldsymbol{u}}_2)}.$$

By using the Wronskian formula and the asymptotic expansion for the exact WKB solution, we can calculate the coefficients approximately in h as follows.

Lemma 3.3. *Suppose Assumption 1. The connection coefficients c_3 and $\tilde{c}_2$ satisfy*

$$c_3 = 1 + \mathcal{O}(h), \quad \tilde{c}_2 = 1 + \mathcal{O}(h)$$

as $h \to 0$ respectively.

Proof. We apply directly the Wronskian formula and obtain

$$c_2 = \frac{w^+_{even}(x_3, h; x_1)}{w^+_{even}(x_3, h; x_2)}, \quad \widetilde{c}_3 = \frac{w^+_{even}(x_4, h; x_2)}{w^+_{even}(x_3, h; x_2)}. \tag{3.2.3}$$

Let $\Gamma(x_i, x_j)$ be a canonical path from x_i to x_j for $(i, j) = (1, 3), (2, 3)$ and $(2, 4)$ as in Figure 3.3. Notice that $\mathrm{Re}z(x, \cdot)$ increases in the direction indicated in Figure 3.3.

According to Proposition 2.4, we obtain, as $h \to 0$,

$$w^+_{even}(x_3, h; x_1) = 1 + \mathcal{O}(h), \quad w^+_{even}(x_3, h; x_2) = 1 + \mathcal{O}(h),$$
$$w^+_{even}(x_4, h; x_2) = 1 + \mathcal{O}(h),$$

and hence, from (3.6), it follows that the argument holds. □

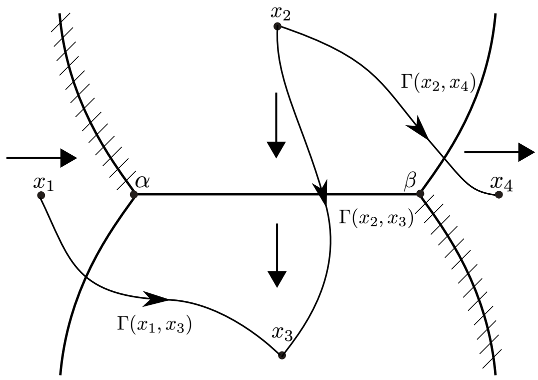


Figure 3.3: Examples of $\Gamma(x_i, x_j)$. Arrows indicate directions along which $\mathrm{Re}z(x)$ increases.

In order to calculate c_2 and d_3, it is convenient to introduce some notation defined by [10].

Definition 3.4. We denote $\mathcal{R}(\gamma, \theta)$ by the operator acting through rotation around γ by θ so that, e.g., $\mathcal{R}(0, \theta)x = e^{i\theta}x$. We also denote $\hat{x}$ by the point obtained when rotating x anticlockwise once around γ , i.e.,

$$\hat{x} = \mathcal{R}(\gamma, 2\pi)x.$$

The point over x that is obtained when rotating x clockwise once around γ will be denoted by $\check{y}_\ell$, i.e.,

$$\check{x} = \mathcal{R}(\gamma, -2\pi)x.$$

Lemma 3.5. *Let γ be the simple zeros of $A(x)^2 - \lambda^2$, and $\hat{x}$ and $\check{x}$ be defined as above. The exact WKB solutions $\boldsymbol{u}^{\pm}(x, h; \gamma, x_0)$ satisfy in the case $A(\gamma) = \lambda$,*

$$\boldsymbol{u}^{\pm}(x, h; \gamma, x_0) = -i\boldsymbol{u}^{\mp}(\hat{x}, h; \gamma, \hat{x_0}) = i\boldsymbol{u}^{\mp}(\check{x}, h; \gamma, \check{x_0}).$$

On the other hand, in the case $A(\gamma) = -\lambda$,

$$\boldsymbol{u}^{\pm}(x, h; \gamma, x_0) = i\boldsymbol{u}^{\mp}(\hat{x}, h; \gamma, \hat{x_0}) = -i\boldsymbol{u}^{\mp}(\check{x}, h; \gamma, \check{x_0}).$$

Proof. We rewrite the solutions $\boldsymbol{u}^{\pm}(x, h; \gamma, x_0)$ in terms of $\hat{x}$. Since $A(x) \pm \lambda$ is analytic and $x = e^{-2\pi i}\hat{x}$, we have

$$A(x) \pm \lambda = e^{-2\pi i}(A(\hat{x}) \pm \lambda).$$

This implies that when $A(\alpha) = \lambda$,

$$\sqrt{A(x) - \lambda} = \sqrt{e^{-2\pi i}\,(A(\hat{x}) - \lambda)} = -\sqrt{A(\hat{x}) - \lambda}.$$

On the other hand, when $A(\alpha) = -\lambda$,

$$\sqrt{A(x)+\lambda} = \sqrt{e^{-2\pi i}\left(A(\hat{x})+\lambda\right)} = -\sqrt{A(\hat{x})+\lambda}.$$

That is, there is a sign change

$$+z(x;\gamma) = -z(\hat{x};\gamma) \tag{3.2.4}$$

in both cases $A(\gamma) = \pm\lambda$.

The representation of the function $H(x)$ is different between the cases $A(\gamma) = \lambda$ and $A(\gamma) = -\lambda$. When $A(\gamma) = \lambda$,

$$H(x) = \left(\frac{A(x)+\lambda}{A(x)-\lambda}\right)^{\frac{1}{4}} = \left(\frac{1}{e^{-2\pi i}}\frac{A(\hat{x})+\lambda}{A(\hat{x})-\lambda}\right)^{\frac{1}{4}} = e^{\frac{\pi i}{2}}\left(\frac{A(\hat{x})+\lambda}{A(\hat{x})-\lambda}\right)^{\frac{1}{4}}.$$

On the contrary, when $A(\alpha) = -\lambda$,

$$H(x) = \left(\frac{A(x)+\lambda}{A(x)-\lambda}\right)^{\frac{1}{4}} = \left(e^{-2\pi i}\frac{A(\hat{x})+\lambda}{A(\hat{x})-\lambda}\right)^{\frac{1}{4}} = e^{-\frac{\pi i}{2}}\left(\frac{A(\hat{x})+\lambda}{A(\hat{x})-\lambda}\right)^{\frac{1}{4}}.$$

Namely, $H(x) = \mp iH(\hat{x})$ holds when $A(\gamma) = \pm\lambda$ respectively. In addition, this leads to

$$Q(x) = \pm iQ(\hat{x})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2.5}$$

Since $H(x) = \mp iH(\hat{x})$, we also see that

$$c(x) = \frac{H'(x)}{H(x)} = \frac{H'(\hat{x})}{H(\hat{x})} = c(\hat{x}) \tag{3.2.6}$$

in both cases $A(\gamma) = \pm\lambda$. According to the sign of $z(x)$ changes and $c(x) = c(\hat{x})$, we find from the recurrence equation (2.1.3) that

$$\boldsymbol{w}^{+}(x,h;x_2) = \boldsymbol{w}^{-}(\hat{x},h;\hat{x_2}). \tag{3.2.7}$$

From (3.2.4), (3.2.7), and (3.2.5), we obtain

$$\boldsymbol{u}(x,h;\gamma,x_0) = i\boldsymbol{u}^{\mp}(\hat{x},h;\gamma,\hat{x_0}),$$

in the case $A(\gamma) = \lambda$, and

$$\boldsymbol{u}(x,h;\gamma,x_0) = -i\boldsymbol{u}^{\mp}(\hat{x},h;\gamma,\hat{x_0}),$$

in the case $A(\gamma) = -\lambda$.

We can also obtain the representation in terms of $\check{x}$ in the same way as above. □

Lemma 3.6. *Suppose Assumption 1. In the case $A^{'}(\alpha_0)A^{'}(\beta_0) > 0$, the connection coefficients c_3 and d_2 satisfy*

$$c_3\tilde{c}_2 = -1 + \mathcal{O}(h)$$

as $h \to 0$. On the other hand, in the case $A^{'}(\alpha_0)A^{'}(\beta_0) < 0$,

$$c_3\tilde{c}_2 = 1 + \mathcal{O}(h)$$

as $h \to 0$.

Proof. We calculate the Wronskians $\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and $\mathcal{W}(\boldsymbol{u}_4, \widetilde{\boldsymbol{u}_3})$. In order to calculate them, we use the representations in terms of $\hat{x}$ or $\check{x}$ to the solutions so that we can apply the asymptotic properties. For example, we rewrite $\boldsymbol{u}_2$ in terms of $\check{x}$, i.e.,

$$\boldsymbol{u}_2 = \pm i\boldsymbol{u}^-(\check{x}, h; \alpha, \check{x_2}),$$

and obtain

$$\begin{aligned}\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2) &= \mathcal{W}\left(\boldsymbol{u}^+(x, h; \alpha, x_1), \pm i\boldsymbol{u}^-(\check{x}, h; \alpha, \check{x_2})\right) \\ &= \mp 4iw^+_{even}(\check{x_2}, h; x_1).\end{aligned}$$

in the case $A(\alpha) = \pm\lambda$ respectively.

In the same way, we also represent $\widetilde{\boldsymbol{u}_3}$ in terms of $\check{x}$ to calculate $\mathcal{W}(\boldsymbol{u}_4, \widetilde{\boldsymbol{u}}_3)$. When $A(\beta) = \pm\lambda$, $\widetilde{\boldsymbol{u}_3}$ is rewritten respectively as

$$\widetilde{\boldsymbol{u}_3} = \pm i\boldsymbol{u}^+(\check{x}, h; \beta, \check{x_3}).$$

Therefore, $\mathcal{W}(\boldsymbol{u}_4, \widetilde{\boldsymbol{u}}_3)$ is calculated as

$$\begin{aligned}\mathcal{W}(\boldsymbol{u}_4, \widetilde{\boldsymbol{u}}_3) &= -\mathcal{W}\left(\pm i\boldsymbol{u}^+(\check{x}, h; \beta, \check{x_3}), \boldsymbol{u}^-(x, h; \beta, x_4)\right) \\ &= \pm 4iw^+_{even}(x_4, h; \check{x_3}).\end{aligned}$$

We can find canonical paths $\Gamma(x_1, \check{x_2})$ and $\Gamma(\check{x_3}, x_4)$ along which $\mathrm{Re}z(x, \cdot)$ is strictly increasing, and obtain by Proipodition 2.4 that as $h \to 0$,

$$w^+_{even}(\check{x_2}, h; x_1) = 1 + \mathcal{O}(h), \quad w^+_{even}(x_4, h; \check{x_3}) = 1 + \mathcal{O}(h).$$

As consequence, we obtain that when $A(\alpha)A(\beta) > 0$,

$$c_3\tilde{c}_2 = i^2 \frac{w^+_{even}(\check{x_2}, h; x_1)}{w^+_{even}(x_3, h; x_2)} \frac{w^+_{even}(x_4, h; \check{x_3})}{w^+_{even}(x_3, h; x_2)} = -1 + \mathcal{O}(h)$$

and when $A(\alpha)A(\beta) < 0$,

$$c_3\tilde{c}_2 = -i^2 \frac{w^+_{even}(\check{x_2}, h; x_1)}{w^+_{even}(x_3, h; x_2)} \frac{w^+_{even}(x_4, h; \check{x_3})}{w^+_{even}(x_3, h; x_2)} = 1 + \mathcal{O}(h)$$

as $h \to 0$. □

From Lemma 3.3 and 3.6, we see that in both cases $A(\alpha) = \pm A(\beta)$,

$$\frac{c_2\widetilde{c}_3}{\widetilde{c}_2 c_3} = \mp 1 + \mathcal{O}(h)$$

as $h \to 0$. This means that the term $\log\left(-\frac{c_2\widetilde{c}_3}{\widetilde{c}_2 c_3}\right)$ in (3.2.2) satisfies in the case $A(\alpha) = A(\beta)$,

$$\log\left(-\frac{c_2\widetilde{c}_3}{\widetilde{c}_2 c_3}\right) = \mathcal{O}(h),$$

and in the case $A(\alpha) = -A(\beta)$,

$$\log\left(-\frac{c_2\widetilde{c}_3}{\widetilde{c}_2 c_3}\right) = \frac{\pi i}{2} + \mathcal{O}(h),$$

as $h \to 0$ respectively. In conclusion, the quantization condition for eigenvalues λ is given by

$$I(\lambda) = \left(k + \frac{1}{2}\right)\pi h + \mathcal{O}(h^2) \qquad \text{as } h \to 0,$$

in the case $A(\alpha)A(\beta) > 0$ and

$$I(\lambda) = k\pi h + \mathcal{O}(h^2) \qquad \text{as } h \to 0,$$

in the case $A(\alpha) = -A(\beta)$.

3.3 Real eigenvalues of the $\mathcal{PT}$ symmetric perturbation

From this section, we consider the eigenvalue problem:

$$L_\varepsilon \boldsymbol{u}(x) = \lambda \boldsymbol{u}(x), \quad \lambda \in \mathbb{C} \tag{3.3.1}$$

for the operator

$$L_\varepsilon := \begin{pmatrix} ih\dfrac{d}{dx} & -iA_\varepsilon(x) \\ iA_\varepsilon(x) & -ih\dfrac{d}{dx} \end{pmatrix}$$

with $A_\varepsilon(x) = A(x) + i\varepsilon B(x)$ and $\varepsilon > 0$. Here we only assume that $A(x)$ satisfies Assumption 1 and that $B(x)$ is real-valued, analytic and bounded on $\mathbb{R}$.

Let $D(\lambda_0, \varepsilon_0) = \{x \in \mathbb{C}; |x - \lambda_0| < \varepsilon_0\}$ for a positive ε_0. Under Assumption 1, for all $\lambda \in D(\lambda_0, \varepsilon_0)$ and $\varepsilon \in (0, \varepsilon_0]$, there exist zeros of $A_\varepsilon(x)^2 - \lambda^2$,

$\alpha(\lambda, \varepsilon)$ and $\beta(\lambda, \varepsilon)$ such that $\alpha(\lambda_0, 0) = \alpha_0$ and $\beta(\lambda_0, 0) = \beta_0$. We simply write them as α_ε and β_ε, and define the integral $I(\lambda, \varepsilon)$ by

$$I(\lambda, \varepsilon) := \int_{\alpha_\varepsilon(\lambda)}^{\beta_\varepsilon(\lambda)} \sqrt{\lambda^2 - A_\varepsilon(t)^2} dt. \tag{3.3.2}$$

It is holomorphic in regard to λ and ε, and we choose a branch of the square root $\sqrt{\lambda^2 - A(x)^2}$ such that it is positive on $[\alpha(\lambda), \beta(\lambda)]$ when $\varepsilon = 0$ and λ near λ_0 is real.

First we consider the quantization condition for the eigenvalues of L_ε. The exact WKB solutions to the equation (3.3.1) are given by replacing $A(x)$ with $A_\varepsilon(x)$ in (2.2.5), and we denote these solutions by $\boldsymbol{u}^{\pm}(x, h, \varepsilon; \gamma, x_0)$.

We choose α_ε and β_ε for the base points of the phase function $z(x, \varepsilon)$ and draw the stokes lines. They pass through the points α_ε and β_ε for small but non zero ε are drown in Figure 3.4. In this case, the Stokes lines change continuously with respect to ε from the case of $\varepsilon = 0$, since α_ε, β_ε and $z(x, \varepsilon)$ are continuous with respect to ε.

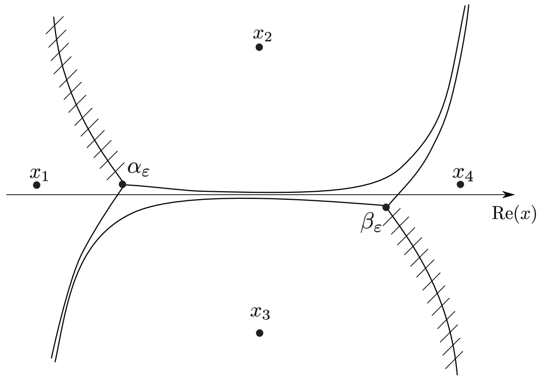


Figure 3.4: The Stokes lines for a sufficiently small ε.

Here, we assume that ε is sufficiently small. We take base points as in Figure 3.4, and define the exact WKB solutions as

$$\begin{cases} \boldsymbol{u}_1 = \boldsymbol{u}^{+}(x, h, \varepsilon; \alpha_\varepsilon, x_1), \\ \boldsymbol{u}_2 = \boldsymbol{u}^{+}(x, h, \varepsilon; \alpha_\varepsilon, x_2), \quad \widetilde{\boldsymbol{u}}_2 = \boldsymbol{u}^{+}(x, h, \varepsilon; \beta_\varepsilon, x_2), \\ \boldsymbol{u}_3 = \boldsymbol{u}^{-}(x, h, \varepsilon; \alpha_\varepsilon, x_3), \quad \widetilde{\boldsymbol{u}}_3 = \boldsymbol{u}^{-}(x, h, \varepsilon; \beta_\varepsilon, x_3), \\ \boldsymbol{u}_4 = \boldsymbol{u}^{-}(x, h, \varepsilon; \beta_\varepsilon, x_4), \end{cases}$$

where we put branch cuts and choose the branches of $\sqrt{A_\varepsilon(x)^2 - \lambda^2}$ and $H(x, \varepsilon)$ such that they are both positive when λ is real, $\varepsilon = 0$ and $x > \beta(\lambda)$. Then, we can derive the quantization conditions for eigenvalues of L_ε just in the same way as in the previous section.

Lemma 3.7. *Suppose Assumption 1, and let $B(x)$ be real-valued, analytic and bounded on $\mathbb{R}$. In the case $A(\alpha_0) = A(\beta_0)$, there exist positive constants ε_0 and h_0, and a function $r_+(\lambda, \varepsilon, h)$ bounded on $D(\lambda_0, \varepsilon_0) \times (0, \varepsilon_0] \times (0, h_0]$ such that $\lambda \in D(\lambda_0, \varepsilon_0)$ is an eigenvalue of L_ε for $\varepsilon \in (0, \varepsilon_0]$ and $h \in (0, h_0]$ if and only if*

$$I(\lambda, \varepsilon) = \left(k + \frac{1}{2}\right)\pi h + h^2 r_+(\lambda, \varepsilon, h) \tag{3.3.3}$$

holds for some integer k. In the case $A(\alpha_0) = -A(\beta_0)$, there exist positive constants ε_0 and h_0, and a function $r_-(\lambda, \varepsilon, h)$ bounded on $D(\lambda_0, \varepsilon_0) \times (0, \varepsilon_0] \times (0, h_0]$ such that $\lambda \in D(\lambda_0, \varepsilon_0)$ is an eigenvalue of L_ε for $\varepsilon \in (0, \varepsilon_0]$ and $h \in (0, h_0]$ if and only if

$$I(\lambda, \varepsilon) = k\pi h + h^2 r_-(\lambda, \varepsilon, h) \tag{3.3.4}$$

holds for some integer k.

Let us define a function $S(\lambda, \varepsilon, h)$ as

$$S(\lambda, \varepsilon, h) = I(\lambda, \varepsilon) + h^2 r_\pm(\lambda, \varepsilon, h).$$

This is a map from neighborhood of λ_0 to one of $S(\lambda_0, \varepsilon, h)$. In addition, H is locally injective near λ_0 for sufficiently small ε and h, since $\frac{\partial I}{\partial \lambda}(\lambda_0, 0) \neq 0$. To prove Theorem 3.2, it is enough to show that $S(\lambda, \varepsilon, h)$ is real-valued for $\lambda \in \mathbb{R}$.

We now assume Assumption 2 for $A_\varepsilon(x)$. Since $A(x)$ and $B(x)$ are real-valued analytic functions on $\mathbb{R}$, one has

$$\overline{A(x)} = A(\bar{x}), \quad \overline{B(x)} = B(\bar{x}),$$

for complex x near the real axis, and hence

$$\overline{A_\varepsilon(x)} = \overline{A(x)} - i\varepsilon\overline{B(x)} = A(\bar{x}) - i\varepsilon B(\bar{x}).$$

Then thanks to the Assumption 2, we see that, in the case (3.1.5),

$$\overline{A_\varepsilon(x)} = A_\varepsilon(-\bar{x}), \tag{3.3.5}$$

and in the case (3.1.6),

$$\overline{A_\varepsilon(x)} = -A_\varepsilon(-\bar{x}). \tag{3.3.6}$$

This relation results in a symmetry of the action integral $I(\lambda, \varepsilon)$.

Lemma 3.8. *Under Assumption 2, the action integral $I(\lambda, \varepsilon)$ is equal to the complex conjugate of $I(\overline{\lambda}, \varepsilon)$:*

$$I(\lambda, \varepsilon) = \overline{I(\overline{\lambda}, \varepsilon)}.$$

Proof. From the relation (3.3.5) and (3.3.6), we have

$$\overline{A_\varepsilon(x)^2} = A_\varepsilon(-\overline{x})^2,$$

that is,

$$\overline{A_\varepsilon(x)^2 - \lambda^2} = A_\varepsilon(-\overline{x})^2 - \overline{\lambda^2}.$$

By substituting $x = \alpha_\varepsilon$ and $x = \beta_\varepsilon$, we see that

$$A_\varepsilon(-\overline{\alpha_\varepsilon})^2 - \overline{\lambda^2} = 0 \quad \text{and} \quad A_\varepsilon(-\overline{\beta_\varepsilon})^2 - \overline{\lambda^2} = 0$$

respectively. In addition, since $A(x)$ is even or odd,

$$-\overline{\alpha_\varepsilon(\lambda)} \to -\overline{\alpha}(\lambda) = \beta(\lambda) \quad \text{and} \quad -\overline{\beta_\varepsilon(\lambda)} \to -\overline{\beta}(\lambda) = \alpha(\lambda)$$

as $\varepsilon \to 0$. Therefore, $I(\overline{\lambda}, \varepsilon)$ is represented by

$$I(\overline{\lambda}, \varepsilon) = \int_{-\overline{\beta_\varepsilon(\lambda)}}^{-\overline{\alpha_\varepsilon(\lambda)}} \sqrt{\overline{\lambda}^2 - A_\varepsilon(t)^2}dt.$$

We take the complex conjugate of this, and obtain that

$$\overline{I(\overline{\lambda}, \varepsilon)} = \int_{-\overline{\beta_\varepsilon(\lambda)}}^{-\overline{\alpha_\varepsilon(\lambda)}} \sqrt{\lambda^2 - \overline{A_\varepsilon(t)}^2}d\overline{t}.$$

Then, we change the variable from t to $-\overline{t}$,

$$\overline{I(\overline{\lambda}, \varepsilon)} = \int_{\alpha_\varepsilon(\lambda)}^{\beta_\varepsilon(\lambda)} \sqrt{\lambda^2 - A_\varepsilon(t)^2}dt.$$

This is just the action integral. □

To complete the proof of Theorem 3.2, we should show that the functions $r_+(\lambda, \varepsilon, h)$ and $r_-(\lambda, \varepsilon, h)$ are real-valued for real λ . Recall that these functions are represented by

$$r_\pm(\lambda, \varepsilon, h) = \frac{1}{2ih} \log \frac{c_2\widetilde{c_3}}{c_3\widetilde{c_2}},$$

where c_j and $\widetilde{c_j}$ (j=2,3) are connection coefficients:

$$\boldsymbol{u}_1 = c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3, \quad \boldsymbol{u}_4 = \widetilde{c_2}\boldsymbol{u}_2 + \widetilde{c_3}\boldsymbol{u}_3.$$

For this, we consider the symmetry property of the exact WKB solutions with respect to complex conjugation. Here, we specify the dependence on λ of the exact WKB solutions and write $\boldsymbol{u}^\pm(x, h, \varepsilon; \gamma, x_0; \lambda)$. In addition, let

y_1 and y_2 be small enough positive numbers, and take the base points of the amplitude as

$$x_1 = -y_1, \quad x_4 = y_1, \quad \text{and} \quad x_2 = iy_2, \quad x_3 = -iy_2$$

for the solutions $\boldsymbol{u}_j$ and $\widetilde{\boldsymbol{u}}_j$.

We also define the exact WKB solution to the equation $L\boldsymbol{v} = \overline{\lambda}\boldsymbol{v}$ by

$$\begin{cases} \boldsymbol{v}_1 = \boldsymbol{u}^+(x, h, \varepsilon; -\overline{\beta_\varepsilon}, -y_1; \overline{\lambda}), & \\ \boldsymbol{v}_2 = \boldsymbol{u}^+(x, h, \varepsilon; -\overline{\beta_\varepsilon}, iy_2; \overline{\lambda}), & \widetilde{\boldsymbol{v}}_2 = \boldsymbol{u}^+(x, h, \varepsilon; -\overline{\alpha_\varepsilon}, iy_2; \overline{\lambda}), \\ \boldsymbol{v}_3 = \boldsymbol{u}^-(x, h, \varepsilon; -\overline{\beta_\varepsilon}, -iy_2; \overline{\lambda}), & \widetilde{\boldsymbol{v}}_3 = \boldsymbol{u}^-(x, h, \varepsilon; -\overline{\alpha_\varepsilon}, -iy_2; \overline{\lambda}), \\ \boldsymbol{v}_4 = \boldsymbol{u}^-(x, h, \varepsilon; -\overline{\alpha_\varepsilon}, y_1; \overline{\lambda}). & \end{cases}$$

Under Assumption 2, those solutions can be represented by the complex conjugate of $\boldsymbol{u}_j$ and $\widetilde{\boldsymbol{u}}_j$, according to the following symmetry relation.

Lemma 3.9. *Suppose Assumption 2. When $A_\varepsilon(x) = \pm\overline{A_\varepsilon(\overline{x})}$, the exact WKB solutions $\boldsymbol{v_1}$ and $\boldsymbol{v_4}$ satisfy*

$$\boldsymbol{v}_1 = C_\pm \overline{\boldsymbol{u}_4}|_{x=-\overline{x}}, \qquad \boldsymbol{v}_4 = \pm C_\pm \overline{\boldsymbol{u}_1}|_{x=-\overline{x}},$$

where $C_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $C_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. More precisely, the exact WKB solutions $\boldsymbol{u}^\pm(x, h, \varepsilon; \gamma, x_0; \lambda)$ satisfy in the case (3.1.5),

$$\boldsymbol{u}^\pm(x, h, \varepsilon; \gamma, x_0; \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{\boldsymbol{u}^\mp(-\overline{x}, h, \varepsilon; -\overline{\gamma}, -\overline{x_0}; \overline{\lambda})}, \tag{3.3.7}$$

and in the case (3.1.6),

$$\boldsymbol{u}^\pm(x, h, \varepsilon; \gamma, x_0; \lambda) = \mp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\boldsymbol{u}^\mp(-\overline{x}, h, \varepsilon; -\overline{\gamma}, -\overline{x_0}; \overline{\lambda})}. \tag{3.3.8}$$

Proof. Let us assume (3.1.5), i.e. $A_\varepsilon(x) = \overline{A_\varepsilon(-\overline{x})}$, and we show the relation (3.3.7) and (3.3.8). It is easy to check

$$z(x, \varepsilon; \gamma; \lambda) = -\overline{z(-\overline{x}, \varepsilon; -\overline{\gamma}; \overline{\lambda})}, \tag{3.3.9}$$

and

$$H(x, \varepsilon; \lambda) = \overline{H(-\overline{x}, \varepsilon; \overline{\lambda})} \tag{3.3.10}$$

by taking the complex conjugate and changing the variable x to $-\overline{x}$. Hence, the matrix function $Q(x, \varepsilon; \lambda)$ satisfies

$$Q(x, \varepsilon; \lambda) = \overline{Q(-\overline{x}, \varepsilon; \overline{\lambda})} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3.11}$$

To prove the first relation, we should show that

$$\boldsymbol{w}^{\pm}(x,h,\varepsilon;x_0;\lambda)=\overline{\boldsymbol{w}^{\mp}(-\overline{x},h,\varepsilon;-\overline{x_0};\overline{\lambda})}. \tag{3.3.12}$$

Recall that the series $\boldsymbol{w}^{\pm}(x,h,\varepsilon;x_0;\lambda)$ are constructed by

$$\begin{cases} w_{-1}^{\pm}\equiv 0, \quad w_0^{\pm}\equiv 1, \\ \dfrac{d}{dx}w_{2n}^{\pm}(x,h,\varepsilon;\lambda)=\dfrac{H'(x,\varepsilon;\lambda)}{H(x,\varepsilon;\lambda)}w_{2n-1}^{\pm}(x,h,\varepsilon;\lambda), \\ \left(\dfrac{d}{dx}\pm\dfrac{2}{h}z'(x,\varepsilon;\lambda)\right)w_{2n-1}^{\pm}(x,h,\varepsilon;\lambda)=\dfrac{H'(x,\varepsilon;\lambda)}{H(x,\varepsilon;\lambda)}w_{2n-2}^{\pm}(x,h,\varepsilon;\lambda), \end{cases}$$

with the initial conditions

$$w_n^{\pm}|_{x=x_0}=0 \quad (n\geq 1).$$

From (3.3.9) and (3.3.10), we have

$$z'(x,\varepsilon;\lambda)=\overline{z'(-\overline{x},\varepsilon;\overline{\lambda})}$$

and

$$H'(x,\varepsilon;\lambda)=-\overline{H'(-\overline{x},\varepsilon;\overline{\lambda})}.$$

Therefore, we see that $\boldsymbol{w}^{\pm}(x,h,\varepsilon;x_0;\lambda)$ also satisfy

$$\begin{cases} \dfrac{d}{dx}\overline{w_{2n}^{\pm}(-\overline{x},h,\varepsilon;\overline{\lambda})}=\dfrac{H'(x,\varepsilon;\lambda)}{H(x,\varepsilon;\lambda)}\overline{w_{2n-1}^{\pm}(-\overline{x},h,\varepsilon;\overline{\lambda})}, \\ \left(\dfrac{d}{dx}\mp\dfrac{2}{h}z'(x,\varepsilon;\lambda)\right)\overline{w_{2n-1}^{\pm}(-\overline{x},h,\varepsilon;\overline{\lambda})}=\dfrac{H'(x,\varepsilon;\lambda)}{H(x,\varepsilon;\lambda)}\overline{w_{2n-2}^{\pm}(-\overline{x},h,\varepsilon;\overline{\lambda})}, \end{cases}$$

by taking the complex conjugate, and changing the variable x to $-\overline{x}$ and λ to $\overline{\lambda}$ for the recurrence equations. This means that $\boldsymbol{w}^{\pm}$ satisfy (3.3.12). According to (3.3.9), (3.3.11), and (3.3.12), we obtain the first relation

$$\boldsymbol{u}^{\pm}(x,h,\varepsilon;\gamma,x_0;\lambda)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\overline{\boldsymbol{u}^{\mp}(-\overline{x},h,\varepsilon;-\overline{\gamma},-\overline{x_0};\overline{\lambda})}.$$

Similary, if $\overline{A_\varepsilon(-\overline{x})}=-A_\varepsilon(x)$, then we find that

$$z(x,\varepsilon;\gamma;\lambda)=-\overline{z(-\overline{x},\varepsilon;-\overline{\gamma};\overline{\lambda})},$$

and

$$H(x,\varepsilon;\lambda)=\overline{H(-\overline{x},\varepsilon;\overline{\lambda})^{-1}}.$$

Moreover, $H(x,\varepsilon;\lambda)$ also satisfies

$$\frac{H'(x,\varepsilon;\lambda)}{H(x,\varepsilon;\lambda)} = \overline{\left(\frac{H'(-\overline{x},\varepsilon;\overline{\lambda})}{H(-\overline{x},\varepsilon;\overline{\lambda})}\right)}.$$

From this property, we obtain

$$Q(x,\varepsilon;\lambda) = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{Q(-\overline{x},\varepsilon;\overline{\lambda})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$\boldsymbol{w}^{\pm}(x,h,\varepsilon;x_0;\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{\boldsymbol{w}^{\mp}(-\overline{x},h,\varepsilon;-\overline{x_0};\overline{\lambda})}.$$

Therefore, the second relation also follows. □

Lemma 3.10. *Suppose Assumption 2. When* $A_\varepsilon(x) = \pm\overline{A_\varepsilon(\overline{x})}$*, the exact WKB solutions* $\boldsymbol{v_j}$ *and* $\widetilde{\boldsymbol{v_j}}$ $(j=2,3)$ *satisfy*

$$\boldsymbol{v}_j = iC_\pm \overline{\widetilde{\boldsymbol{u}}_j}|_{x=-\overline{x}}, \quad \widetilde{\boldsymbol{v}}_j = \pm iC_\pm \overline{\boldsymbol{u}_j}|_{x=-\overline{x}},$$

More precisely, the exact WKB solutions $\boldsymbol{u}^{\pm}(x,h,\varepsilon;\gamma,x_0;\lambda)$ *satisfy in the case* (3.1.5)*,*

$$\boldsymbol{u}^{\pm}(x,h,\varepsilon;\gamma,x_0;\lambda) = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{\boldsymbol{u}^{\pm}(-\overline{x},h,\varepsilon;-\overline{\gamma},-\overline{x_0};\overline{\lambda})},$$

and in the case (3.1.6)*,*

$$\boldsymbol{u}^{\pm}(x,h,\varepsilon;\gamma,x_0;\lambda) = \pm i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\boldsymbol{u}^{\pm}(-\overline{x},h,\varepsilon;-\overline{\gamma},-\overline{x_0};\overline{\lambda})}.$$

The proof is almost the same as in Lemma 3.9, except the changes for multi-valued functions $z(x,\varepsilon;\gamma;\lambda)$:

$$z(x,\varepsilon;\gamma;\lambda) = \overline{z(-\overline{x},\varepsilon;-\overline{\gamma};\overline{\lambda})},$$

and $H(x,\varepsilon;\lambda)$:

$$H(x,\varepsilon;\lambda) = \begin{cases} -i\overline{H(-\overline{x},\varepsilon;\overline{\lambda})} & \text{when } A_\varepsilon(x) = \overline{A_\varepsilon(-\overline{x})}, \\ -i\overline{H(-\overline{x},\varepsilon;\overline{\lambda})^{-1}} & \text{when } A_\varepsilon(x) = -\overline{A_\varepsilon(-\overline{x})}. \end{cases}$$

Proof of Theorem 3.2. We represent $\boldsymbol{v_1}$ and $\boldsymbol{v_4}$ as

$$\boldsymbol{v}_1 = d_2\boldsymbol{v}_2 + d_3\boldsymbol{v}_3, \quad \boldsymbol{v}_4 = \widetilde{d}_2\widetilde{\boldsymbol{v}}_2 + \widetilde{d}_3\widetilde{\boldsymbol{v}}_3.$$

Then, the quantization condition for the complex conjugate of λ near λ_0 is also given in the form of Lemma 3.7. In particular, the functions $r_\pm(\overline{\lambda}, \varepsilon, h)$ are described by the connection coefficients d_j and $\widetilde{d}_j$:

$$r_\pm(\overline{\lambda}, \varepsilon, h) = \frac{1}{2ih} \log \frac{d_2 \widetilde{d}_3}{d_3 \widetilde{d}_2}.$$

For those coefficients, we have

$$d_2 \widetilde{d}_3 = \frac{\mathcal{W}(\boldsymbol{v}_1, \boldsymbol{v}_3)\mathcal{W}(\widetilde{\boldsymbol{v}}_2, \boldsymbol{v}_4)}{\mathcal{W}(\boldsymbol{v}_2, \boldsymbol{v}_3)\mathcal{W}(\widetilde{\boldsymbol{v}}_2, \widetilde{\boldsymbol{v}}_3)}, \quad d_3 \widetilde{d}_2 = \frac{\mathcal{W}(\boldsymbol{v}_2, \boldsymbol{v}_1)\mathcal{W}(\boldsymbol{v}_4, \widetilde{\boldsymbol{v}}_3)}{\mathcal{W}(\boldsymbol{v}_2, \boldsymbol{v}_3)\mathcal{W}(\widetilde{\boldsymbol{v}}_2, \widetilde{\boldsymbol{v}}_3)}.$$

According to lemma 3.9 and 3.10, we find that in the case $A_\varepsilon(x) = \overline{A_\varepsilon(-\overline{x})}$,

$$\begin{aligned} \mathcal{W}(\boldsymbol{v}_1, \boldsymbol{v}_3) &= -i\overline{\mathcal{W}(\boldsymbol{u}_4, \widetilde{\boldsymbol{u}}_3)}, \quad \mathcal{W}(\widetilde{\boldsymbol{v}_2}, \boldsymbol{v}_4) = -i\overline{\mathcal{W}(\boldsymbol{u}_2, \boldsymbol{u}_1)}, \\ \mathcal{W}(\boldsymbol{v}_2, \boldsymbol{v}_1) &= -i\overline{\mathcal{W}(\widetilde{\boldsymbol{u}}_2, \boldsymbol{u}_4)}, \quad \mathcal{W}(\boldsymbol{v}_4, \widetilde{\boldsymbol{v}_3}) = -i\overline{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_3)}, \end{aligned}$$

and

$$\mathcal{W}(\boldsymbol{v}_2, \boldsymbol{v}_3) = \overline{\mathcal{W}(\widetilde{\boldsymbol{u}}_2, \widetilde{\boldsymbol{u}}_3)}, \quad \mathcal{W}(\widetilde{\boldsymbol{v}}_2, \widetilde{\boldsymbol{v}}_3) = \overline{\mathcal{W}(\boldsymbol{u}_2, \boldsymbol{u}_3)}.$$

That is, we obtain

$$d_2 \widetilde{d}_3 = -\overline{c_3 \widetilde{c}_2}, \quad d_3 \widetilde{d}_2 = -\overline{c_2 \widetilde{c}_3}.$$

On the other hand, in the case $A_\varepsilon(x) = -\overline{A_\varepsilon(-\overline{x})}$,

$$d_2 \widetilde{d}_3 = \overline{c_3 \widetilde{c}_2}, \quad d_3 \widetilde{d}_2 = \overline{c_2 \widetilde{c}_3}$$

In any case, those connection coefficients have the symmetry relation

$$\frac{d_2 \widetilde{d}_3}{d_3 \widetilde{d}_2} = \overline{\left(\frac{c_3 \widetilde{c}_2}{c_2 \widetilde{c}_3} \right)}$$

under Assumption 2. This implies that the functions $r_\pm(\lambda, \varepsilon, h)$ satisfy

$$r_\pm(\lambda, \varepsilon, h) = \overline{r_\pm(\overline{\lambda}, \varepsilon, h)}.$$

From this relation and Lemma 3.8, $S(\lambda, \varepsilon, h)$ becomes real-valued for real λ, and it completes the proof of Theorem 3.2. □

Chapter 4

Eigenvalues of the non-self-adjoint Zakharov-Shabat operator and complex splitting

We consider the eigenvalue problem

$$L_-\boldsymbol{u}(x) = \lambda \boldsymbol{u}(x), \quad \lambda \in \mathbb{C} \tag{4.0.1}$$

for the first order 2×2 differential system on the line:

$$L_- = \begin{pmatrix} ih\dfrac{d}{dx} & -iA(x) \\ -iA(x) & -ih\dfrac{d}{dx} \end{pmatrix},$$

where h is a positive small parameter and $A(x)$ is a real-valued potential. Although this operator is non-self-adjoint, all of complex eigenvalues are purely imaginary when $A(x)$ is in some cases. In 1974, Satsuma and Yajima studied this operator with $A(x) = A_0 \operatorname{sech}(x),\ A_0 > 0$ and computed explicitly all the eigenvalues in [20]. They solved the equation (4.0.1) by reducing it to the hypergeometric equation, and found that if $h = h_N = A_0/N$, there are exactly N purely imaginary eigenvalues λ_k given by

$$\lambda_k = ih_N\left(N - k - \frac{1}{2}\right), \qquad k = 0, \ldots, N-1.$$

In the non-semiclassical case (i.e. $h = 1$), Klaus and Shaw studied the eigenvalues of this operator with a "single-lobe" potential, and found that all the eigenvalues are purely imaginary (see [16]). Here, it is said that $A(x)$

is single-lobe if it is a non-negative, piecewise smooth, bounded L^1 function on the real line which is nondecreasing for $x < 0$ and nonincreasing for $x > 0$. In addition, Klaus and Shaw also show that there are no purely imaginary eigenvalues when $A(x)$ is L^1 and odd function ([17]).

Suppose that $A(x)$ is smooth and decays at $x = \pm\infty$. As mentioned in [18, 19], the continuous spectrum of the operator L_- consists of the whole of the real axis.On the other hand, the discrete spectrum consists of eigenvalues appearing near the imaginary axis and accumulating on the axis as $h \to 0$. For example, in the case $A(x) = \mathrm{sech}(2x)$, there exist eigenvalues accumulating on the interval $[0, i]$ as $h \to 0$ (see [4]).

We observe that fact from the numerical range of the operator L_-. Let $L_-(x, \xi)$ be the semiclassical symbol of the operator L_-, that is,

$$L_-(x,\xi) = \begin{pmatrix} \xi & -iA(x) \\ -iA(x) & -\xi \end{pmatrix},$$

where I is 2×2 identity matrix. We also denote the closure of the set of eigenvalues of L_- by

$$\Sigma(L_-) = \overline{\{\lambda \in \mathbb{C} : \exists (x,\xi) \in T^*\mathbb{R},\ \det(L_-(x,\xi) - \lambda I) = 0\}}.$$

Since $\det(L_-(x,\xi) - \lambda I) = 0$ is equivalent to $\lambda = \pm\sqrt{\xi^2 - A(x)^2}$, one see that

$$\Sigma(L_-) = \mathbb{R} \cup i[-A_0, A_0],$$

where $A_0 := \max_{x \in \mathbb{R}} |A(x)|$. According to [10, Proposition 2.1], the operator L_- has discrete spectrum accumulating on $\Sigma(L_-)$.

In this chapter, we consider the quantization condition of eigenvalues in a complex neighborhood of a fixed $\lambda_0 \in i(-A_0, A_0)$ in two cases. One is that there are two simple turning points (i.e., zeros of $A(x)^2 + \lambda^2$), and the other is that there are four simple turning points. We also mention some properties for eigenvalues of L_- from the quantization condition.

All of the results in this Chapter are in collaboration with J.Wittsten.

4.1 Eigenvalues for a simple well potential

In this section, we consider the case where $A(x)$ satisfies the following condition.

Assumption 3. The function $A(x)$ is real-valued on $\mathbb{R}$ and analytic in $D := \{z \in \mathbb{C}; |\mathrm{Im} z| < \delta\}$ for some $\delta > 0$, and $\mu_0 (= -i\lambda_0)$ a positive real number satisfying the following conditions:

1. There exist two real numbers, α_0 and β_0 ($\alpha_0 < \beta_0$) such that $|A(x)| = \mu_0, x \in \mathbb{R}$ if and only if $x = \alpha_0,\ \beta_0$.

2. $A'(\alpha_0) > 0$ and $A'(\beta_0) < 0$.

3. $A(x) > \mu_0$ for $\alpha_0 < x < \beta_0$, and $|A(x)| < \mu_0$ for $x < \alpha_0$ and $x > \beta_0$.

4. $\limsup_{|x|\to\infty} |A(x)| < \mu_0$.

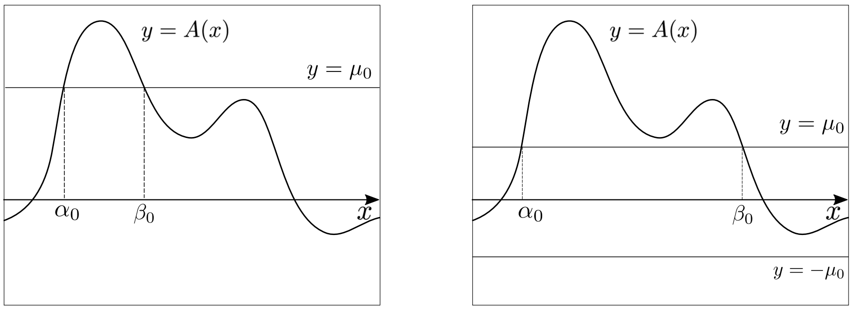


Figure 4.1: Examples of the potential $A(x)$ and μ_0.

On the interval $[\alpha(\mu), \beta(\mu)]$ and $\mu \in \mathbb{R}$, the solutions are oscillating since the phase function

$$z(x) = i\int^x \sqrt{A(t)^2 + \lambda^2}dt = i\int^x \sqrt{A(t)^2 - \mu^2}dt$$

is purely imaginary for $x \in [\alpha(\mu), \beta(\mu)]$. This implies the interval $[\alpha(\mu), \beta(\mu)]$ corresponds to the classically allowed region in the Schrödinger case, and we say this type of potential *simple well*.

Under this condition, we will show that the eigenvalues are purely imaginary for sufficiently small h. In order to prove that, we derive the quantization condition for the eigenvalues. For $\mu \in D(\mu_0, \varepsilon) = \{\mu \in \mathbb{C};\ |\mu - \mu_0| < \varepsilon\}$, there exist two simple zeros of $A(x)^2 - \mu^2$ close to α_0 and β_0, and we denote them by $\alpha(\mu)$ and $\beta(\mu)$. We define the action integral by

$$I(\mu) = \int_{\alpha(\mu)}^{\beta(\mu)} \sqrt{A(x)^2 - \mu^2}dx.$$

Then, the quantization condition are described as follows.

Theorem 4.1. *Suppose Assumption 3. Then, there exist positive constants δ and h_0, and a function $r(\mu, h)$ bounded on $D(\mu_0, \delta) \times (0, h_0]$ such that $\lambda = i\mu$, $\mu \in D(\mu_0, \delta)$ is an eigenvalue of L_- for $h \in (0, h_0]$ if and only if*

$$I(\lambda) = \left(k + \frac{1}{2}\right)\pi h + h^2 r(\mu, h) \tag{4.1.1}$$

holds for some integer k.

We can prove this in the same way as in Section 2.3. The Stokes lines are defined by

$$\left\{x \in D;\ \operatorname{Re} z(x;\gamma) = \operatorname{Im} \int_\gamma^x \sqrt{A(t)^2-\mu^2}dt = \text{const.}\right\}.$$

In particular, the Stokes lines passing through the turning point $\alpha(\mu)$ (resp. $\beta(\mu)$) have angles $0, 2\pi/3$, and $4\pi/3$ (resp. $\pi/3, \pi$ and $5\pi/3$) under Assumption 3, as in Figure 3.2. Here, we also put a branch cut on the Stokes lines emanating from α at angle $2\pi/3$ and another branch cut on the Stokes lines emanating from β at angle $5\pi/3$, and choose the branches such that $\sqrt{A(x)^2-\mu^2}$ and $H(x)$ are both positive when $\alpha(\mu) < x < \beta(\mu)$ for real μ.

Proof of Theorem 4.1. We take base points α and β for the phase function, and x_1, x_2, x_3, x_4, for the amplitude function in each sector as in Figure 3.2, and define the following 6 exact WKB solutions:

$$\begin{cases} \boldsymbol{u}_1 = \boldsymbol{u}^+(x,h;\alpha,x_1), \\ \boldsymbol{u}_2 = \boldsymbol{u}^+(x,h;\alpha,x_2), \quad \tilde{\boldsymbol{u}}_2 = \boldsymbol{u}^+(x,h;\beta,x_2), \\ \boldsymbol{u}_3 = \boldsymbol{u}^-(x,h;\alpha,x_3), \quad \tilde{\boldsymbol{u}}_3 = \boldsymbol{u}^-(x,h;\beta,x_3), \\ \boldsymbol{u}_4 = \boldsymbol{u}^-(x,h;\beta,x_4). \end{cases}$$

We represent $\boldsymbol{u}_1$ as a linear combination of $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$:

$$\boldsymbol{u}_1 = c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3,$$

and $\boldsymbol{u}_4$ as

$$\boldsymbol{u}_4 = \tilde{c}_2\tilde{\boldsymbol{u}}_2 + \tilde{c}_3\tilde{\boldsymbol{u}}_3.$$

Then, the quantization condition is equivalent to $\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_4) = 0$, i.e.,

$$I(\mu) + \frac{h}{2i}\log\left(-\frac{c_2\tilde{c}_3}{\tilde{c}_3 c_2}\right) = \left(k + \frac{1}{2}\right)\pi h.$$

By the Wronskian formula, each connection coefficient is represented as

$$c_2 = \frac{w^+_{even}(x_3,h;x_1)}{w^+_{even}(x_3,h;x_2)}, \qquad \tilde{c}_3 = \frac{w^+_{even}(x_4,h;x_2)}{w^+_{even}(x_3,h;x_2)},$$
$$c_3 = i\frac{w^+_{even}(\check{x}_2,h;x_1)}{w^+_{even}(x_3,h;x_2)}, \qquad \tilde{c}_2 = i\frac{w^+_{even}(x_4,h;\check{x}_3)}{w^+_{even}(x_3,h;x_2)},$$

where, $\check{x}_2$ and $\check{x}_3$ are defined in accordance with Definition 3.4. Hence,

$$c_2 = 1 + \mathcal{O}(h), \qquad \tilde{c}_3 = 1 + \mathcal{O}(h)$$
$$c_3 = i + \mathcal{O}(h), \qquad \tilde{c}_2 = i + \mathcal{O}(h)$$

as $h \to 0$, which completes the proof. □

From the quantization condition, we have the following.

Theorem 4.2. *Suppose Assumption 3. Then there exist positive constants h_0 and δ such that $\sigma_p(L_-)\cap\{\lambda\in\mathbb{C}:|\lambda-\lambda_0|<\delta\}\subset i\mathbb{R}$ when $0<h\le h_0$.*

Proof. It is enough to show that the integral $I(\mu)$ and the function $r(\mu,h)$ is real-valued for real μ. We can easily check that $\overline{I(\overline{\mu})}=I(\mu)$ in the same way as the proof of Lemma 3.8. In order to prove $\overline{r(\overline{\mu},h)}=r(\mu,h)$, let us define six exact WKB solutions v_1,v_2,v_3,v_4 and $\tilde{v}_2,\tilde{v}_3$ by

$$\begin{cases} \boldsymbol{v}_1=\boldsymbol{u}^+(x,h;\bar{\alpha},\bar{x}_1;\bar{\mu}), \\ \boldsymbol{v}_2=\boldsymbol{u}^+(x,h;\bar{\alpha},\bar{x}_3;\bar{\mu}), \quad \tilde{\boldsymbol{v}}_2=\boldsymbol{u}^+(x,h;\bar{\beta},\bar{x}_3;\bar{\mu}), \\ \boldsymbol{v}_3=\boldsymbol{u}^-(x,h;\bar{\alpha},\bar{x}_2;\bar{\mu}), \quad \tilde{\boldsymbol{v}}_3=\boldsymbol{u}^-(x,h;\bar{\beta},\bar{x}_2;\bar{\mu}), \\ \boldsymbol{v}_4=\boldsymbol{u}^-(x,h;\bar{\beta},\bar{x}_4;\bar{\mu}). \end{cases}$$

Then, we can express $r(\overline{\mu},h)$ by

$$r(\bar{\mu},h)=\frac{1}{2ih}\log\left(-\frac{\mathcal{W}(\boldsymbol{v}_1,\boldsymbol{v}_3)\mathcal{W}(\tilde{\boldsymbol{v}}_2,\boldsymbol{v}_4)}{\mathcal{W}(\boldsymbol{v}_2,\boldsymbol{v}_1)\mathcal{W}(\boldsymbol{v}_4,\tilde{\boldsymbol{v}}_3)}\right).$$

Here, these solutions have some symmetry as follows.

Lemma 4.3. *Suppose Assumption 3. The exact WKB solutions $\boldsymbol{v_j}$ and $\tilde{\boldsymbol{v}}_j$ $(j=2,3)$ satisfy*

$$\boldsymbol{v}_2=\overline{\boldsymbol{u}_3}|_{x=\overline{x}}, \quad \boldsymbol{v}_3=\overline{\boldsymbol{u}_2}|_{x=\overline{x}},$$
$$\tilde{\boldsymbol{v}}_2=\overline{\tilde{\boldsymbol{u}}_3}|_{x=\overline{x}}, \quad \tilde{\boldsymbol{v}}_3=\overline{\tilde{\boldsymbol{u}}_2}|_{x=\overline{x}}.$$

Similarly, $\boldsymbol{v_1}$ and $\boldsymbol{v}_4$ also have

$$\boldsymbol{v}_1=i\overline{\boldsymbol{u}_1}|_{x=\overline{x}}, \quad \boldsymbol{v}_4=i\overline{\boldsymbol{u}_4}|_{x=\overline{x}}.$$

More precisely, the exact WKB solutions $\boldsymbol{u}^\pm(x,h;\gamma,x_0;\mu)$ satisfy

$$\boldsymbol{u}^\pm(x,h;\gamma,x_0;\mu)=\overline{\boldsymbol{u}^\mp(\overline{x},h;\overline{\gamma},\overline{x_0};\overline{\mu})},$$

Proof. It is easy to check that

$$\overline{z(\overline{x};\overline{\gamma};\overline{\mu})}=-z(x;\gamma;\mu)$$

by a simple calculation. We can also check that $H(\overline{x};\overline{\mu})=H(x;\mu)$ and

$$\overline{Q(\overline{x};\overline{\mu})}=\begin{pmatrix}\overline{H(\overline{x};\overline{\mu})^{-1}} & \overline{H(\overline{x};\overline{\mu})^{-1}} \\ \overline{iH(\overline{x};\overline{\mu})} & \overline{-iH(\overline{x};\overline{\mu})}\end{pmatrix}=Q(x;\mu)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Since $\overline{z^{'}(\overline{x};\overline{\gamma};\overline{\mu})}=-z^{'}(x;\gamma;\mu)$ and $\overline{c(\overline{x};\overline{\mu})}=c(x;\mu)$, we have

$$\overline{\boldsymbol{w}^\pm(\overline{x},h;\overline{x_0};\overline{\mu})}=\boldsymbol{w}^\mp(x,h;x_0;\mu).$$

from the recurrence equations. Therefore, the statement holds. ☐

According to Lemma 4.3, we see that $\overline{r(\overline{\mu}, h)} = r(\mu, h)$. Indeed, we have

$$\overline{\mathcal{W}(\boldsymbol{v}_1, \boldsymbol{v}_3)\mathcal{W}(\tilde{\boldsymbol{v}}_2, \boldsymbol{v}_4)} = -\mathcal{W}(\boldsymbol{u}_2, \boldsymbol{u}_1)\mathcal{W}(\boldsymbol{u}_4, \tilde{\boldsymbol{u}}_3)$$

and

$$\overline{\mathcal{W}(\boldsymbol{v}_2, \boldsymbol{v}_1)\mathcal{W}(\boldsymbol{v}_4, \tilde{\boldsymbol{v}}_3)} = -\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_3)\mathcal{W}(\tilde{\boldsymbol{u}}_2, \boldsymbol{u}_4).$$

Since $\overline{I(\mu)} = I(\mu)$ and $\overline{r(\mu, h)} = r(\mu, h)$, we can show that $\mu = -i\lambda$ near μ_0 is real by adapting same argument in the proof of Theorem 3.2. $\square$

4.2 Eigenvalues for a double well potential

We consider the eigenvalues near an energy level $\lambda_0 = i\mu_0$, $0 < \mu_0 < A_0$ such that $A(x)^2$ has double wells. More precisely, we assume

Assumption 4. The function $A(x)$ is a real-valued on $\mathbb{R}$ and analytic in $D := \{z \in \mathbb{C}; |\mathrm{Imz}| < \delta\}$ for some $\delta > 0$, and $\mu_0 (= -i\lambda_0)$ a positive real number satisfying the following conditions:

1. There exist four real numbers, $\alpha_{0,\ell},\ \beta_{0,\ell}$ and $\beta_{0,r},\ \alpha_{0,r}$ $(\alpha_{0,\ell} < \beta_{0,\ell} < \beta_{0,r} < \alpha_{0,r})$ such that $|A(x)| = \mu_0, x \in \mathbb{R}$ if and only if $x = \alpha_{0,\bullet},\ \beta_{0,\bullet}$, where $\bullet = \ell,\ r$.
2. $A^{'}(\alpha_{0,\bullet})A^{'}(\beta_{0,\bullet}) \neq 0$.
3. $|A(x)| > \mu_0$ for $x \in (\alpha_{0,\ell}, \beta_{0,\ell}) \cup (\beta_{0,r}, \alpha_{0,r})$, otherwise $|A(x)| < \mu_0$.
4. $\limsup_{|x|\to\infty} |A(x)| < \mu_0$.

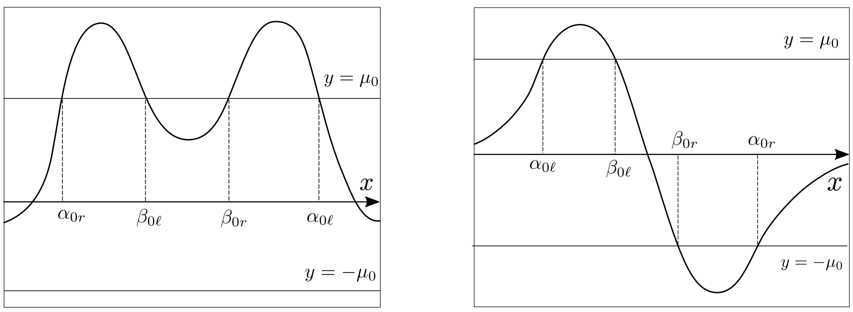


Figure 4.2: Examples of the potential $A(x)$ and μ_0 under Assumption 4.

Under this assumption, there are two types of potentials according to the sign of the product $A^{'}(\beta_{0,\ell})A^{'}(\beta_{0,r})$ is positive or negative.

In this section, we aim to describe the quantization condition under these cases, and estimate the eigenvalue splitting when $A(x)$ is even or odd function.

For $\mu \in D(\mu_0, \varepsilon)$ with a small $\varepsilon > 0$, there exist exactly four roots of $A(x)^2 - \mu^2 = 0$, and we simply write them as $\alpha_\ell(\mu), \beta_\ell(\mu), \beta_r(\mu), \alpha_r(\mu)$. Let us define the action integrals by

$$I_\ell(\mu) = \int_{\alpha_\ell(\mu)}^{\beta_\ell(\mu)} \sqrt{A(x)^2 - \mu^2}dx, \quad I_r(\mu) = \int_{\beta_r(\mu)}^{\alpha_r(\mu)} \sqrt{A(x)^2 - \mu^2}dx,$$

and

$$J(\mu) = \int_{\beta_\ell(\mu)}^{\beta_r(\mu)} \sqrt{\mu^2 - A(x)^2}dx.$$

Notice that these functions are positive for positive μ.

Under Assumption 4, the Stokes lines emanating from turning points are drawn as Figure 4.3. Let us take real numbers x_ℓ and x_r such that $x_\ell < \mathrm{Re}\alpha_\ell$

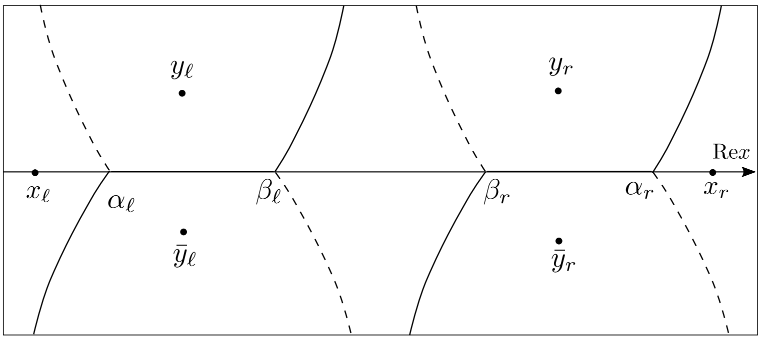


Figure 4.3: The Stokes lines emanating from four turning points.

and $\mathrm{Re}\beta_r < x_r$. We define two exact WKB solutions $\tilde{\boldsymbol{u}}_\ell$ and $\tilde{\boldsymbol{u}}_r$

$$\tilde{\boldsymbol{u}}_\ell = \boldsymbol{u}^+(x; \alpha_\ell, x_\ell; \mu), \quad \tilde{\boldsymbol{u}}_r = \boldsymbol{u}^-(x; \alpha_r, x_r; \mu),$$

and express them as linear combinations of the solutions $u^+(x; \beta_\bullet, y_\bullet, \mu)$ and $u^-(x; \beta_\bullet, \bar{y}_\bullet, \mu)$, $\bullet = \ell, r$, i.e.,

$$\tilde{\boldsymbol{u}}_\ell = c_{11}\boldsymbol{u}^+(x; \beta_\ell, y_\ell, \mu) + c_{12}\boldsymbol{u}^-(x; \beta_\ell, \bar{y}_\ell, \mu), \tag{4.2.1}$$
$$\tilde{\boldsymbol{u}}_r = c_{21}\boldsymbol{u}^+(x; \beta_r, y_r, \mu) + c_{22}\boldsymbol{u}^-(x; \beta_r, \bar{y}_r, \mu). \tag{4.2.2}$$

Remark 4.4. The coefficients c_{jk} are represented as

$$c_{11} = e^{iI_\ell/h}\tau_\ell^+, \qquad c_{12} = ie^{-iI_\ell/h}\tau_\ell^-,$$
$$c_{21} = \pm ie^{-iI_r/h}\tau_r^+, \qquad c_{22} = e^{iI_r/h}\tau_r^-,$$

with

$$\tau_\ell^+ = \frac{w_{even}^+(\bar{y}_\ell; x_\ell)}{w_{even}^+(\bar{y}_\ell; y_\ell)}, \qquad \tau_\ell^- = \frac{w_{even}^+(\check{y}_\ell; x_\ell)}{w_{even}^+(\bar{y}_\ell; y_\ell)}$$
$$\tau_r^+ = \frac{w_{even}^+(x_r; \check{\bar{y}}_r)}{w_{even}^+(\bar{y}_r; y_r)}, \qquad \tau_r^- = \frac{w_{even}^+(x_r; y_r)}{w_{\rm even}^+(\bar{y}_r; y_r)},$$

and all of $\tau_{l,r}^\pm$ are $1 + \mathcal{O}(h)$ as $h \to 0$. The sign of c_{21} is minus (resp. plus) when $A(\beta_\ell) = A(\beta_r)$ (resp. $A(\beta_\ell) = -A(\beta_r)$).

We set the solution $\boldsymbol{u}_\ell$ by

$$\boldsymbol{u}_\ell = \frac{1}{2}\left(\tilde{\boldsymbol{u}}_\ell - i\tilde{\boldsymbol{u}}_\ell^*\right).$$

Here, the asterisk means to take a complex conjugate and change variables from x to $\bar{x}$ and μ to $\bar{\mu}$. (For example, $c^*(\mu) = \overline{c(\bar{\mu})}$, $f^*(x,\mu) = \overline{f(\bar{x},\bar{\mu})}$.) More precisely, $\boldsymbol{u}_\ell$ are written as

$$\boldsymbol{u}_\ell = e^{iI_\ell/h}\tau_\ell \boldsymbol{u}^+(x;\beta_\ell, y_\ell;\mu) - ie^{-iI_\ell/h}\tau_\ell^* \boldsymbol{u}^-(x;\beta_\ell,\bar{y}_\ell;\mu),$$

where $\tau_\ell = \frac{1}{2}\left(\tau_\ell^+ + \left(\tau_\ell^-\right)^*\right)$, and notice that $\tau_\ell = 1 + \mathcal{O}(h)$ as $h \to 0$.

We also set $\boldsymbol{u}_r$ in the same way. Because of the difference of the sign of c_{21}, we take $\boldsymbol{u}_r = \frac{1}{2}\left(\boldsymbol{u}_r - i\boldsymbol{u}_r^*\right)$ in the case $A(\beta_\ell) = A(\beta_r)$, or $\boldsymbol{u}_r = \frac{1}{2}\left(\boldsymbol{u}_r + i\boldsymbol{u}_r^*\right)$ in the case $A(\beta_\ell) = -A(\beta_r)$, i.e.,

$$\boldsymbol{u}_r = \mp ie^{-iI_r/h}\tau_r^* \boldsymbol{u}^+(x;\beta_r, y_r;\mu) + e^{iI_r/h}\tau_r \boldsymbol{u}^-(x;\beta_r,\bar{y}_r;\mu),$$

where $\tau_r = \frac{1}{2}(\tau_r^- + (\tau_r^+)^*)$. Remark that we have $\boldsymbol{u}_\ell(x) \to 0$ as $x \to -\infty$ and $\boldsymbol{u}_r(x) \to 0$ as $x \to \infty$.

We introduce the four solutions $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$, and $\boldsymbol{u}_4$ so that

$$\boldsymbol{u}_\ell = -ie^{iI_\ell/h}\boldsymbol{u}_1 + ie^{-iI_\ell/h}\boldsymbol{u}_2, \tag{4.2.3}$$
$$\boldsymbol{u}_r = -ie^{iI_r/h}\boldsymbol{u}_3 + ie^{-iI_r/h}\boldsymbol{u}_4, \tag{4.2.4}$$

that is, we take

$$\boldsymbol{u}_1 = i\tau_\ell \boldsymbol{u}^+(x;\beta_\ell, y_\ell;\mu), \qquad \boldsymbol{u}_2 = \tau_\ell^* \boldsymbol{u}^-(x;\beta_\ell,\bar{y}_\ell;\mu), \tag{4.2.5}$$
$$\boldsymbol{u}_3 = i\tau_r \boldsymbol{u}^-(x;\beta_r, y_r,\mu), \qquad \boldsymbol{u}_4 = \pm\tau_r^* \boldsymbol{u}^+(x;\beta_r,\bar{y}_r,\mu). \tag{4.2.6}$$

By those solutions, we define the central solutions $\boldsymbol{v}_\ell$ and $\boldsymbol{v}_r$ in the same way as [23]:

$$\boldsymbol{v}_\ell = e^{-J/h}\frac{1}{2i}(\boldsymbol{u}_3 - \boldsymbol{u}_4), \quad \boldsymbol{v}_r = e^{-J/h}\frac{1}{2i}(\boldsymbol{u}_1 - \boldsymbol{u}_2). \tag{4.2.7}$$

Remark 4.5. The central solutins $\boldsymbol{v}_\ell$ and $\boldsymbol{v}_r$ are linearly independent if h is sufficiently small. Indeed, we have

$$\mathcal{W}(\boldsymbol{v}_\ell, \boldsymbol{v}_r) = -ie^{-2J/h}\big(\tau_\ell\tau_r w^+_{even}(\bar{y}_r, y_\ell) + \tau_\ell\tau_r^* w^+_{even}(\check{y}_\ell, y_r) \\ + \tau_\ell^*\tau_r w^+_{even}(\bar{y}_r, \check{\bar{y}}_\ell) + \tau_\ell^*\tau_r^* w^+_{even}(\bar{y}_\ell, y_r)\big),$$

and we see that the right-hand side is $4 + \mathcal{O}(h)$ as $h \to 0$.

We also write $\boldsymbol{v}_\ell$ and $\boldsymbol{v}_r$ as

$$\boldsymbol{v}_\ell = d_{11}\boldsymbol{u}_1 + d_{12}\boldsymbol{u}_2, \quad \boldsymbol{v}_r = d_{21}\boldsymbol{u}_3 + d_{22}\boldsymbol{u}_4. \tag{4.2.8}$$

Then, from (4.2.7) and (4.2.8), we have the representation

$$\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\end{pmatrix} = D_\ell\begin{pmatrix}\boldsymbol{v}_\ell\\ \boldsymbol{v}_r\end{pmatrix}, \quad D_\ell = \frac{1}{d_{11}+d_{12}}\begin{pmatrix}1 & 2ie^{J/h}d_{12}\\ 1 & -2ie^{J/h}d_{11}\end{pmatrix} \tag{4.2.9}$$

and

$$\begin{pmatrix}\boldsymbol{u}_3\\ \boldsymbol{u}_4\end{pmatrix} = D_r\begin{pmatrix}\boldsymbol{v}_\ell\\ \boldsymbol{v}_r\end{pmatrix}, \quad D_r = \frac{1}{d_{21}+d_{22}}\begin{pmatrix}2ie^{J/h}d_{22} & 1\\ -2ie^{J/h}d_{21} & 1\end{pmatrix}. \tag{4.2.10}$$

Here, we recall that $\mu = -i\lambda$ near μ_0 is an eigenvalue of L_- if and only if

$$\mathcal{W}(\boldsymbol{u}_\ell, \boldsymbol{u}_r) = 0.$$

By the representation (4.2.3)-(4.2.4), the arguments (4.2.9)-(4.2.10), and Remark 4.5, we see that this condition is also equivalent to

$$\det\begin{pmatrix}\begin{pmatrix}-ie^{iI_\ell/h} & ie^{-iI_\ell/h}\end{pmatrix} D_\ell\\ \begin{pmatrix}-ie^{iI_r/h} & ie^{-iI_r/h}\end{pmatrix} D_r\end{pmatrix} = 0,$$

i.e.,

$$\frac{1}{i}\left(e^{iI_\ell/h} - e^{-iI_\ell/h}\right)\frac{1}{i}\left(e^{iI_r/h} - e^{-iI_r/h}\right) \\ - 4e^{2J/h}\left(e^{iI_\ell/h}d_{12} + e^{-iI_\ell/h}d_{11}\right)\left(e^{iI_r/h}d_{22} + e^{-iI_r/h}d_{21}\right) = 0.$$

Theorem 4.6. *Suppose Assumption 4. There exist positive constants δ, h_0 and functions $d_{jk}(\mu, h)$, $j,k = 1,2$, defined on $D(\mu_0;\delta)\times(0,h_0]$ with asymptotic behavior as $h \to 0$*

$$d_{11} = -1 + \mathcal{O}(h), \qquad d_{12} = -1 + \mathcal{O}(h),$$
$$d_{21} = 1 + \mathcal{O}(h), \qquad d_{22} = 1 + \mathcal{O}(h),$$

in the case $A(\beta_\ell) = A(\beta_r)$ and

$$d_{11} = -1 + \mathcal{O}(h), \qquad d_{12} = -1 + \mathcal{O}(h),$$
$$d_{21} = -1 + \mathcal{O}(h), \qquad d_{22} = -1 + \mathcal{O}(h),$$

in the case $-A(\beta_\ell) = A(\beta_r)$, *such that* $\lambda = i\mu$, $\mu \in D(\mu;\delta)$ *is an eigenvalue of* L_- *for* $h \in (0, h_0]$ *if and only if*

$$\left(d_{12}e^{iI_\ell/h} + d_{11}e^{-iI_\ell/h}\right)\left(d_{22}e^{iI_r/h} + d_{21}e^{-iI_r/h}\right) + e^{-2J/h}\sin(I_\ell/h)\sin(I_r/h) = 0.$$

Remark 4.7. From this quantization condition, the eigenvalues $\lambda = i\mu$ for $\mu \in D(\mu_0;\delta)$ are given modulo exponentially small error by the roots of the equation

$$\left(d_{12}e^{iI_\ell/h} + d_{11}e^{-iI_\ell/h}\right)\left(d_{22}e^{iI_r/h} + d_{21}e^{-iI_r/h}\right) = 0.$$

This is equivalent to the two Bohr-Sommerfeld quantization conditions corresponding to each potential well:

$$\frac{d_{12}}{d_{11}}e^{2iI_\ell/h} = -1, \quad \frac{d_{22}}{d_{21}}e^{2iI_\ell/h} = -1.$$

which can also be written in the form

$$I_\ell(\mu) = \left(k + \frac{1}{2}\right)\pi h + h^2 r_\ell(\mu, h), \quad I_r(\mu) = \left(k + \frac{1}{2}\right)\pi h + h^2 r_r(\mu, h),$$

where

$$r_\ell = \frac{1}{2ih}\log\left(\frac{d_{12}}{d_{11}}\right), \quad r_r = \frac{1}{2ih}\log\left(\frac{d_{22}}{d_{21}}\right)$$

are both bounded when h goes to 0.

Thus we conclude that the set of eigenvalues produced by a double well potential is a union of the sets of eigenvalues produced by each potential well. This is a well known fact for the Schrödinger equation.

For the calculation of d_{jk}, we summarize the result of the Wronskians as follows.

Lemma 4.8. *Let* $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ *and* $\boldsymbol{u}_4$ *be defined by* (4.2.5)-(4.2.6)*. Then,*

(i) $\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_3) = -4ie^{J/h}\tau_\ell\tau_r w^+_{even}(\bar{y}_r; y_\ell)$,

(ii) $\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_4) = 4ie^{J/h}\tau_\ell\tau_r^* w^+_{even}(\check{y}_\ell; y_r)$,

(ii) $\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_2) = -4ie^{J/h}\tau_\ell^*\tau_r w^+_{even}(\bar{y}_r; \check{\bar{y}}_\ell)$,

(iv) $\mathcal{W}(\boldsymbol{u}_4, \boldsymbol{u}_2) = 4ie^{J/h}\tau_\ell^*\tau_r^* w^+_{even}(\bar{y}_\ell; y_r)$,

(v) $\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2) = i \cdot 4i\tau_\ell\tau_\ell^* w^+_{even}(\bar{y}_\ell; y_\ell)$,

(vi) $\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_4) = \mp i \cdot 4i\tau_r\tau_r^* w^+_{even}(\bar{y}_r; y_r)$,

in the cases $A(\beta_\ell) = \pm A(\beta_r)$. *Moreover, all amplitude functions appearing on the right are* $1 + \mathcal{O}(h)$ *as* $h \to 0$.

This is proven by adapting the argument in [10, Sec.5]. We then calculate the coefficients d_{jk} by using them.

Proof of Theorem 4.6. Note that each coefficient d_{jk} is represented as

$$d_{11} = \frac{\mathcal{W}(\boldsymbol{v}_\ell, \boldsymbol{u}_2)}{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2)}, \qquad d_{12} = \frac{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{v}_\ell)}{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2)},$$
$$d_{21} = \frac{\mathcal{W}(\boldsymbol{v}_r, \boldsymbol{u}_4)}{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_4)}, \qquad d_{22} = \frac{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{v}_r)}{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_4)},$$

that is,

$$d_{11} = \frac{e^{-J/h}}{2i}\frac{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_2) - \mathcal{W}(\boldsymbol{u}_4, \boldsymbol{u}_2)}{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2)}, \quad d_{12} = \frac{e^{-J/h}}{2i}\frac{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_3) - \mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_4)}{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_2)},$$
$$d_{21} = \frac{e^{-J/h}}{2i}\frac{\mathcal{W}(\boldsymbol{u}_1, \boldsymbol{u}_4) - \mathcal{W}(\boldsymbol{u}_2, \boldsymbol{u}_4)}{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_4)}, \quad d_{22} = \frac{e^{-J/h}}{2i}\frac{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_1) - \mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_2)}{\mathcal{W}(\boldsymbol{u}_3, \boldsymbol{u}_4)}.$$

By applying Lemma 4.8, we obtain

$$d_{11} = -\frac{\tau_r w^+_{even}(\bar{y}_\ell, \hat{\bar{y}}_r) + \tau_r^* w^+_{even}(\bar{y}_\ell, y_r)}{2\tau_\ell w^+_{\mathrm{even}}(\bar{y}_\ell, y_\ell)},$$
$$d_{12} = -\frac{\tau_r w^+_{even}(\bar{y}_r, y_\ell) + \tau_r^* w^+_{even}(\hat{y}_r, y_\ell)}{2\tau_\ell^* w^+_{\mathrm{even}}(\bar{y}_\ell, y_\ell)},$$
$$d_{21} = \pm\frac{\tau_\ell w^+_{even}(\hat{y}_r, y_\ell) + \tau_\ell^* w^+_{even}(\bar{y}_\ell, y_r)}{2\tau_r w^+_{even}(\bar{y}_r, y_r)},$$
$$d_{22} = \pm\frac{\tau_\ell w^+_{even}(\bar{y}_r, y_\ell) + \tau_\ell^* w^+_{even}(\bar{y}_\ell, \hat{\bar{y}}_r)}{2\tau_r^* w^+_{even}(\bar{y}_r, y_r)}$$

in the cases $A(\beta_\ell) = \pm A(\beta_r)$. Here, we know that $\tau_\bullet^\pm$ ($\bullet = \ell, r$) and all amplitude functions are $1 + \mathcal{O}$ as $h \to 0$, and it completes the proof. □

Remark 4.9. The coefficients d_{jk} also satisfy

$$d_{11} = d_{12}^*, \quad d_{21} = d_{22}^*.$$

In order to show this, we check the conjugate of the amplitude function $w^+_{even}(y_1; y_2; \mu)$. Indeed,

$$\left(w^+_{even}(y_1; y_2; \mu)\right)^* = \overline{w^+_{even}(y_1; y_2; \bar{\mu})} = w^-_{even}(\bar{y_1}; \bar{y_2}; \mu).$$

By the Wronskian formula, we have

$$w_{even}^{-}(\bar{y_1};\bar{y_2};\mu) = w_{even}^{+}(\bar{y_2};\bar{y_1};\mu),$$

that is,

$$\left(w_{even}^{+}(y_1;y_2;\mu)\right)^{*} = w_{even}^{+}(\bar{y_2};\bar{y_1};\mu).$$

Now we assume a symmetry for the potential and study the splitting phenomenon of eigenvalues which is well known for the Schrödinger operator.

Assumption 5. In addition to Assumption 4, $A(x)$ is either even or odd function.

Now, we retake the base points of the solutions so that $x_\ell = -x_r$ and $y_\ell = -\bar{y}_r$. Under this assumption, we obtain some symmetry for the elements on the quantization condition.

Lemma 4.10. *Under Assumption 5, the action integrals satisfy*

$$I_\ell(\mu) = \overline{I_\ell(\overline{\mu})} = \overline{I_r(\overline{\mu})} = I_r(\mu),$$

and

$$J(\mu) = \overline{J(\overline{\mu})}.$$

We can prove them in the same way as Lemma 3.8.

Lemma 4.11. *Under Assumption 5, the connection coefficients d_{jk} have symmetry*

$$d_{12}(\mu) = \overline{d_{21}(\overline{\mu})}, \quad d_{11}(\mu) = \overline{d_{22}(\overline{\mu})}.$$

Proof. It is sufficient to show that $\tau_\ell = \tau_r$. Recall that τ_ℓ is in the form of

$$\tau_\ell = \frac{1}{2}\left(\tau_\ell^{+} + (\tau_\ell^{-})^{*}\right),$$

where,

$$\tau_\ell^{+} = \frac{w_{even}^{+}(\bar{y}_\ell;x_\ell;\mu)}{w_{even}^{+}(\bar{y}_\ell;y_\ell;\mu)}, \quad \tau_\ell^{-} = \frac{w_{even}^{+}(\check{y}_\ell;x_\ell;\mu)}{w_{even}^{+}(\bar{y}_\ell;y_\ell;\mu)}.$$

Under the condition $A(x) = \pm A(-x)$, the amplitude $w_{even}^{+}(y_1;y_2;\mu)$ have the symmetry $w_{even}^{+}(y_1;y_2;\mu) = w_{even}^{+}(-y_2;-y_1;\mu)$ from the recurrence equations. Hence,

$$\tau_\ell^{+} = \frac{w_{even}^{+}(-x_\ell;-\bar{y}_\ell;\mu)}{w_{even}^{+}(-y_\ell;-\bar{y}_\ell;\mu)} = \frac{w_{even}^{+}(x_r;y_r;\mu)}{w_{even}^{+}(\bar{y}_r;y_r;\mu)},$$

that is, $\tau_\ell^{+} = \tau_r^{-}$. One can obtain $\tau_\ell^{-} = \tau_r^{+}$ in the same calculation, and this gives the symmetry $\tau_\ell = \tau_r$ as a result. □

We rewrite the quantization condition as

$$\left(\gamma_\ell e^{iI_\ell/h} + \gamma_\ell^* e^{-iI_\ell/h}\right)\left(\gamma_r e^{iI_r/h} + \gamma_r^* e^{-iI_r/h}\right) \mp e^{-2J/h}\sin(I_\ell/h)\sin(I_r/h) = 0,$$

where $\gamma_\bullet$ and $\gamma_\bullet^*$ are $1+\mathcal{O}(h)$ as $h\to 0$, from Theorem 4.6. Moreover, we write

$$\gamma_\ell = (\gamma_\ell\gamma_\ell^*)^{1/2}(\gamma_\ell/\gamma_\ell^*)^{1/2} := \rho_\ell e^{i\theta_\ell},$$
$$\gamma_\ell^* = (\gamma_\ell\gamma_\ell^*)^{1/2}(\gamma_\ell/\gamma_\ell^*)^{-1/2} := \rho_\ell e^{-i\theta_\ell},$$

and

$$\gamma_r = (\gamma_r\gamma_r^*)^{1/2}(\gamma_r/\gamma_r^*)^{1/2} := \rho_r e^{i\theta_r},$$
$$\gamma_r^* = (\gamma_r\gamma_r^*)^{1/2}(\gamma_r/\gamma_r^*)^{-1/2} := \rho_r e^{-i\theta_r},$$

where we choose branches of the square roots and the logarithm in such a way that $\rho_\bullet = 1+\mathcal{O}(h)$ and $\theta_\bullet = \mathcal{O}(h)$. Then, our quantization condition is rewritten as

$$\rho_\ell\rho_r\left(e^{i\tilde{I}_\ell/h} + e^{-i\tilde{I}_\ell/h}\right)\left(e^{i\tilde{I}_r/h} + e^{-i\tilde{I}_r/h}\right) \mp e^{-2J/h}\sin(I_\ell/h)\sin(I_r/h) = 0,$$

that is,

$$\rho_\ell\rho_r\cos(\tilde{I}_\ell/h)\cos(\tilde{I}_r/h) \mp 4e^{-2J/h}\sin(I_\ell/h)\sin(I_r/h) = 0,$$

where $\tilde{I}_\bullet = I + h\theta_\bullet$. Note that $\rho_\bullet^* = \rho_\bullet$, $\theta_\bullet^* = \theta_\bullet$ and $\tilde{I}_\bullet^* = \tilde{I}_\bullet$ from Remark 4.9 and Lemma 4.10. If Assumption 5 holds, we have $\rho_\ell = \rho_r$, $\theta_\ell = \theta_r$ and $\tilde{I}_\ell = \tilde{I}_r$ from Lemmas 4.10 and 4.11. Thus, the quantization condition is described in the case $A(x) = \pm A(-x)$,

$$\rho^2\cos^2(\tilde{I}/h) \mp 4e^{-2J/h}\sin^2(I/h) = 0. \tag{4.2.11}$$

Here, $I := I_\bullet$ ($\tilde{I} := \tilde{I}_\bullet$) and $\rho := \rho_\bullet$.

This means that the eigenvalues produced by each potential well satisfy the same Bohr-Sommerfeld quantization condition modulo exponentially small error:

$$\tilde{I}(\mu) = \left(k + \frac{1}{2}\right)\pi h. \tag{4.2.12}$$

We are going to show, in the following theorem, the so-called splitting phenomenon of these eigenvalues (which is already well known in the self-djoint Schrödinger case), thanks to the explicit exponential error term in (4.2.11).

Theorem 4.12. *Suppose Assumption 5. Let $\mu_k^0(h)$ be the (unique) root of the equation (4.2.12) near μ_0. Then the two eigenvalues $i\mu_k^+(h)$, $i\mu_k^-(h)$ approximated by $i\mu_k^0(h)$ satisfy the following asymptotic formulas as $h\to 0$:*

1. In the case where $A(x)$ is an even function,

$$\mu_k^{\pm}(h) - \mu_k^0(h) = \pm e^{-J/h}\left(\frac{2h}{I'(\mu_k^0)} + \mathcal{O}(h^2)\right).$$

2. In the case where $A(x)$ is an odd function,

$$\mu_k^{\pm}(h) - \mu_k^0(h) = \pm i e^{-J/h}\left(\frac{2h}{I'(\mu_k^0)} + \mathcal{O}(h^2)\right).$$

Remark 4.13. We see from this theorem that the eigenvalues split vertically in the even case, whereas they split horizontally in the odd case. This implies in particular that there are no purely imaginary eigenvalues in the odd case.

Proof. From the quantization condition, we have

$$\cos(\tilde{I}(\mu)/h) = \pm 2\rho^{-1} e^{-J/h} \sin\left(I(\mu)/h\right) \tag{4.2.13}$$

in the even case $A(x) = A(-x)$, and

$$\cos(\tilde{I}(\mu)/h) = \pm 2i\rho^{-1} e^{-J/h} \sin\left(I(\mu)/h\right) \tag{4.2.14}$$

in the odd case $A(x) = -A(-x)$.

Hence it is easy to see that $|\mu - \mu_k^0| = \mathcal{O}(he^{-J/h})$ when μ satisfies (4.2.13) or (4.2.14). But we want more precise estimates. Since $\tilde{I}(\mu) = I(\mu) + h\theta = I(\mu) + \mathcal{O}(h^2)$, we have

$$\sin\left(I(\mu_k^{\pm})/h\right) = \pm 1 + \mathcal{O}(h),$$

where the sign of the RHS depends on the parity of k (but not on the sign of the LHS).

On the other hand,

$$\begin{aligned}\tilde{I}(\mu_k^{\pm}) &= \tilde{I}(\mu_k^0) + \tilde{I}'(\mu_k^0)(\mu_k^{\pm} - \mu_k^0) + \mathcal{O}(h^2 e^{-2J/h}) \\ &= \tilde{I}(\mu_k^0) + I'(\mu_k^0)(\mu_k^{\pm} - \mu_k^0) + \mathcal{O}(h^3 e^{-J/h}),\end{aligned}$$

and hence

$$\cos(\tilde{I}(\mu_k^{\pm})/h) = \pm I'(\mu_k^0)\frac{\mu_k^{\pm} - \mu_k^0}{h} + \mathcal{O}(h^2 e^{-J/h}).$$

The theorem follows from these estimates. ∎

Bibliography

[1] C.M. Bender : *Introduction to $\mathcal{PT}$-symmetric Quantum Theory,* Contemporary Physics **46**(4), 277-292 (2005).

[2] C.M. Bender and S. Boettcher : *Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry,* Physical Review Letters **80**(24), (1998).

[3] N. Boussekkine and N. Mecherout : *$\mathcal{PT}$-symmetry and potentiel well. The simple well case,* Mathematische Nachrichten **289**(1), 13-27 (2016).

[4] J.C. Bronski : *Semiclassical eigenvalue distribution of the Zakharov-Shabat eigenvalue problem,* Physica D **97**(4), 376-397 (1996).

[5] E.B. Davies : *Pseudospectra of differential operators,* Journal of Operator Theory **43**(2), 243-262 (2000).

[6] E.B. Davies : *Non-self-adjoint differential operators,* Bulletin of the London Mathematical Society **34**(5), 513-532 (2002).

[7] N. Dencker : *The pseudospectrum of systems of semiclassical operators,* Analysis & PDE **1**, 323-373 (2008).

[8] N. Dencker, J. Sjöstrand, M. Zwroski : *Pseudospectra of semiclassical (pseudo-) differential operators,* Communications on Pure and Applied Mathematics **57**(3), 384-415 (2004).

[9] S. Fujiié, C. Lasser, and L. Nédélec : *Semiclassical resonances for a two-level Schrödinger operator with a conical intersection,* Asymptotic Analysis **65**(1-2), 17-58 (2009).

[10] S. Fujiié and J. Wittsten : *Quantization conditions of eigenvalues for semiclassical Zakharov-Shabat systems on the circle,* preprint (2017), arXiv:1703.08352.

[11] S. Fujiié and M. Zerzeri : *Bohr-Sommerfeld quantization condition derived by a microlocal WKB method,* Vietnam Journal of Mathematics **32** 153-160 (2004).

[12] C. Gérard and A. Grigis : *Precise estimates of tunneling and eigenvalues near a potential barrier,* Journal of differential equations **72**(1), 149-177 (1988).

[13] K. Hirota : *Real eigenvalues of a non-self-adjoint perturbation of the self-adjoint Zakharov-Shabat operator,* Journal of Mathematical Physics **58**(10), 102108 (2017).

[14] K. Hirota and J. Wittsten : *Complex eigenvalue splitting for the Zakharov-Shabat operator,* in preparation.

[15] S. Kamvissis, K.T.-R. McLaughlin and P.D Miller : *Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation,* Princeton University Press, (2003).

[16] M. Klaus and J.K. Shaw : *Purely imaginary eigenvalues of Zakharov-Shabat systems,* Physical Review E **65**, 036607 (2002).

[17] M. Klaus and J.K. Shaw : *On the Eigenvalues of Zakharov-Shabat Systems,* SIAM Journal on Mathematical Analysis **34**(4), 759-773 (2003).

[18] P.D Miller : *Riemann-Hilbert Problems with Lots of Discrete Spectrum,* Contemporary Mathematics **458**, (2008).

[19] P.D Miller and S. Kamvissis : *On the semiclassical limit of the focusing nonlinear Schrodinger equation,* Physics Letters A **247**, 75-86 (1998).

[20] J. Satsuma and N. Yajima : *Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media,* Progress of Theoretical Physics **55**, 284-306 (1974).

[21] L.N Trefethen : *Pseudospectra of linear operators,* SIAM Review **39**(3), 383-406 (1997).

[22] L.N Trefethen and M. Embree : *Spectra and Pseudospectra: the Behavior of Nonnormal Matrices and Operators,* Princeton University Press, (2005).

[23] N. Mecherout, N. Boussekkine, T. Ramond and J. Sjöstrand : *$\mathcal{PT}$-symmetry and Schrödinger operators. The double well case,* Mathematische Nachrichten **289**(7), 854-887 (2016).

[24] J.v. Neumann : The Mathematical Foundations of Quantum Mechanics, Investigations in Physics **2**, (1955).

[25] V.E. Zakharov and A.B. Shabat : *Exact Theory of Two-dimensional Self-focusing and One-dimensional Self-modulation of Wave in Nonlinear Media,* Journal of Experimental and Theoretical Physics **34**(1), 62-69 (1972).