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One Loop Analyses of Superstring Vacua on Asymmetric Orbifolds
(非対称オービフォールド上の超弦真空の１ループ解析)

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Abstract

In this thesis, we study the type II string vacua on asymmetric orbifolds compactified by the twists with the orbifold group generated by a single element by means of worldsheet conformal field theory approach. We focus on the non-supersymmetric string vacua with vanishing cosmological constant, which are difficult to be constructed by symmetric orbifolds. These orbifolds are analyzed by evaluating the partition function of the worldsheet corresponding to one-loop vacuum amplitude of the string. We find these partition functions by requiring the modular invariance of two-dimensional torus of the worldsheet. It is not obvious whether the spectrum of the string is unitary for asymmetric orbifolds generally. We confirm that the asymmetric orbifolds discussed in this thesis are consistent. In addition, we show the stability of these vacua, namely, the tachyonic states do not arise when the vacuum energy density vanishes by the bose-fermi degeneracy.

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1 Introduction

In string theory, we observe the phenomena which cannot be explained from a point particle viewpoint, which are called "stringy" effects. This is because string theory is the only framework for addressing quantum gravity, and a structure of string geometry is richer than that of the ordinary Riemannian geometry in many aspects.

A simple example is provided by orbifolds [1–3], which are defined as the quotient of space by a discrete group. The orbifold is consistent with the modular invariance owing to the existence of the twisted sectors in string theory. Although the geometry of orbifold generally includes singular points, we can evaluate the physical states, which is impossible in the ordinary manifold.

In addition, the string geometry allows an asymmetric structure where the right and left-movers of the string can propagate on different spaces. For example, in heterotic superstring theory [4], right-movers propagate on ten-dimensional supersymmetric space, and left-movers on 26-dimensional non-supersymmetric space. We may consider the transformation of coordinates separately on the right and left on certain geometries. This conception makes it possible to find that the string theory for specific toroidal compactification [5] exhibits an isomorphism between the large radius and small radius tori, which is known as T-duality. There is no geometrical interpretation classically.

The asymmetric orbifolds are also studied in string description, combing two stringy phenomena mentioned above [6]. For asymmetric orbifold, the orbifold group acts asymmetrically on the geometries where the right and left-movers live. This does not mean that the string spectrum is considered separately. The spectrum is the product of the states of the right and left-movers. They are restricted by the level matching condition and by the orbifold group. Thus, the orbifold action on the right-movers affects the left-moving sector as well as the right-moving one, and the spectrum of string are not obvious, especially in the twisted sectors.

A typical asymmetric orbifold is defined by the twist including the simple T-duality transformation, namely, the chiral reflection. This type of the string vacua have been studied by worldsheet conformal field theory (CFT) approach e.g. in [7–14]. In this method, we can accurately evaluate α'-corrections of the strings, namely, this is not the low energy effective theory $\alpha' \to 0$, where the parameter α' is the slope of the Regge trajectory. This analysis is suitable for investigation of the string vacuum with the stringy phenomena.

In this paper, we focus on the type II string vacua with interesting characteristics realized by stringy effects: the one-loop vacuum energy density vanishes by the bose-fermi cancellations in spite of the absence of the spacetime supersymmetry. Superstring models with such characteristics have already been found in the earlier studies [15–20][1]. These string vacua are realized in asymmetric orbifolds constructed by the twist with two non-commutative elements, which are called "non-abelian orbifolds". Therefore, it would be almost too complicated to investigate them in detail, and hard to extend them to the other more realistic models or the other applications. On the other hand, the

[1] In heterotic string theory, there are also the studies of non-supersymmetric model building [21–24].

orbifold models studied in this paper are defined by the orbifold groups with a singe element [25, 26], which are simpler than those of the previous works.

Instead, we introduce the "$\mathbb{Z}_4$-chiral reflection" as well as $\mathbb{Z}_2$ as orbifold action [27]. Here, "$\mathbb{Z}_4$-chiral reflection" means the chiral reflection of order four in untwisted sector[2]. A chiral reflection must act on the worldsheet fermions as well as the bosons in order to preserve the symmetry of the worldsheet CFT in superstring theory. Then, the chiral reflection also acts unclearly on the spin fields. The square of this $\mathbb{Z}_4$-chiral reflection becomes the action of the sign inversion of the states of the right or left-moving Ramond sector. Therefore, when the orbifold action includes the right (left)-moving $\mathbb{Z}_4$-chiral reflection, the right (left)-moving supercurrents of the spacetime are not constructed by the spin fields that exist in the original space at least[3]. In our method, this $\mathbb{Z}_4$-chiral reflection is necessary to construct the non-supersymmetric string vacuum keeping the bose-fermi degeneracy.

When the orbifold action includes the certain action such as the sign inversion of spacetime fermions, the GSO projection potentially remains the tachyonic states in twisted sectors even if the vacuum is stable in the original background before orbifolding. Therefore, it is important to check the lightest excitations in the twisted sectors. The construction of the orbifolds we study are so simple that we can explicitly calculate the partition functions of the worldsheet corresponding to one-loop vacuum amplitude of the string. The suitable partition functions can be obtained by requiring the modular invariance. From these results, we directly investigate the spectra of the strings in these string vacua in detail. We actually confirm the presence or absence of the tachyonic states and the spacetime supercurrents composed by the massless fermions in twisted sectors from the results of the partition functions.

One of the motivation of studying such models is that these models may provide the hints of the solutions of the physical problems such as cosmological constant problem [28]. It would be important to show that the variety of such vacua can be constructed in string geometry, especially due to the stringy effects. We expect that there is still a lot of room for improvement or new directions regarding this discussion. Of course, these models still do not overcome primordial problem, namely, the number of fermionic fields is the same as bosonic one in each mass level. However, the bose-fermi degeneracy might be necessary for the vacuum energy to vanish perturbatively in the flamework of string theory.

Nevertheless, these vacua are interesting. For example, when we consider the certain particle physics with the mechanism of the mass gaps in the low energy region in vacua derived from these string models, we do not have to find the superpartners of the particles. In addition, it is simple to picture the scenario of the cosmological constant vanishing completely in the perturbative analysis, but small cosmological constant arising from the contribution of non-perturbative effects.

[2] In general, the chiral reflection of order two or four in untwisted sector becomes the action of order more than four in twisted sectors, which relates to the level matching condition.

[3] The right (left)-moving supercurrents are potentially constructed by the spin fields in the twisted sectors.

Organization of the thesis

The structure of the present paper is summarized as follows:

In section 2, we review the calculations of the partition function of the orbifold defined by the twist of the translation acting on a single boson, and explain the basic concept of our method to construct the asymmetric orbifolds in this paper. In addition, we introduce the chiral reflection acting on the coordinates of certain tori and the sign inversion of the right or left-moving fermionic states there.

In section 3, we study the type II string vacua on various asymmetric orbifolds. Combining the building blocks introduced in subsection 3.1, we practically calculate the partition functions of asymmetric orbifolds in the remaining subsections. In subsection 3.2, we mainly investigate the non-supersymmetric asymmetric orbifolds with vanishing cosmological constant based on [25], comparing them with supersymmetric ones and with the case of non-vanishing cosmological constant. In order to consider how to break the supersymmetry on these orbifolds, we investigate some other types of non-supersymmetric asymmetric orbifolds in subsection 3.3. Furthermore, a variety of such asymmetric orbifolds are constructed. We consider the way to efficiently calculate the partition function of these asymmetric orbfolds in subsection 3.4. Writing down the q-expansions of the partition function explicitly, we can confirm that these orbifolds we study are consistent with unitarity. In addition, we can find that the tachyonic states do not arise if the vacuum energy vanishes by bose-fermi cancellation in each twisted sectors. We make these reasons clear in subsection 3.5.

2 String analysis by the calculation of partition functions

We discuss the compactifications on the backgrounds that have the remarkable characteristic of string geometry, which are not described in the flame of the ordinary Riemannian geometry. In order to study these string vacua, we use the traditional way of string theory, the world-sheet CFT approach.

We study the string vacua by calculating the one-loop amplitudes for the closed string, especially ones that have no vertex operators, that is, the partition functions on the SCFTs for two-dimensional torus. The results of these calculations are helpful to evaluate the physical states for the string vacua more than the vacuum energy density. In this section, we introduce the methods to calculate the partition functions on worldsheet to make preparations for the analyses of the string vacua discussed later.

2.1 Partition function for worldsheet

We describe the two-dimensional torus of the worldsheet as complex plane z with metric $ds^2 = dzd\bar{z}$ and identifications

$$z \simeq z + 2\pi \simeq z + 2\pi\tau, \tag{2.1}$$

where $\tau = \tau_1 + i\tau_2$ $(\tau_1, \tau_2 \in \mathbb{R})$ is the modular parameter. Two moduli τ_1 and τ_2 parametrize the family of distinct tori, and they have the physical degree of freedom for string theory. In addition, the identifications (2.1) means that the complex structure given by τ is equivalent to ones given by

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{2.2}$$

Above $\tau \to \tau'$ is called modular transformation, and generated by two elements

$$\mathrm{T} : \tau \to \tau + 1, \tag{2.3}$$
$$\mathrm{S} : \tau \to -\frac{1}{\tau}. \tag{2.4}$$

Reversing the sign of all of a, b, c, d in (2.2) does not change τ, so the modular group is $\mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2$. Modular transformation is regarded as discrete gauge transformation, and useful to lead to the correct physical quantity for string amplitudes.

As a first example of the partition function, we shall discuss S^1 compact boson $X(z, \bar{z})$ with radius R living on the worldsheet. The partition function is given by the path integral

$$\int \mathcal{D}X e^{-S[X]}, \tag{2.5}$$

where the action of the free boson $S = \frac{1}{2\pi} \int \partial X \bar{\partial} X$ $(\partial := \partial/\partial z, \bar{\partial} := \partial/\partial \bar{z})$. Through this paper, we use the self-dual radius $\alpha' = 1$ convention.

by path integral

We can directly calculate the partition function of S^1 boson by means of a Lagrangian formulation. For this functional integration, we need to consider all instanton sectors labeled by the spatial and temporal winding numbers $w, m \in \mathbb{Z}$. We define them so that, with the identifications (2.1), the boson of the (w, m)-sector satisfies the boundary condition

$$X(z + 2\pi, \bar{z} + 2\pi) = (\mathcal{T}_{2\pi R})^w X(z, \bar{z}) := X(z, \bar{z}) + 2\pi R w, \tag{2.6}$$
$$X(z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}) = (\mathcal{T}_{2\pi R})^m X(z, \bar{z}) := X(z, \bar{z}) + 2\pi R m, \tag{2.7}$$

where $\mathcal{T}_{2\pi R}$ is defined as translation operator $X \to X + 2\pi R$. It is easy to understand this condition intuitively by string picture. For example, fig.1 shows the time-slice of the worldsheet corresponding to a string with winding number $w = 2$ living on S^1.

The classical solution $X_{\mathrm{cl}}^{(w,m)}$ to equations of motion $\partial\bar{\partial}X = 0$ is

$$X_{\mathrm{cl}}^{(m,w)} = \frac{R}{2i\tau_2} \left(m(z - \bar{z}) + w(\tau\bar{z} - \bar{\tau}z) \right). \tag{2.8}$$

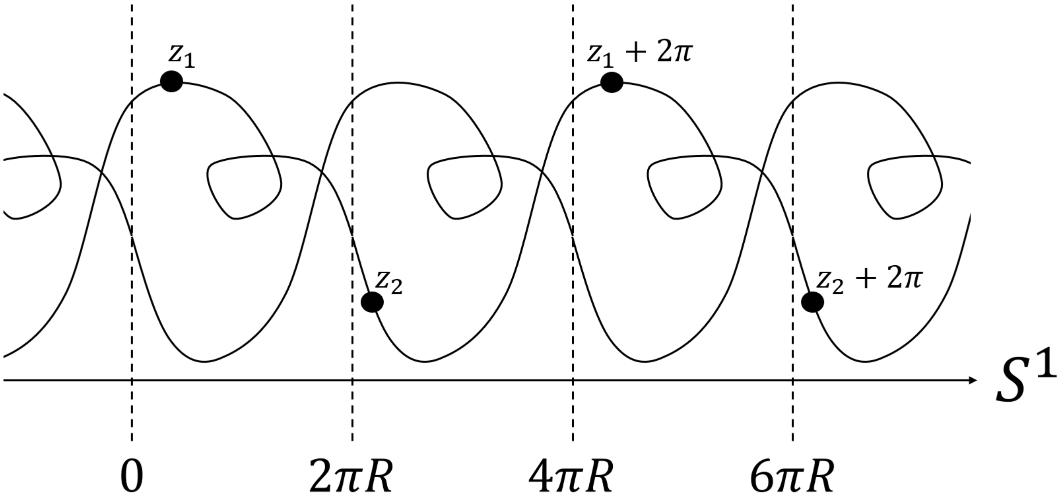

Figure 1: A string with winding number $w = 2$ on S^1.

The partition function is obtained by calculating the determinant $-\partial\bar{\partial}$ from the gaussian functional integral. The result of the integral of (2.5) is [4]

$$
\begin{aligned}
Z_R^{S^1}(\tau,\bar{\tau}) &= 2\pi R \cdot \frac{1}{2\pi\sqrt{\tau_2}|\eta(\tau)|^2} \cdot \sum_{m,w\in\mathbb{Z}} \exp\left(-\frac{\pi R^2 |m - w\tau|^2}{\alpha' \tau_2}\right), \\
&=: \sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}),
\end{aligned} \tag{2.9}
$$

where we write the partition function as the summation of the partition blocks sorted by (w, m)-sector

$$
Z_{R(w,m)}(\tau,\bar{\tau}) := \frac{R}{\sqrt{\tau_2}|\eta(\tau)|^2} e^{-\frac{\pi R^2}{\tau_2}|w\tau+m|^2} \quad (w, m \in \mathbb{Z}). \tag{2.10}
$$

Then, this is transformed as follows under the modular transformation (2.3) and (2.4),

$$
Z_{R(w,m)}(\tau,\bar{\tau})|_{\mathrm{T}} = Z_{R(w,m+w)}(\tau,\bar{\tau}), \tag{2.11}
$$

$$
Z_{R(w,m)}(\tau,\bar{\tau})|_{\mathrm{S}} = Z_{R(m,-w)}(\tau,\bar{\tau}), \tag{2.12}
$$

though the partition function (2.9) is invariant by the summation over w, m. This result seems natural by the definition of modular transformation and winding numbers, and is important when we consider the construction of orbifold. Remember that each (w, m)-sector is defined by the boundary condition (2.3) and (2.4). From these conditions, we

[4]Using the normalization conventions of [29], an overall normalization is consistent with the result in Hamiltonian formalism.

obtain

$$X(z+2\pi\tau+2\pi,\bar{z}+2\pi\bar{\tau}+2\pi)=(\mathcal{T}_{2\pi R})^{w}(\mathcal{T}_{2\pi R})^{m}X(z,\bar{z}). \tag{2.13}$$

Defining the modular parameter as $\tau'=\tau+1$, we can regard (2.13) as (2.7) in $(w,m+w)$-sector, namely,

$$X(z+2\pi\tau',\bar{z}+2\pi\bar{\tau}')=(\mathcal{T}_{2\pi R})^{m+w}X(z,\bar{z}). \tag{2.14}$$

Therefore, $(w,m+w)$-sector is obtained by the transformation $\tau\to\tau+1$ of (w,m)-sector. This is consistent with (2.11).

Similarly, we would like to consider the redefinition $\tau'=-1/\tau$. The modular transformation S :$\tau\to-1/\tau$ means not only switching the boundary conditions in spacial and temporal directions but also rescaling coordinate z by τ. The total system should not be changed by rescaling of z on CFT, but generic fields are potentially changed. Defining $X'(z,\bar{z})=X(\tau z,\bar{\tau}\bar{z})$, we can guess that

$$X'(z+2\pi,\bar{z}+2\pi)=(\mathcal{T}_{2\pi R})^{m}X'(z,\bar{z}), \tag{2.15}$$
$$X'(z+2\pi\tau',\bar{z}+2\pi\bar{\tau}')=(\mathcal{T}_{2\pi R})^{-w}X'(z,\bar{z}), \tag{2.16}$$

from the boundary condition (2.6) and (2.7). Rather from (2.12), we confirm that the transformation $\tau\to-1/\tau$ of (w,m)-sector gives $(m,-w)$-sector defined by the boundary condition (2.6) and (2.7). As a result, we might define (w,m)-sector so that $(w,0)$-sector is decided by (2.6), and $(m,-w)$ and $(w,m+w)$-sectors are defined as (w,m)-sectors transformed by the S and T-modular transformation (2.4), (2.3).

by Hilbert space

In order to observe a spectrum of the string, we shall shift the interpretation of the partition function (2.5) from path integral to Hilbert space. In operator formalism, a partition function of a bosonic field with the temporal boundary condition $X(z+2\pi\tau,\bar{z}+2\pi\bar{\tau})=\hat{B}X(z,\bar{z})$ can be schematically written as

$$\mathrm{Tr}\left[\hat{B}q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right]=\frac{1}{|\eta|^2}\mathrm{Tr}\left[q^{\frac{1}{4}p_L^2}\bar{q}^{\frac{1}{4}p_R^2}\right], \tag{2.17}$$

where $L_n(n\in\mathbb{Z})$ is Virasoro algebra, c is central charge, and p_L, p_R are left and right-moving momentum zero modes. Performing a Poisson resummation (B.1), we can rewrite (2.9) as

$$\begin{aligned} Z_R^{S^1}(\tau,\bar{\tau}) &= \frac{1}{|\eta|^2}\sum_{w,n\in\mathbb{Z}}\exp\left[-\pi\tau_2\left(\frac{n^2}{R^2}+w^2R^2\right)+2\pi i\tau_1 nw\right] \\ &= \frac{1}{|\eta|^2}\sum_{w,n\in\mathbb{Z}}q^{\frac{1}{4}(Rw+\frac{n}{R})^2}\bar{q}^{\frac{1}{4}(Rw-\frac{n}{R})^2}. \end{aligned} \tag{2.18}$$

The summation over the temporal winding m in (2.9) can be exchanged for a sum over a Kaluza-Klein momentum n in operator formalism. Comparing (2.17) with (2.18), the momentum zero modes are

$$p_L = \frac{n}{R} + wR, \tag{2.19}$$

$$p_R = \frac{n}{R} - wR. \tag{2.20}$$

Thus the contribution from zero modes to the square of the mass of string is

$$\frac{1}{2}\left(p_L^2 + p_R^2\right) = \frac{n^2}{R^2} + w^2 R^2, \tag{2.21}$$

in string theory. This is invariant under

$$R \to \frac{1}{R}, n \to w, w \to n. \tag{2.22}$$

In other words, $p_L \to p_L,\ p_R \to -p_R$, so separating left and right -movings, this transformation acts as

$$X(z,\bar{z}) = X_L(z) + X_R(\bar{z}) \to X'(z,\bar{z}) = X_L(z) - X_R(\bar{z}). \tag{2.23}$$

Obviously, the partition function $Z_R^{S^1}(\tau,\bar{\tau})$ is also invariant under this transformation. It is known as "T-duality" in string theory.

In ordinary particle physics, S^1 compact boson with any radius R has $U(1)$-gauge symmetry. Otherwise, there are special compactification radii R at which the gauge symmetry is enhanced in string theory. If we fix the radius to self-dual $R = 1$, the partition function is

$$Z_{R=1}^{S^1}(\tau,\bar{\tau}) = \frac{1}{|\eta|^2} \sum_{w,n\in\mathbb{Z}} q^{\frac{1}{4}(w+n)^2} \bar{q}^{\frac{1}{4}(w-n)^2} \tag{2.24}$$

$$= \frac{1}{|\eta|^2} \sum_{w,n\in\mathbb{Z}} \frac{1}{2}\left(1 + (-1)^{m+n}\right) q^{\frac{1}{2}m^2} \bar{q}^{\frac{1}{2}n^2} \tag{2.25}$$

$$= \frac{1}{|\eta|^2} \sum_{w,n\in\mathbb{Z}} \frac{1}{2}\left(q^{n^2}\bar{q}^{m^2} + q^{(n+\frac{1}{2})^2}\bar{q}^{(m+\frac{1}{2})^2}\right) \tag{2.26}$$

$$= \left|\frac{\Theta_{0,1}(\tau)}{\eta}\right|^2 + \left|\frac{\Theta_{1,1}(\tau)}{\eta}\right|^2 = |\chi_0(\tau)|^2 + |\chi_1(\tau)|^2, \tag{2.27}$$

where $\chi_0(\tau),\ \chi_1(\tau)$ are characters of a level one $SU(2)$ current algebra with spin $0, \frac{1}{2}$. We emphasize that this implication of $SU(2)$-gauge symmetry is easily given by deforming the expression of the partition function.

In addition, we look at the product of two $Z^{S^1}_{R=1}(\tau,\bar{\tau})$.

$$Z^{S^1}_{R=1}(\tau,\bar{\tau}) \times Z^{S^1}_{R=1}(\tau,\bar{\tau}) = \left(\frac{1}{2}\left|\frac{\theta_3(2\tau)}{\eta}\right|^2 + \frac{1}{2}\left|\frac{\theta_2(2\tau)}{\eta}\right|^2\right)^2 \tag{2.28}$$

$$= \frac{1}{2}\left(\left|\frac{\theta_3(\tau)}{\eta}\right|^4 + \left|\frac{\theta_4(\tau)}{\eta}\right|^4 + \left|\frac{\theta_2(\tau)}{\eta}\right|^4\right) \tag{2.29}$$

$$:= Z^{T^2[SO(4)]}(\tau,\bar{\tau}), \tag{2.30}$$

where we use the relation (A.2), (A.3), (A.4). $Z^{T^2[SO(4)]}(\tau,\bar{\tau})$ is the partition function of the two-dimensional torus with the $SO(4)$-symmetry enhancement. From this, we can conform the equivalence $T^2[SO(4)] \simeq S^1[SU(2)] \times S^1[SU(2)]$.

We can also derive the partition function $Z^{S^1}_R(\tau,\bar{\tau})$ directly from the momentum zero modes if we can find the appropriate ones in operator formalism. However, the modular invariance is not obvious such as (2.18).

by twisting

Actually, we can mechanically compute the partition function by the method other than investigating the momentum zero modes in operator formalism. We start with non-compact theory $X \in \mathbb{R}$, and consider the identification $X \sim X + 2\pi R$. This identification causes Hilbert spaces of which spacial boundary conditions are different from a original CFT. We call them "twisted sector". These correspond to the sectors such that $w \neq 0$ in Lagrangian formalism.

At first, we consider the states for original sector $w = 0$, called "untwisted sector". The identification $X \sim X + 2\pi R$ means that the spectrum of string must be invariant under m times $2\pi R$ translation

$$X \to X' = X + 2\pi R m =: (\mathcal{T}_{2\pi R})^m X. \tag{2.31}$$

To realize that, we insert the projection operator of these translation in the trace (2.17), and we obtain the partition function for untwisted sector,

$$\mathrm{Tr}_{\text{untwisted}}\left[\sum_{m\in\mathbb{Z}} (\mathcal{T}_{2\pi R})^m q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}}\right] = \sum_{m\in\mathbb{Z}} Z_{R(0,m)}(\tau,\bar{\tau}), \tag{2.32}$$

where $Z_{R(0,m)}(\tau,\bar{\tau})$ is in (2.10), and we regulate the infinite order of the projection.

Next, we consider the twisted sector. In order to make modular invariance clear, we define w-twisted sector as the Hilbert space defined by $S : \tau \to -1/\tau$ of $(0,w)$-sector in terms of Lagrangian formalism. We can find w-twisted sector's counterpart of the identification (2.31) by the relation (2.14), that is, we can know the relevant projection operator by reading the variation of the temporal boundary condition under $\tau \to \tau + 1$. In the current case, the projection for any w-twisted sector is the same as untwisted sector. Therefore, we can write the partition function for w-twisted sector as

$$\mathrm{Tr}_{w-\mathrm{twisted}}\left[\sum_{m\in\mathbb{Z}}(\mathcal{T}_{2\pi R})^m q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right]=\sum_{m\in\mathbb{Z}}Z_{R(w,m)}(\tau,\bar{\tau}). \tag{2.33}$$

Eventually, once we calculate $Z_{R(0,m)}(\tau,\bar{\tau})$ by introducing the projection, we can calculate the partition function of other sectors by the modular transformation (2.11) and (2.12). The total partition function (2.9) is obtained by the summation over them. The partition function derived in this way is necessarily modular invariant. This construction is known as "twisting".

2.2 Partition function of orbifold

2.2.1 Twisting for orbifold

We would like to consider a calculation method of the partition function of a general orbifold by an identification of spacetime coordinate. It is useful to apply an idea of twisting to orbifold construction. We call it "orbifolding". We express the identification by using the operator $\hat{B}$ acting on spacetime coordinate X on $\mathcal{M}$ as follows.

$$X\simeq X'=\hat{B}X \tag{2.34}$$

We think of $\hat{B}$ as an element of a discrete group action $B:\mathcal{M}\to\mathcal{M}$. We investigate the orbifold $\mathcal{M}/B$.

At first, we consider the untwisted sector. We assume that the partition function of the original coordinate before the identification is described as

$$Z(\tau,\bar{\tau})=\mathrm{Tr}_{\mathrm{untwisted}}\left[q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right],\quad q=\exp(2\pi i\tau) \tag{2.35}$$

which satisfies modular invariance $Z(\tau,\bar{\tau})=Z(\tau+1,\bar{\tau}+1)=Z(-1/\tau,-1/\bar{\tau})$. The identification (2.34) means that the spectrum of string must be invariant under $X\to\hat{B}X$. By inserting the orbifold projection $\mathbf{P}_{\mathrm{orb}}$ in this trace, we can remove the states variant under the action of $\hat{B}$ in untwisted sector such that the partition function of the orbifold for untwisted sector is

$$\mathrm{Tr}_{\mathrm{untwisted}}\left[\mathbf{P}_{\mathrm{orb}}q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right]=\mathrm{Tr}_{\mathrm{untwisted}}\left[\frac{1}{M}\sum_{b\in\mathbb{Z}_M}(\hat{B})^b q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] \tag{2.36}$$

$$=:\frac{1}{M}\sum_{b\in\mathbb{Z}_M}Z_{(0,b)}(\tau,\bar{\tau}), \tag{2.37}$$

where M is the order of the action of $\hat{B}$.

Modular invariance requires the twisted sectors in compensation for the orbifold projection. If an original coordinate is non-compact $X\in\mathbb{R}$, the term $Z_{(0,b)}(\tau,\bar{\tau})$ corresponds to the boson with the boundary conditions

$$X(z+2\pi, \bar{z}+2\pi) = X(z, \bar{z}), \tag{2.38}$$

$$X(z+2\pi\tau, \bar{z}+2\pi\bar{\tau}) = (\hat{B})^b X(z, \bar{z}), \tag{2.39}$$

in terms of Lagrangian formalism. However, we cannot say that for a generic original X. For example, when X is expressed by fermion description, the modular transformations of $Z_{(0,b)}(\tau, \bar{\tau})$ is non-obvious from the boundary condition. It is difficult to define the twisted sectors by the spacial boundary conditions of the fields on worldsheet, though it is important to investigate them. Then, we define a-twisted sector as a Hilbert space for which the partition function is written as $Z_{(a,0)}(\tau, \bar{\tau})$, where $Z_{(a,0)}(\tau, \bar{\tau})$ denotes the one obtained by the S-modular transformation (2.4) of $Z_{(0,a)}(\tau, \bar{\tau})$. In general, $Z_{(a,0)}(\tau, \bar{\tau})$ is potentially not invariant under the T-modular transformation (2.3). This means that the states in a-twisted sector are incomplete or missing to satisfy the identification (2.34) or the consistency, so we need to find the other terms. We also define $Z_{(a,b+a)}(\tau, \bar{\tau})$ and $Z_{(b,-a)}(\tau, \bar{\tau})$ as the T and S-modular transformation (2.3), (2.4) of $Z_{(a,b)}(\tau, \bar{\tau})$. We write the partition function for a-twisted sector as

$$\mathrm{Tr}_{a-\mathrm{twisted}} \left[\frac{1}{M} \sum_{b \in \mathbb{Z}_{M_a}} (\hat{B}_a)^b q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} \right] = \frac{1}{M} \sum_{b \in \mathbb{Z}_{M_a}} Z_{(a,b)}(\tau, \bar{\tau}), \tag{2.40}$$

where M_a is the order of the action of $\hat{B}_a$. We actually meet the situation that $M \neq M_a$ in our convention of $Z_{(a,b)}(\tau, \bar{\tau})$ in calculating asymmetric orbifolds. This is because difference between the left and right-moving zero point energy causes the phase factor in the trace under the T-modular transformation (2.3). Eventually, the extra action of the order $\frac{M_a}{M}$ in $\hat{B}_a$ corresponds to the restriction on the level matching condition. In our method, any terms which possess an order more than M periodicity cancel each other. Thus we can easily write the partition function in a-twisted sector as

$$\sum_{b \in \mathbb{Z}_{M_a}} Z_{(a,b)}(\tau, \bar{\tau}) = \sum_{b \in \mathbb{Z}_M} Z'_{(a,b)}(\tau, \bar{\tau}), \tag{2.41}$$

after summing over the phase factors. In this way, we can take appropriate matching with a coefficient $\frac{1}{M}$ in (2.40). However, it is helpful to use the fact that the $\hat{B}'$-variant states are removed when we actually investigate the states as string excitations in w-twisted sector. This procedure is similar to studying the Gepner model related to spectral flow [30].

As a simple example of orbifold by twisting, we compute the partition function of orbifold $S^1/\mathbb{Z}_2$ from the original manifold S^1 by an identification $X \simeq X' = -X$ as explained above. This identification is realized by the discrete group $\mathbb{Z}_2 : S^1 \to S^1$ defined by the generator $-\mathbf{1}|_X : X \to -X$. This group action has fixed points at $X = 0$ and $X = \pi R$. The partition function of the original S^1 is (2.18). Inserting the orbifold projection operator in untwisted sector, we obtain

$$\mathrm{Tr}_{\mathrm{untwisted}}\left[\frac{1+(-\mathbf{1}|_X)}{2}q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] \tag{2.42}$$

$$=\frac{1}{2}Z_R^{S^1}(\tau,\bar{\tau})+\frac{1}{2}\mathrm{Tr}_{\mathrm{untwisted}}\left[(-\mathbf{1}|_X)q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] \tag{2.43}$$

$$\equiv\frac{1}{2}Z_{R(0,0)}^{S^1/\mathbb{Z}_2}+\frac{1}{2}Z_{R(0,1)}^{S^1/\mathbb{Z}_2}. \tag{2.44}$$

The momentum zero modes in $Z_{R(0,1)}^{S^1/\mathbb{Z}_2}$ are $p_L=0$, $p_R=0$, and other terms cancel each other, so the second term in (2.44) is

$$Z_{R(0,1)}^{S^1/\mathbb{Z}_2}\equiv\mathrm{Tr}_{\mathrm{untwisted}}\left[(-\mathbf{1}|_X)q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] \tag{2.45}$$

$$=q^{-\frac{1}{24}}\bar{q}^{-\frac{1}{24}}\cdot\frac{1}{\prod_{n=1}^{\infty}(1+q^n)\prod_{n'=1}^{\infty}(1+\bar{q}^{n'})} \tag{2.46}$$

$$=2\left|\frac{\eta}{\theta_2}\right|, \tag{2.47}$$

where η is eta function, $\theta_2:=\theta_2(\tau,0)$ is theta function explained in appendix A. Once we calculate (2.47), we can immediately investigate the terms in twisted sector $Z_{R(1,0)}^{S^1/\mathbb{Z}_2}$ and $Z_{R(1,1)}^{S^1/\mathbb{Z}_2}$ by the S and T-modular transformation (2.4), (2.3) as follows,

$$Z_{R(1,0)}^{S^1/\mathbb{Z}_2}\equiv Z_{R(0,1)}^{S^1/\mathbb{Z}_2}|_S=2\left|\frac{\eta}{\theta_4}\right|, \tag{2.48}$$

$$Z_{R(1,1)}^{S^1/\mathbb{Z}_2}\equiv Z_{R(1,0)}^{S^1/\mathbb{Z}_2}|_T=2\left|\frac{\eta}{\theta_3}\right|. \tag{2.49}$$

The total partition function is

$$Z_R^{S^1/\mathbb{Z}_2}=\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}Z_{R(a,b)}^{S^1/\mathbb{Z}_2} \tag{2.50}$$

$$=\frac{1}{2}Z_R^{S^1}+\left|\frac{\eta}{\theta_2}\right|+\left|\frac{\eta}{\theta_4}\right|+\left|\frac{\eta}{\theta_3}\right|. \tag{2.51}$$

This result is the same as calculated on path integral or trace.

2.2.2 Asymmetric twist

Next, we consider the orbifold twist which acts on left and right-movers asymmetrically. At first, we introduce the chiral reflection, namely, a simple T-duality transformation

$$-\mathbf{1}_L: X=X_L+X_R\to X'=-X_L+X_R, \tag{2.52}$$

$$-\mathbf{1}_R: X=X_L+X_R\to X'=X_L-X_R. \tag{2.53}$$

We shall apply it to the two-dimensional torus with the $SO(4)$-symmetry enhancement $T^2[SO(4)]$. As confirmed in (2.30), $T^2[SO(4)]$ is the product of two circles with self-dual radius $S^1[SU(2)]_1 \times S^1[SU(2)]_2$. We express the two coordinates by

$$X^1 = X_L^1 + X_R^1 \in S^1[SU(2)]_1, \; X^2 = X_L^2 + X_R^2 \in S^1[SU(2)]_2. \tag{2.54}$$

We would like to consider the action of right chiral reflection on X^1, X^2-directions

$$(-\mathbf{1}_R)^{\otimes 2} : X_R^a \to -X_R^a \; (a = 1, 2). \tag{2.55}$$

$$\left\{(-\mathbf{1}_R)^{\otimes 2}\right\}^2 = \mathbf{1}, \tag{2.56}$$

where $\mathbf{1}$ is identity operator, so we assume that the chiral reflection has the order two periodicity here. Another type of chiral reflection is discussed later. We focus on X_1, and define a sub-sector of X_1 such that

$$\mathrm{Tr}_{\mathrm{sub}\ X_1}\left[q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] = Z^{S^1}_{R=1}(\tau, \bar{\tau}), \tag{2.57}$$

where $Z^{S^1}_{R=1}(\tau, \bar{\tau})$ is the partition function of a circle with self-dual radius $R = 1$ in (2.27). The momentum zero mode is changed such that $p_L \to p_L, p_R \to -p_R$, namely, $n \to w, w \to n$ under the action $(-\mathbf{1}_R)$. Inserting $(-\mathbf{1}_R)$ in the trace, the terms are removed except for $w = n$, and we obtain

$$\mathrm{Tr}_{\mathrm{sub}\ X_1}\left[(-\mathbf{1}_R)q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] = \frac{q^{\frac{-1}{24}}\bar{q}^{\frac{-1}{24}}}{\prod_{n=1}^{\infty}(1-q^n)\prod_{n'=1}^{\infty}(1+\bar{q}^{n'})}\sum_{n\in\mathbb{Z}} q^{\frac{1}{4}(2n)^2} \tag{2.58}$$

$$= \frac{\theta_3(2\tau)}{\eta} \cdot \overline{\sqrt{2}\left(\sqrt{\frac{\eta}{\theta_2}}\right)} \tag{2.59}$$

$$= \frac{\theta_3(2\tau)}{\eta} \cdot \overline{\left(\sqrt{\frac{\theta_3\theta_4}{\eta}}\right)}. \tag{2.60}$$

Similarly, we define an untwisted sector of (X_1, X_2) such that

$$\mathrm{Tr}_{\mathrm{untwisted}}\left[q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] = \left\{Z^{S^1}_{R=1}(\tau, \bar{\tau})\right\}^2. \tag{2.61}$$

By the result of (2.60), we obtain

$$\mathrm{Tr}_{\mathrm{untwisted}}\left[(-\mathbf{1}_R)^{\otimes 2}q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right] = \left\{\frac{\theta_3(2\tau)}{\eta} \cdot \overline{\sqrt{2}\left(\sqrt{\frac{\theta_3\theta_4}{\eta}}\right)}\right\}^2 \tag{2.62}$$

$$= \frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2 + \left(\frac{\theta_4}{\eta}\right)^2\right\}\overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)} \tag{2.63}$$

$$\equiv Z^{T^2[SO(4)]}_{[0,2](0,1)}(\tau, \bar{\tau}). \tag{2.64}$$

In this way, the asymmetric twists can act on certain string geometries.

When we examine the action of the asymmetric twists, the other descriptions of bosonic fields can give the aspects which it is difficult to find in original one as the coordinates of spacetime. For example, a description of $T^2[SO(4)]$ is also given by the fermionization introducing free fermions, which correspond to compact bosons with radius $\sqrt{2}$ as follows.

We introduce the bosons Y^1, Y^2 so that

$$(Y^1, Y^2) = (Y_L^1 + Y_R^1, Y_L^2 + Y_R^2) \in T^2[SO(4)], \tag{2.65}$$

and $Y^\mu \simeq Y^\mu + 2\sqrt{2}\pi$ $(\mu = 1, 2)$. They can be described as Majorana fermions λ_L^i, λ_R^i $(i = 1, 2, 3, 4)$. In these fermion description, we can immediately derive the result (2.64) from the original (2.30). On the other hand, the coordinates of $S^1_{\text{self-dual}} \times S^1_{\text{self-dual}}$ are given by

$$X^1 \equiv \frac{1}{\sqrt{2}}(Y^1 + Y^2), \tag{2.66}$$

$$X^2 \equiv \frac{1}{\sqrt{2}}(Y^1 - Y^2), \tag{2.67}$$

which are compactified as $X^a \sim X^a + 2\pi$ $(a = 1, 2)$. This is matching with the result of (2.30).

In considering the action of right chiral reflection in (2.56), it is assumed that the square of the chiral reflection $(-\mathbf{1}_R)$ acts as the identity over the Hilbert space of the untwisted sectors of bosons. However, it is not necessary in this case.

The chiral reflection $(-\mathbf{1}_R)^{\otimes 2} : (X_R^1, X_R^2) \to (-X_R^1, -X_R^2)$, equivalently, $(Y_R^1, Y_R^2) \to (-Y_R^1, -Y_R^2)$, is interpretable as the sign inversion of two of λ_R^1,...,λ_R^4 in terms of the fermion description. Moreover, when we consider the bosonization of these fermions, we have two options of $\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = 1$ or $\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = (-1)^{F_R}|_\lambda$ as follows. Here, $(-1)^{F_R}|_\lambda$ acts as the sign inversion of any states in the Ramond sector of λ_R^i in $T^2[SO(4)]$.

(a) $\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = \mathbf{1}$

In this case, we bosonize $\lambda^{1,\cdots,4}$ such that

$$\lambda_R^1 \pm \lambda_R^3 \equiv \sqrt{2} e^{\pm i H_R^1}, \quad \lambda_R^2 \pm \lambda_R^4 \equiv \sqrt{2} e^{\pm i H_R^2}, \tag{2.68}$$

and define the spin fields such that

$$S_{\epsilon_1, \epsilon_2} \equiv e^{\frac{i}{2}\sum_{i=1}^2 \epsilon_i H_R^i}, \quad (\epsilon_i = \pm 1). \tag{2.69}$$

When $(-\mathbf{1}_R)^{\otimes 2}$ acts as

$$(-\mathbf{1}_R)^{\otimes 2} : (\lambda_R^1, \lambda_R^2, \lambda_R^3, \lambda_R^4) \to (\lambda_R^1, \lambda_R^2, -\lambda_R^3, -\lambda_R^4), \tag{2.70}$$

we find

$$(-\mathbf{1}_R)^{\otimes 2} : (H_R^1, H_R^2) \to (-H_R^1, -H_R^2), \tag{2.71}$$

so $\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = \mathbf{1}$ for all the states in the untwisted sector.

(b) $\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = (-1)^{F_R}|_\lambda$

In this case, we bosonize $\lambda^{1,\ldots,4}$ such that

$$\lambda_R^1 \pm \lambda_R^2 \equiv \sqrt{2}e^{\pm iH'^1_R}, \quad \lambda_R^3 \pm \lambda_R^4 \equiv \sqrt{2}e^{\pm iH'^2_R}, \tag{2.72}$$

and define the spin fields such that

$$S'_{\epsilon_1,\epsilon_2} \equiv e^{\frac{i}{2}\sum_{i=1}^2 \epsilon_i H'^i_R}, \quad (\epsilon_i = \pm 1). \tag{2.73}$$

When $(-\mathbf{1}_R)^{\otimes 2}$ acts as

$$(-\mathbf{1}_R)^{\otimes 2} : (\lambda_R^1, \lambda_R^2, \lambda_R^3, \lambda_R^4) \to (\lambda_R^1, \lambda_R^2, -\lambda_R^3, -\lambda_R^4), \tag{2.74}$$

we find

$$(-\mathbf{1}_R)^{\otimes 2} : (H_R^1, H_R^2) \to (H_R^1, H_R^2 + \pi), \tag{2.75}$$

so $\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = -\mathbf{1}$ for the R sector, while $\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = \mathbf{1}$ for the NS sector in the fermion description. We express this action by $(-1)^{F_R}|_\lambda$. Namely,

$$\{(-\mathbf{1}_R)^{\otimes 2}\}^2 = (-1)^{F_R}|_\lambda : Z^{T^2[SO(4)]}(\tau,\bar{\tau}) \to \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^4 + \left|\frac{\theta_4}{\eta}\right|^4 - \left|\frac{\theta_2}{\eta}\right|^4\right\}. \tag{2.76}$$

In this way, we can define $(-\mathbf{1}_R)^{\otimes 2}$ as $\mathbb{Z}_4$ action on the untwisted R sector in the fermionized bosonic sector as well as the $\mathbb{Z}_2$ action. Then, the orbifold projection can be written as

$$\frac{\mathbf{1} + (-\mathbf{1}_R)^{\otimes 2} + (-1)^{F_R}|_\lambda + (-\mathbf{1}_R)^{\otimes 2}\cdot(-1)^{F_R}|_\lambda}{4}. \tag{2.77}$$

We can also define $(-\mathbf{1}_L)^{\otimes 4}$ as $\mathbb{Z}_4$ action as well as $\mathbb{Z}_2$ similarly.

Next, we shall attempt to apply twisting by $\{(-1)^{F_R}|_\lambda\}^{\otimes 2}$ to four-dimensional torus $T^2[SO(4)]_1 \times T^2[SO(4)]_2$. At first, the partition function of $T^2[SO(4)]_1 \times T^2[SO(4)]_2$ is the square of $T^2[SO(4)]$'s one as follows.

$$Z^{T^2[SO(4)]_1\times T^2[SO(4)]_2}(\tau,\bar{\tau}) \equiv \left\{Z^{T^2[SO(4)]}(\tau,\bar{\tau})\right\}^2 \tag{2.78}$$

$$= \frac{1}{2}\left\{Z^{T^4[SO(8)]}(\tau,\bar{\tau}) + \left|\frac{\theta_3\theta_4}{\eta^2}\right|^4 + \left|\frac{\theta_4\theta_2}{\eta^2}\right|^4 + \left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}, \tag{2.79}$$

where $Z^{T^4[SO(8)]}$ is the partition function of four dimensional torus with the $SO(8)$-symmetry enhancement $T^4[SO(8)]$,

$$Z^{T^4[SO(8)]}(\tau,\bar{\tau}) = \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8\right\}. \tag{2.80}$$

This implies $T^2[SO(4)] \times T^2[SO(4)] \simeq T^4[SO(8)]/\mathbb{Z}_2$. Inserting the orbifold projection, by (2.76), the partition function for untwisted sector can be written as

$$\frac{1}{2}\left\{Z^{T^2[SO(4)]}(\tau,\bar\tau)\right\}^2 + \frac{1}{2}\cdot\frac{1}{2}\left\{Z^{T^4[SO(8)]}(\tau,\bar\tau) + \left|\frac{\theta_3\theta_4}{\eta^2}\right|^4 - \left|\frac{\theta_4\theta_2}{\eta^2}\right|^4 - \left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}. \quad (2.81)$$

By the modular transformation of the second term, we can find the partition function for twisted sectors.

$$\begin{aligned}
&\frac{1}{2}\left\{Z^{T^4[SO(8)]}(\tau,\bar\tau) + \left|\frac{\theta_3\theta_4}{\eta^2}\right|^4 - \left|\frac{\theta_4\theta_2}{\eta^2}\right|^4 - \left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}\\
\xrightarrow{S}\ &\frac{1}{2}\left\{Z^{T^4[SO(8)]}(\tau,\bar\tau) - \left|\frac{\theta_3\theta_4}{\eta^2}\right|^4 + \left|\frac{\theta_4\theta_2}{\eta^2}\right|^4 - \left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}\\
\xrightarrow{T}\ &\frac{1}{2}\left\{Z^{T^4[SO(8)]}(\tau,\bar\tau) - \left|\frac{\theta_3\theta_4}{\eta^2}\right|^4 - \left|\frac{\theta_4\theta_2}{\eta^2}\right|^4 + \left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}
\end{aligned}$$

Summing over them, we find the partition function of this orbifold is $Z^{T^4[SO(8)]}(\tau,\bar\tau)$ eventually. Namely, the partition function of $T^4[SO(8)]$ can be derived from $T^2[SO(4)] \times T^2[SO(4)]$ by asymmetric twisting $(-1)^{F_R}|_\lambda$ [19].

3 Asymmetric orbifolds on string theory

We discuss type II string vacua constructed from asymmetric orbifolds of the ten-dimensional flat background given by

$$M^{1,3} \times \mathcal{M}, \quad (3.1)$$

where $M^{1,3}$ ($X^{0,1,2,3}$-directions) is the four-dimensional Minkowski spacetime. In this paper, we mainly address the simple cases $\mathcal{M} = \mathbb{R} \times T^5$ or $\mathcal{M} = T^6$. In the former case, we consider the orbifolding defined by the twist operator $\mathcal{T}_{2\pi R}$ acting on $\mathbb{R}$ together with something acting on the other directions in the torus T^5 at the same time.

We schematically write the one-loop vacuum energy density for type II string vacua as

$$\int_F \frac{d\tau d\bar\tau}{4\tau_2}\,|\eta(\tau)|^4\, Z_l(\tau,\bar\tau) Z(\tau,\bar\tau). \quad (3.2)$$

F denotes the fundamental region for the moduli space of the torus, and $Z_l(\tau,\bar\tau)$ as the partition function of the contribution from a temporal direction X^0 and a spacial direction X^1 of the spacetime. The factor $\frac{d\tau d\bar\tau}{4\tau_2}|\eta(\tau)|^4 Z_l(\tau,\bar\tau)$ in (3.2) is modular invariant, so the remaining part $Z(\tau,\bar\tau)$ must also be invariant. Moreover, we consider $Z(\tau,\bar\tau)$ to decompose it into the non-compact bosonic sector $Z^{X^{2,3}}(\tau,\bar\tau)$ for the transverse contribution for four-dimensional Minkovski coordinate and others as follows.

$$Z(\tau,\bar\tau) = Z^{X^{2,3}}(\tau,\bar\tau) Z^{X_{\mathcal{M}}}(\tau,\bar\tau) Z^{\psi_L}(\tau) Z^{\psi_R}(\bar\tau). \quad (3.3)$$

Here, $Z^{X_{\mathcal{M}}}(\tau,\bar{\tau})$ corresponds to the compact bosonic part, but this original coordinates before orbifolding do not have to be compact. $Z^{\psi_L}(\tau)$, $Z^{\psi_R}(\bar{\tau})$ denote left and right-moving fermionic partition functions.

When we analyze the partition function of the orbifold, we need mainly calculate the twisting of $Z^{X_{\mathcal{M}}}(\tau,\bar{\tau})$, $Z^{\psi_L}(\tau)$ and $Z^{\psi_R}(\bar{\tau})$. In this paper, we take the orbifold twists as the combinations of the left and right-chiral reflection $-\mathbf{1}_L$, $-\mathbf{1}_R$, the sign flip of Ramond sector $(-1)^{F_R}$, and translation $\mathcal{T}_{2\pi R}$. Such a orbifold twist acts on each sector at the same time, so it is convenient to investigate the behavior of the partition function under the modular transformation for each. In computing the partition function of the orbifold defined by a twisting σ acting on $\mathcal{M}$, we start with the calculation of the building blocks for each sector $Z^{X_{\mathcal{M}}}_{(a,b)}(\tau,\bar{\tau})$, $Z^{\psi_L}_{(a,b)}(\tau)$ and $Z^{\psi_R}_{(a,b)}(\bar{\tau})$. We assume that M is a maximal order of σ in all sectors, and require the modular covariance such as

$$Z^{X_{\mathcal{M}}}_{(a,b)}(\tau,\bar{\tau})|_S = Z^{X_{\mathcal{M}}}_{(b,-a)}(\tau,\bar{\tau}), \tag{3.4}$$

$$Z^{X_{\mathcal{M}}}_{(a,b)}(\tau,\bar{\tau})|_T = Z^{X_{\mathcal{M}}}_{(a,a+b)}(\tau,\bar{\tau}). \tag{3.5}$$

The partition function of the orbifold can be written as

$$Z(\tau,\bar{\tau}) = \frac{1}{M}\sum_{a,b\in\mathbb{Z}_M} Z^{X^{3,4}}(\tau,\bar{\tau}) Z^{X_{\mathcal{M}}}_{(a,b)}(\tau,\bar{\tau}) Z^{\psi_L}_{(a,b)}(\tau) Z^{\psi_R}_{(a,b)}(\bar{\tau}). \tag{3.6}$$

This is clearly modular invariant. Therefore, we shall calculate the building blocks before investigating the orbifolds.

3.1 Building blocks

3.1.1 Bosonic sector

We begin with the building block of the translation $\mathcal{T}_{2\pi R}$ acting on a non-compact $\mathbb{R}$. Actually, we have already calculated it in (2.10)-(2.12). Namely, this building block is written as

$$Z_{R(w,m)}(\tau,\bar{\tau}) = \frac{R}{\sqrt{\tau}_2|\eta(\tau)|^2} e^{-\frac{\pi R^2}{\tau_2}|w\tau+m|^2} \quad (w,m\in\mathbb{Z}), \tag{3.7}$$

$$Z_{R(w,m)}(\tau,\bar{\tau})|_{\mathrm{T}} = Z_{R(w,m+w)}(\tau,\bar{\tau}), \quad Z_{R(w,m)}(\tau,\bar{\tau})|_{\mathrm{S}} = Z_{R(m,-w)}(\tau,\bar{\tau}). \tag{3.8}$$

Next, we consider the building block $\mathbf{Z}^{T^2[SO(4)]}_{(a,b)}(\tau,\bar{\tau})$ which corresponds to the twist $(-1)^{F_R}|_\lambda$ acting on the two-dimensional torus with the $SO(4)$-symmetry enhancement $T^2[SO(4)]$ as mentioned in subsection 2.2. We define $\mathbf{Z}^{T^2[SO(4)]}_{(0,0)}(\tau,\bar{\tau}) = Z^{T^2[SO(4)]}(\tau,\bar{\tau})$, and from (2.76),

$$\mathbf{Z}^{T^2[SO(4)]}_{(0,1)}(\tau,\bar{\tau}) = \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^4 + \left|\frac{\theta_4}{\eta}\right|^4 - \left|\frac{\theta_2}{\eta}\right|^4\right\}.$$

Then, calculating the modular transformation

$$\mathbf{Z}^{T^2[SO(4)]}_{(a,b)}(\tau,\bar{\tau})|_S = \mathbf{Z}^{T^2[SO(4)]}_{(b,-a)}(\tau,\bar{\tau}), \tag{3.9}$$

$$\mathbf{Z}^{T^2[SO(4)]}_{(a,b)}(\tau,\bar{\tau})|_T = \mathbf{Z}^{T^2[SO(4)]}_{(a,a+b)}(\tau,\bar{\tau}), \tag{3.10}$$

we obtain

$$\mathbf{Z}^{T^2[SO(4)]}_{(a,b)}(\tau,\bar{\tau}) \equiv \begin{cases} \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^4 + \left|\frac{\theta_4}{\eta}\right|^4 + \left|\frac{\theta_2}{\eta}\right|^4\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^4 + \left|\frac{\theta_4}{\eta}\right|^4 - \left|\frac{\theta_2}{\eta}\right|^4\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\ \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^4 - \left|\frac{\theta_4}{\eta}\right|^4 + \left|\frac{\theta_2}{\eta}\right|^4\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left\{-\left|\frac{\theta_3}{\eta}\right|^4 + \left|\frac{\theta_4}{\eta}\right|^4 + \left|\frac{\theta_2}{\eta}\right|^4\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1). \end{cases} \tag{3.11}$$

We can also consider N-dimensional ones $\mathbf{Z}^{T^N[SO(2N)]}_{(a,b)}(\tau,\bar{\tau})$ as follows.

$$\begin{aligned} &\mathbf{Z}^{T^N[SO(2N)]}_{(a,b)}(\tau,\bar{\tau}) \\ &\equiv \begin{cases} \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N} + \left|\frac{\theta_4}{\eta}\right|^{2N} + \left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N} + \left|\frac{\theta_4}{\eta}\right|^{2N} - \left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\ \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N} - \left|\frac{\theta_4}{\eta}\right|^{2N} + \left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left\{-\left|\frac{\theta_3}{\eta}\right|^{2N} + \left|\frac{\theta_4}{\eta}\right|^{2N} + \left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1). \end{cases} \end{aligned} \tag{3.12}$$

Next, we introduce the notation of the partition function, $Z^{T^N[SO(2N)]}_{[\alpha,\beta](a,b)}(\tau,\bar{\tau})$. This denotes the building blocks obtained by the asymmetric action of the sign inversion

$$\sigma_{[\alpha,\beta]}|_{T^N[SO(2N)]} = (-\mathbf{1}_L)^{\otimes\alpha} \otimes (-\mathbf{1}_R)^{\otimes\beta}|_{T^N[SO(2N)]}, \tag{3.13}$$

acting on $T^N[SO(2N)]$ bosonic sector with assumption $\sigma^2_{[\alpha,\beta]} = \mathbf{1}$. At first, we introduce

$$Z^{T^N[SO(2N)]}_{[\alpha,\beta](0,0)}(\tau,\bar{\tau}) = \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N} + \left|\frac{\theta_4}{\eta}\right|^{2N} + \left|\frac{\theta_2}{\eta}\right|^{2N}\right\}. \tag{3.14}$$

Then, we define

$$\begin{aligned} &Z^{T^N[SO(2N)]}_{[\alpha,\beta](0,1)}(\tau,\bar{\tau}) = Z^{T^N[SO(2N)]}_{[\alpha,\beta](0,-1)}(\tau,\bar{\tau}) \\ =&\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^{\frac{\alpha}{2}}\left(\frac{\theta_4}{\eta}\right)^{\frac{1}{2}(2N-\alpha)}\left(\frac{\overline{\theta_3}}{\eta}\right)^{\frac{\beta}{2}}\left(\frac{\overline{\theta_4}}{\eta}\right)^{\frac{1}{2}(2N-\beta)} + \left(\frac{\theta_4}{\eta}\right)^{\frac{\alpha}{2}}\left(\frac{\theta_3}{\eta}\right)^{\frac{1}{2}(2N-\alpha)}\left(\frac{\overline{\theta_4}}{\eta}\right)^{\frac{\beta}{2}}\left(\frac{\overline{\theta_3}}{\eta}\right)^{\frac{1}{2}(2N-\beta)}\right\} \end{aligned} \tag{3.15}$$

and

$$Z^{T^N[SO(2N)]}_{[\alpha,\beta](1,0)}(\tau,\bar\tau) = Z^{T^N[SO(2N)]}_{[\alpha,\beta](-1,0)}(\tau,\bar\tau) \tag{3.16}$$
$$=\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^{\frac{\alpha}{2}}\left(\frac{\theta_2}{\eta}\right)^{\frac{1}{2}(2N-\alpha)}\overline{\left(\frac{\theta_3}{\eta}\right)^{\frac{\beta}{2}}\left(\frac{\theta_2}{\eta}\right)^{\frac{1}{2}(2N-\beta)}}+\left(\frac{\theta_2}{\eta}\right)^{\frac{\alpha}{2}}\left(\frac{\theta_3}{\eta}\right)^{\frac{1}{2}(2N-\alpha)}\overline{\left(\frac{\theta_2}{\eta}\right)^{\frac{\beta}{2}}\left(\frac{\theta_3}{\eta}\right)^{\frac{1}{2}(2N-\beta)}}\right\}.$$

We may also define $\mathbf{Z}^{T^N[SO(2N)]}_{[\alpha,\beta](a,b)}(\tau,\bar\tau)$ as the building blocks obtained by the twist operator (3.13) with assumption $\sigma^2_{[\alpha,\beta]} = (-1)^{F_R}|_{\tilde\lambda}$. It can be written as

$$\mathbf{Z}^{T^N[SO(2N)]}_{[\alpha,\beta](a,b)}(\tau,\bar\tau) \equiv \begin{cases} \mathbf{Z}^{T^N[SO(2N)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ Z^{T^N[SO(2N)]}_{[\alpha,\beta](a,b)}(\tau,\bar\tau) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases} \tag{3.17}$$

Here, we look at building blocks corresponding to twist operator $\sigma_{[0,2]}$ acting on $T^2[SO(4)]$ as mentioned in subsection 2.2. Now we have two options $\sigma^2_{[0,2]} = \mathbf{1}$ or $\sigma^2_{[0,2]} = (-1)^{F_R}|_\lambda$. By the modular transformation, the building block corresponding to $\sigma^2_{[0,2]} = \mathbf{1}$ is described as

$$Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)$$
$$\equiv \begin{cases} \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^4+\left|\frac{\theta_4}{\eta}\right|^4+\left|\frac{\theta_2}{\eta}\right|^4\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ e^{\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2+(-1)^{\frac{a}{2}}\left(\frac{\theta_4}{\eta}\right)^2\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_3}{\eta^2}\right)}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2+(-1)^{\frac{b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_4}{\eta^2}\right)}\frac{1}{2}\left\{\left(\frac{\theta_4}{\eta}\right)^2-i(-1)^{\frac{a+b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1). \end{cases} \tag{3.18}$$

The building blocks corresponding to $\sigma^2_{[0,2]} = (-1)^{F_R}|_\lambda$ are

$$\mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)$$
$$\equiv \begin{cases} \mathbf{Z}^{T^2[SO(4)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases} \tag{3.19}$$

Similarly, the building blocks corresponding to twist operator $\sigma_{[0,4]}$ acting on $T^4[SO(8)]$

are as follows,

$$
\begin{aligned}
&Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)\\
&\equiv \begin{cases}
\frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^8+\left|\frac{\theta_4}{\eta}\right|^8+\left|\frac{\theta_2}{\eta}\right|^8\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z})\\
(-1)^{\frac{a}{2}}\overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)^2}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^4+\left(\frac{\theta_4}{\eta}\right)^4\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1)\\
(-1)^{\frac{b}{2}}\overline{\left(\frac{\theta_2\theta_3}{\eta^2}\right)^2}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^4+\left(\frac{\theta_2}{\eta}\right)^4\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z})\\
e^{-\frac{i\pi}{2}ab}\overline{\left(\frac{\theta_2\theta_4}{\eta^2}\right)^2}\frac{1}{2}\left\{\left(\frac{\theta_4}{\eta}\right)^4-\left(\frac{\theta_2}{\eta}\right)^4\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1),
\end{cases}
\end{aligned}
\tag{3.20}
$$

or

$$
\begin{aligned}
&\mathbf{Z}^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)\\
&\equiv \begin{cases}
\mathbf{Z}^{T^4[SO(8)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z})\\
Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1).
\end{cases}
\end{aligned}
\tag{3.21}
$$

We can also consider the building blocks corresponding to the twist operator $\sigma_{[2,4]}$ acting on $T^4[SO(8)]$. Namely,

$$
\sigma_{[2,4]} : (X^6, X^7, X^8_R, X^9_R) \to (-X^6, -X^7, -X^8_R, -X^9_R). \tag{3.22}
$$

It gives the building blocks (C.8) and (C.9). These building blocks are independent of which two bosons of four on $T^4[SO(8)]$ the action of the left chiral flips $(-\mathbf{1}_L)^{\otimes 2}$ acts on.

In addition, the building blocks defined by the twist operator $\sigma_{[N,N]}$ which acts on $T^N[SO(2N)]$ are as follows,

$$
Z^{T^N[SO(2N)]}_{[N,N](a,b)}(\tau,\bar\tau) \equiv \begin{cases}
Z^{T^N[SO(2N)]} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z})\\
\left|\frac{\theta_3\theta_4}{\eta^2}\right|^N & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1)\\
\left|\frac{\theta_2\theta_3}{\eta^2}\right|^N & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z})\\
\left|\frac{\theta_2\theta_4}{\eta^2}\right|^N & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1)
\end{cases}
\tag{3.23}
$$

or

$$
\begin{aligned}
&\mathbf{Z}^{T^N[SO(2N)]}_{[N,N](a,b)}(\tau,\bar\tau)\\
&\equiv \begin{cases}
\mathbf{Z}^{T^N[SO(2N)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z})\\
Z^{T^N[SO(2N)]}_{[N,N](a,b)}(\tau,\bar\tau) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1).
\end{cases}
\end{aligned}
\tag{3.24}
$$

In this notation, the building block $Z^{S^1/\mathbb{Z}_2}_{R(a,b)}$ in (2.51) can be written as

$$Z^{S^1/\mathbb{Z}_2}_{R(a,b)} \equiv \begin{cases} Z^{S^1}_R(\tau,\bar{\tau}) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ Z^{T^1[SO(2)]}_{[1,1](a,b)} & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1). \end{cases} \tag{3.25}$$

It may be convenient to utilize these results in our notation when we consider the variety of these products. For example, the twist $(\sigma_{[0,2]})^{\otimes 2}$ acting on $T^2[SO(4)]_1 \times T^2[SO(4)]_2$ has three cases. These building blocks can be written as follows.

(i) $\{(\sigma_{[0,2]})^{\otimes 2}\}^2 = \mathbf{1}$

$$\left(Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar{\tau})\right)^2 \tag{3.26}$$

$$= \begin{cases} \frac{1}{2}\left[Z^{T^4[SO(8)]}(\tau,\bar{\tau}) + \left|\frac{\theta_3\theta_4}{\eta^2}\right|^4 + \left|\frac{\theta_4\theta_2}{\eta^2}\right|^4 + \left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right] & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ \frac{1}{2}\left(Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar{\tau}) + Z^{T^4[SO(8)]}_{[4,4](a,b)}(\tau,\bar{\tau})\right) & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1) \end{cases} \tag{3.27}$$

(ii) $\{(\sigma_{[0,2]})^{\otimes 2}\}^2 = (-1)^{F_R}|_{T^2[SO(4)]_1} \otimes \mathbf{1}|_{T^2[SO(4)]_2}$ or $\mathbf{1}|_{T^2[SO(4)]_1} \otimes (-1)^{F_R}|_{T^2[SO(4)]_2}$

$$\mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar{\tau}) Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar{\tau}) \tag{3.28}$$

(iii) $\{(\sigma_{[0,2]})^{\otimes 2}\}^2 = (-1)^{F_R}|_{T^2[SO(4)]_1} \otimes (-1)^{F_R}|_{T^2[SO(4)]_2}$

$$\left(\mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar{\tau})\right)^2 \tag{3.29}$$

Next, we consider the building blocks defined by the twist operator

$$\sigma_{[2,2]}|_{T^2[SO(4)]_1} \otimes \sigma_{[0,2]}|_{T^2[SO(4)]_2}, \tag{3.30}$$

such that for left-moving,

$$(-\mathbf{1}_L)^{\otimes 2}: \ (X^6_L, X^7_L, X^8_L, X^9_L) \to (-X^6_L, -X^7_L, X^8_L, X^9_L). \tag{3.31}$$

One can find

$$Z^{T^2[SO(4)]}_{[2,2](0,1)}(\tau,\bar{\tau}) Z^{T^2[SO(4)]}_{[0,2](0,1)}(\tau,\bar{\tau}) = \overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)} \left|\frac{\theta_3\theta_4}{\eta^2}\right|^2 \left(\frac{\theta_3(2\tau)}{\eta}\right)^2, \tag{3.32}$$

and obtain three building blocks (C.11), (C.12), and (C.13).

Similarly, we can consider the twist operator

$$\sigma_{[1,2]}|_{T^2[SO(4)]_1} \otimes \sigma_{[1,2]}|_{T^2[SO(4)]_2} \tag{3.33}$$

such that for left-moving,

$$(-\mathbf{1}_L)^{\otimes 2}: \ (X_L^6, X_L^7, X_L^8, X^9) \to (X_L^6, -X_L^7, X_L^8, -X_L^9). \tag{3.34}$$

We find

$$\left\{Z^{T^2[SO(4)]}_{[1,2](0,1)}(\tau,\bar\tau)\right\}^2 = \overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)}\left|\frac{\theta_3\theta_4}{\eta^2}\right|^2\left(\frac{\theta_3(4\tau)}{\eta}\right)^2 \tag{3.35}$$

$$= \overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)}\left|\frac{\theta_3\theta_4}{\eta^2}\right|^2\cdot\frac{1}{2}\left\{\left(\frac{\theta_3(2\tau)}{\eta}\right)^2+\left(\frac{\theta_4(2\tau)}{\eta}\right)^2\right\}, \tag{3.36}$$

and obtain three building blocks (C.15), (C.16), and (C.17).

The difference between the choice of the coordinates (3.31) and (3.34) the reflection acts on yields different results (3.32) and (3.36). In considering $\mathbb{Z}_2$-twist, (3.32) corresponds to $T^2/\mathbb{Z}_2\times T^2$ and (3.36) does $T^2/\mathbb{Z}_2\times T^2/\mathbb{Z}_2$ respectively. This relation reflects (3.36) $\simeq$ (3.32)/$\mathbb{Z}_2$.

In this way, there are many variations of the building blocks by the twisting with $\sigma_{[a,b]}$. Practically, it is helpful to use the following relation in calculating the building block of some twisting on the product of $T^N[SO(2N)]$.

$$\begin{aligned}&Z^{T^{N_1}[SO(2N_1)]}_{[\alpha_1,\beta_1](a,b)}(\tau,\bar\tau)Z^{T^{N_2}[SO(2N_2)]}_{[\alpha_2,\beta_2](a,b)}(\tau,\bar\tau)\\ &=\begin{cases} Z^{T^{N_1}[SO(2N_1)]}(\tau,\bar\tau)Z^{T^{N_2}[SO(2N_2)]}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z})\\ \frac{1}{2}\left(Z^{T^{N_1+N_2}[SO(2N_1+2N_2)]}_{[\alpha_1+\alpha_2,\beta_1+\beta_2](a,b)}(\tau,\bar\tau)+Z^{T^{N_1+N_2}[SO(2N_1+2N_2)]}_{[2N_1-\alpha_1+\alpha_2,2N_1-\beta_1+\beta_2](a,b)}(\tau,\bar\tau)\right) & \\ \qquad (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1) & \end{cases}\end{aligned} \tag{3.37}$$

except for $\alpha_1=\beta_1=0$ or $\alpha_2=\beta_2=0$. We use more building blocks in this paper, but they are given by the similar way above. The remaining building blocks are summarized in appendix C.

3.1.2 Fermionic Sector

Fermion Partition Function

We next consider the fermionic sector. The fermionic part of the partition function of the type II string on ten-dimensional Minkowski coordinate is

$$Z^{\psi,\tilde\psi}_{\text{typeII}}(\tau,\bar\tau) = \frac{1}{4}|\mathcal{J}(\tau)|^2, \tag{3.38}$$

where

$$\mathcal{J}(\tau) \equiv \left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 - \left(\frac{\theta_2}{\eta}\right)^4. \tag{3.39}$$

We can confirm the modular covariance

$$\mathcal{J}(\tau)|_T = -e^{\frac{\pi i}{3}}\mathcal{J}(\tau) \tag{3.40}$$

$$\mathcal{J}(\tau)|_S = \mathcal{J}(\tau), \tag{3.41}$$

so (3.38) is modular invariant.

Desired free fermion chiral blocks are given by having some chiral twist operator act on $\mathcal{J}(\tau)$, and requiring the modular covariance.

Chiral Reflection on Fermionic Sector

In subsubsection 3.1.1, $(-\mathbf{1}_L)$ is defined as chiral reflection for bosonic sector

$$X_L \to -X_L.$$

In superstring theory, we have to take care of superconformal symmetry of the worldsheet. This means that the chiral reflection $(-\mathbf{1}_L)$ acting on worldsheet bosons should be defined as the chiral reflection of the worldsheet fermions which are partners of each boson at the same time,

$$\psi_L \to -\psi_L.$$

We also have two ways to define the chiral reflection $(-\mathbf{1}_L)$ similarly to the fermions in bosonic sector discussed in subsection 2.2. For example, we have to choose between the two, $\{(-\mathbf{1}_L)^{\otimes 2}\}^2 = \mathbf{1}$ and $\{(-\mathbf{1}_L)^{\otimes 2}\}^2 = (-1)^{F_L}|_\psi$. Here, $(-1)^{F_L}|_\psi$ acts as the sign inversion of all the states of the left-moving R sector in fermionic sector.

Chiral blocks

For fermionic sector, the partition function (3.38) consists of the product of the left and right-moving part, so one can construct the building blocks to consider the left and right-parts separately, and it is enough to do the one of them.

Let us study the chiral blocks, namely the left-parts of the building blocks on fermionic sector. It is convenient to first define the chiral block $\mathbf{J}_{(a,b)}(\tau)$, which is defined by $(-1)^{F_L}|_\psi$ like (3.11). We find

$$\mathbf{J}_{(0,b)}(\tau) = \begin{cases} \mathcal{J}(\tau) & (b \in 2\mathbb{Z}) \\ \left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 + \left(\frac{\theta_2}{\eta}\right)^4 & (b \in 2\mathbb{Z}+1). \end{cases} \tag{3.42}$$

Requiring the modular covariance

$$\left[\mathbf{J}_{(a,b)}(\tau)\overline{\mathcal{J}(\tau)}\right]|_S = \left[\mathbf{J}_{(b,-a)}(\tau)\overline{\mathcal{J}(\tau)}\right] \tag{3.43}$$

$$\left[\mathbf{J}_{(a,b)}(\tau)\overline{\mathcal{J}(\tau)}\right]|_T = \left[\mathbf{J}_{(a,a+b)}(\tau)\overline{\mathcal{J}(\tau)}\right], \tag{3.44}$$

we obtain

$$\mathbf{J}_{(a,b)}(\tau) \equiv \begin{cases} \mathcal{J}(\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ \left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 + \left(\frac{\theta_2}{\eta}\right)^4 & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}+1) \\ \left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 - \left(\frac{\theta_2}{\eta}\right)^4 & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}) \\ -\left\{\left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 + \left(\frac{\theta_2}{\eta}\right)^4\right\} & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}+1). \end{cases} \tag{3.45}$$

A modular covariant form of the left-chiral block always arises the phase $-\exp(\frac{\pi i}{3})$ under the modular transformation $T : \tau \to \tau + 1$ such as (3.40), and the phase should be counteracted with right-chiral ones like (3.44). $\mathbf{J}_{(a,b)}$ ($a \in 2\mathbb{Z}+1$, or $b \in 2\mathbb{Z}+1$) are non-vanishing, which signals the SUSY breaking in the left-moving sector.

In the same way as bosonic T^4 sector, we define $Z^{\psi}_{[\alpha](a,b)}$ as the chiral blocks of fermion obtained by $\mathbb{Z}_2$-chiral reflection $(-\mathbf{1}_L)^{\otimes\alpha}$, and $\mathbf{Z}^{\psi}_{[\alpha](a,b)}(\tau)$ as

$$\mathbf{Z}^{\psi}_{[\alpha](a,b)}(\tau) = \begin{cases} \mathbf{J}_{(\frac{a}{2},\frac{b}{2})}(\tau) & (b \in 2\mathbb{Z}) \\ \\ Z^{\psi}_{[\alpha](a,b)}(\tau) & (b \in 2\mathbb{Z}+1). \end{cases} \tag{3.46}$$

Let us look at the chiral block which is defined by $\mathbb{Z}_2$-chiral reflection $(-\mathbf{1}_L)^{\otimes 4}$ such that

$$(-\mathbf{1}_L)^{\otimes 4} : (\psi^6_L, \psi^7_L, \psi^8_L, \psi^9_L) \to (-\psi^6_L, -\psi^7_L, -\psi^8_L, -\psi^9_L), \quad |\mathbf{s}\rangle \to |\mathbf{s}'\rangle \tag{3.47}$$

$$\text{and } \{(-\mathbf{1}_L)^{\otimes 4}\}^2 = \mathbf{1}, \tag{3.48}$$

where we express the Ramond vacuum by

$$|\mathbf{s}\rangle := \exp\left(i \sum_{a=1}^{4} s_a \Psi^a\right), \ \mathbf{s} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}). \tag{3.49}$$

$(-\mathbf{1}_L)^{\otimes 4}$ acting on $\mathcal{J}$, we find

$$Z^{\psi}_{[4](0,b)}(\tau) = \begin{cases} \mathcal{J}(\tau) & (b \in 2\mathbb{Z}) \\ \\ e^{\frac{i\pi}{2}ab}\left\{\left(\frac{\theta_3}{\eta}\right)^2\left(\frac{\theta_4}{\eta}\right)^2 - \left(\frac{\theta_4}{\eta}\right)^2\left(\frac{\theta_3}{\eta}\right)^2 + 0\right\} & (b \in 2\mathbb{Z}+1), \end{cases} \tag{3.50}$$

and by requiring modular covariance, we obtain the chiral block

$$Z^{\psi}_{[4](a,b)}(\tau) \equiv \begin{cases} \mathcal{J}(\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ e^{\frac{i\pi}{2}ab}\left\{\left(\frac{\theta_3}{\eta}\right)^2\left(\frac{\theta_4}{\eta}\right)^2 - \left(\frac{\theta_4}{\eta}\right)^2\left(\frac{\theta_3}{\eta}\right)^2 + 0\right\} & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}+1) \\ e^{\frac{i\pi}{2}ab}\left\{\left(\frac{\theta_3}{\eta}\right)^2\left(\frac{\theta_2}{\eta}\right)^2 + 0 - \left(\frac{\theta_2}{\eta}\right)^2\left(\frac{\theta_3}{\eta}\right)^2\right\} & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}) \\ -e^{\frac{i\pi}{2}ab}\left\{0 + \left(\frac{\theta_2}{\eta}\right)^2\left(\frac{\theta_4}{\eta}\right)^2 - \left(\frac{\theta_4}{\eta}\right)^2\left(\frac{\theta_2}{\eta}\right)^2\right\} & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}+1). \end{cases} \tag{3.51}$$

Vanishing $Z^{\psi}_{[4](a,b)}(\tau)$ for any a, b implies that supersymmetry of spacetime potentially survives in the left-moving sector.

If we modify the chiral reflection so that (3.48) $\to \{(-\mathbf{1}_L)^{\otimes 4}\}^2 = (-1)^{F_L}|_{\psi}$, (3.50) is modified so that

$$\mathbf{Z}^{\psi}_{[4](0,b)}(\tau) = \begin{cases} \mathbf{J}_{(0,\frac{b}{2})}(\tau) & (b \in 2\mathbb{Z}) \\ Z^{\psi}_{[4](0,b)}(\tau) & (b \in 2\mathbb{Z}+1), \end{cases} \tag{3.52}$$

so we obtain the chiral block (C.32).

Similarly, one can also obtain the chiral blocks which are defined by $(-\mathbf{1}_L)^{\otimes 2}$. The chiral block corresponding to $\{(-\mathbf{1}_L)^{\otimes 2}\}^2 = \mathbf{1}$ is (C.33), and the one corresponding to $\{(-\mathbf{1}_L)^{\otimes 2}\}^2 = (-1)^{F_L}|_{\psi}$ is (C.34).

3.2 Non-supersymmetric asymmetric orbifolds

Orbifolds defined by $\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-1)^{F_L}|_{\psi_L} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4}$

Recently, the asymmetric orbifolds defined by twisting with $\mathbb{Z}_4$-chiral reflection have begun to be studied to attempt to construct non-supersymmetric string vacua with vanishing cosmological constant [25, 26]. It may be worth investigating them more. Then, referring to [25], we begin with study of the twists written by

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-1)^{F_L}|_{\psi_L} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4}, \tag{3.53}$$

where $\mathcal{T}_{2\pi R}$ acts on $\mathbb{R}$ (X^5-direction), $(-1)^{F_L}$ on left-moving fermionic sector and $(-\mathbf{1}_R)^{\otimes 4}$ on $T^4[SO(8)]$ ($X^{6,7,8,9}$-direction). Now, $\mathcal{M}$ in (3.1) is

$$\mathcal{M} = S^1 \times \mathbb{R} \times T^4[SO(8)]. \tag{3.54}$$

The twist can be defined by four ways from the description of (3.53) as follows.

$$\{(-\mathbf{1}_R)^{\otimes 4}\}^2 \equiv \mathbf{1} \tag{3.53a}$$

$$\{(-\mathbf{1}_R)^{\otimes 4}\}^2 \equiv (-1)^{F_R}|_{\psi_R} \tag{3.53b}$$

$$\{(-\mathbf{1}_R)^{\otimes 4}\}^2 \equiv (-1)^{F_R}|_{T^4[SO(8)]} \tag{3.53c}$$

$$\{(-\mathbf{1}_R)^{\otimes 4}\}^2 \equiv (-1)^{F_R}|_{T^4[SO(8)]} \otimes (-1)^{F_R}|_{\psi_R}\ , \tag{3.53d}$$

where $(-1)^{F_R}|_{T^4[SO(8)]}$ acts as the sign inversion of any states in the Ramond sector by the fermion description of bosonic sector $T^4[SO(8)]$. These twists construct the different orbifolds respectively. We call them model (a), (b), (c), (d) sorted by (3.53a)-(3.53d).

The twist (3.53) includes the translation, so the order of this action is infinite. Inserting the m-th power of twist operator (3.53) into the trace in the untwisted sector, we obtain

$$Z_{(0,m)}(\tau,\bar{\tau}) \equiv \mathrm{Tr}_{\text{untwisted}}\left[\left(\mathcal{T}_{2\pi R}\otimes(-1)^{F_L}\otimes(-\mathbf{1}_R)^{\otimes 4}\right)^m q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right] \tag{3.55}$$

$$= Z_{R(0,m)}\mathrm{Tr}_{\text{untwisted}}\left[\left((-1)^{F_L}\otimes(-\mathbf{1}_R)^{\otimes 4}\right)^m q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right], \tag{3.56}$$

where $Z_{R(0,m)}$ is a part of building block of S^1 in (2.10). Requiring the modular covariance of (3.56), we obtain the partition function of the general winding sector $Z_{(w,m)}(\tau,\bar{\tau})$. The partition function is given by summing over the winding numbers $w,m\in\mathbb{Z}$ along the X^5-direction,

$$Z(\tau,\bar{\tau}) \sim \sum_{w,m\in\mathbb{Z}} Z_{(w,m)}(\tau,\bar{\tau}). \tag{3.57}$$

Thus, the partition functions of (a)-(d) are written as follows respectively.

$$Z(\tau,\bar{\tau}) = \frac{1}{4}Z^{\text{tr}}_{M^4\times S^1}(\tau,\bar{\tau})\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau})Z^{T^4[SO(8)]}_{[0,4](w,m)}(\tau,\bar{\tau})\mathbf{J}_{(w,m)}(\tau)\overline{Z^{\psi}_{[4](w,m)}(\tau)} \tag{3.57a}$$

$$Z(\tau,\bar{\tau}) = \frac{1}{4}Z^{\text{tr}}_{M^4\times S^1}(\tau,\bar{\tau})\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau})Z^{T^4[SO(8)]}_{[0,4](w,m)}(\tau,\bar{\tau})\mathbf{J}_{(w,m)}(\tau)\overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.57b}$$

$$Z(\tau,\bar{\tau}) = \frac{1}{4}Z^{\text{tr}}_{M^4\times S^1}(\tau,\bar{\tau})\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau})\mathbf{Z}^{T^4[SO(8)]}_{[0,4](w,m)}(\tau,\bar{\tau})\mathbf{J}_{(w,m)}(\tau)\overline{Z^{\psi}_{[4](w,m)}(\tau)} \tag{3.57c}$$

$$Z(\tau,\bar{\tau}) = \frac{1}{4}Z^{\text{tr}}_{M^4\times S^1}(\tau,\bar{\tau})\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau})\mathbf{Z}^{T^4[SO(8)]}_{[0,4](w,m)}(\tau,\bar{\tau})\mathbf{J}_{(w,m)}(\tau)\overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.57d}$$

Here, the overall factor $\frac{1}{4}$ is due to the chiral GSO projections, and $Z^{\text{tr}}_{M^4\times S^1}(\tau,\bar{\tau})$ denotes $Z^{X^{3,4}}(\tau,\bar{\tau})Z^{S^1}(\tau,\bar{\tau})$ in (3.6).

In order to evaluate the partition function in operator formalism, we rewrite them by the Poisson resummation of the relevant partition function with respect to the temporal winding m along X^5-direction. Then, we can make sure that any tachyonic states do not appear in twisted sectors as well as the untwisted sector and that there are not negative norm states by writing down the explicit form of the partition function. Moreover, it is convenient to decompose the partition function with respect to the spatial winding w and the spin structures, and factor out the component of $Z^{\mathrm{tr}}_{M^4\times S^1}$;

$$Z(\tau,\bar{\tau}) = \frac{1}{4} Z^{\mathrm{tr}}_{M^4\times S^1}(\tau,\bar{\tau}) \sum_{\sigma_s,\tilde{\sigma}_s} \sum_{w\in\mathbb{Z}} Z_w^{(\sigma_s,\tilde{\sigma}_s)}(\tau,\bar{\tau}), \tag{3.58}$$

where $(\sigma_s,\tilde{\sigma}_s) = (\mathrm{NS},\mathrm{NS}),\ (\mathrm{NS},\mathrm{R}),\ (\mathrm{R},\mathrm{NS}),\ (\mathrm{R},\mathrm{R})$ denote the spin structures. We calculate the partition function (3.58) to divide into each of sectors, and investigate the states of each vacuum (a)-(d).

Orbifold model (a)

The total partition function (3.57a) vanishes because of $\overline{Z^{\psi}_{[4](w,m)}(\tau)} = 0$ in any w-sector. This reflects the bose-fermi cancellation $Z_w^{(*,\mathrm{NS})}(\tau,\bar{\tau}) = -Z_w^{(*,\mathrm{R})}(\tau,\bar{\tau})$ caused by the right-moving supersymmetry of spacetime in each w-sector. The twist operator commutes with 16 supercharges of the spacetime from the right-mover, so this vacuum preserves 1/4 of the spacetime supersymmetry.

The explicit form of the partition function (3.57a) is as follows.
$w \in 2\mathbb{Z}$

$$\begin{aligned} &Z_w^{(\mathrm{NS},\mathrm{NS})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{R},\mathrm{NS})}(\tau,\bar{\tau}) = Z_w^{(\mathrm{R},\mathrm{R})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{NS},\mathrm{R})}(\tau,\bar{\tau}) \\ &= \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2} \frac{1}{2} \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8 \end{aligned} \tag{3.59}$$

$w \in 2\mathbb{Z}+1$

$$\begin{aligned} Z_w^{(\mathrm{NS},\mathrm{NS})}(\tau,\bar{\tau}) &= -Z_w^{(\mathrm{NS},\mathrm{R})}(\tau,\bar{\tau}) = \sum_{a=0,1} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{2} (-1)^{an} q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2} \\ &\times \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4} \frac{1}{2} \left\{ (-1)^a \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4 + \left(\frac{\theta_2}{\eta}\right)^4 \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 \right\} \end{aligned} \tag{3.60}$$

$$\begin{aligned} Z_w^{(\mathrm{R},\mathrm{R})}(\tau,\bar{\tau}) &= -Z_w^{(\mathrm{R},\mathrm{NS})}(\tau,\bar{\tau}) = \sum_{a=0,1} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{2} (-1)^{an} q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2} \\ &\times \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4} \frac{1}{2} \left\{ (-1)^a \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4 + \left(\frac{\theta_2}{\eta}\right)^4 \right\} \left(\frac{\theta_2}{\eta}\right)^4 \end{aligned} \tag{3.61}$$

We can easily investigate the massless states in untwisted sector by looking for the combinations of left and right-mover which commute with the orbifold action. We exhibit such massless states on table 1. We find 64 bosons and 64 fermions, and confirm bose-

Table 1: Massless spectrum in untwisted sector in model (a)

spin structure	left		right	4D fields (d.o.f)
(NS, NS)	$\psi^{\mu}_{-1/2}\|0\rangle$ ($\mu = 2, ..., 9$)	$\otimes$	$\tilde{\psi}^{\nu}_{-1/2}\|0\rangle$ ($\nu = 2, ..., 5$)	graviton (2), 8 vectors (16), 14 (pseudo) scalars (14)
(R, NS)	$\|\mathbf{s}\rangle$	$\otimes$	$\tilde{\psi}^{\nu}_{-1/2}$ ($\nu = 6, ..., 9$)	16 Weyl fermions (32)
(NS, R)	$\psi^{\mu}_{-1/2}$ ($\mu = 2, ..., 9$)	$\otimes$	$\|\tilde{\mathbf{s}}\rangle$ ($\tilde{s}_3 + \tilde{s}_4 = 0$)	4 gravitini (8) 12 Weyl spinor (24)
(R, R)	$\|\mathbf{s}\rangle$	$\otimes$	$\|\tilde{\mathbf{s}}\rangle$ ($\tilde{s}_3 - \tilde{s}_4 = 0$)	8 vectors (16) 16 (pseudo) scalar (16)

fermi cancellation with the same number of bosons and their superpartners (fermions) at massless level as expected.

Naively, we can guess the massless spectrum does not arise in w-twisted sectors ($w \neq 0$) since non-zero winding w contributes to mass. On the other hand, the orbifold defined by the twist (3.57) is the same type as Scherk-Schwarz compactification [31,32], where the tachyonic states potentially arise in twisted sectors. Then, we focus on the terms of $w = \pm 1$-sector in the partition function (3.60), where the lightest excitations appear. Moreover, the leading contribution from the θ-part comes from $\theta_3(\tau, \frac{a}{2}) = 1 + (-1)^a q^{\frac{1}{2}} + ...$, and the related part in (3.60) is $|\theta_3|^8 - (-1)^n |\theta_4|^8$ after summing over $a = 1, 2$. Thus any states are massive when $n = 0$. The lightest excitation exists when $n = \pm 1$, and then the lowest conformal weight is

$$h_L = \frac{1}{4}\left(R \pm \frac{1}{2R}\right)^2, \ h_R = \frac{1}{2} + \frac{1}{4}\left(R \mp \frac{1}{2R}\right)^2. \tag{3.62}$$

The mass formula is given by

$$\frac{1}{4}m^2 = -\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2}(p_L^2 + p_R^2) = \frac{1}{4}\left(R - \frac{1}{2R}\right)^2 \geq 0. \tag{3.63}$$

The lowest mode is not tachyonic. It is interesting that the massless winding states exist at the special radius $R = 1/\sqrt{2}$.

Orbifold model (b)

The partition function vanishes for each winding sector with $w, m \in \mathbb{Z}$ since $\overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} = 0$ for $^\forall w \in 2\mathbb{Z}+1$ or $^\forall m \in 2\mathbb{Z}+1$, while $Z^{\psi}_{[2](w,m)}(\tau) = 0$ for $^\forall w, m \in 2\mathbb{Z}$. On the other hand, $((-\mathbf{1}_R)^{\otimes 4})^2 = (-1)^{F_R}|_{\tilde{\psi}}$, so any supercharges of the spacetime do not commute with the twist operator. This result means that this orbifold space loses all the spacetime supersymmetry in the same way that an orbifold $\mathbb{R}/\mathbb{Z}_2$ defined by the reflection twist loses the translational symmetry.

The explicit form of the partition function (3.57b) is as follows.

$w \in 4\mathbb{Z}$

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) \tag{3.64}$$
$$= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n}{2R}\right)^2} \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8 \right\} \left| \left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 \right|^2$$

$$Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau)$$
$$= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n+\frac12}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n+\frac12}{2R}\right)^2} \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8 \tag{3.65}$$

$w \in 4\mathbb{Z}+2$

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) = \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n+\frac12}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n+\frac12}{2R}\right)^2}$$
$$\times \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8 \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 \right\} \left\{ \overline{\left(\frac{\theta_3}{\eta}\right)^4} + \overline{\left(\frac{\theta_4}{\eta}\right)^4} \right\} \tag{3.66}$$

$$Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau)$$
$$= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n}{2R}\right)^2} \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8 \tag{3.67}$$

$w \in 2\mathbb{Z}+1$

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau) = \sum_{a=0,1} \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac12 (-1)^{an} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n}{2R}\right)^2}$$
$$\times \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac a2)}{\eta^2}\right)^4} \frac12 \left\{ (-1)^a \left(\frac{\theta_3(\tau,\frac a2)}{\eta}\right)^4 + \left(\frac{\theta_2}{\eta}\right)^4 \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 \right\} \tag{3.68}$$

$$Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) = \sum_{a=0,1} \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac12 (-1)^{an} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n}{2R}\right)^2}$$
$$\times \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac a2)}{\eta^2}\right)^4} \frac12 \left\{ (-1)^a \left(\frac{\theta_3(\tau,\frac a2)}{\eta}\right)^4 + \left(\frac{\theta_2}{\eta}\right)^4 \right\} \left(\frac{\theta_2}{\eta}\right)^4 \tag{3.69}$$

The massless spectrum is displayed in table 2. Indeed, the contribution of 32 bosons and 32 fermions cancel each other at massless level. However, they do not have superpartners, which is different characteristic from model (a). This is non-supersymmetric asymmetric orbifold with vanishing cosmological constant.

In the twisted sector, any tachyonic state do not appear, but the massless winding states exist at two special radii. One radius is the same as (a), namely, the massless point is $R = 1/\sqrt2$ in $w = \pm1$. Another is $R = 1/2\sqrt2$ in $w = \pm2$ as seen in (3.66).

Table 2: Massless spectrum in untwisted sector in model (b)

spin structure	4D fields
(NS, NS)	graviton, 8 vectors, 14 (pseudo) scalars
(R , NS)	16 Weyl fermions

Orbifold model (c)

The partition function vanishes, and the spacetime supersymmetry exists for the same reasons as model (a). The explicit form of the partition function (3.57c) is as follows.

$w \in 4\mathbb{Z}$

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar{\tau}) = Z_w^{(\mathrm{R,R})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar{\tau}) \tag{3.70}$$

$$= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}\frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^8+\left|\frac{\theta_4}{\eta}\right|^8\right\}\left|\frac{\theta_2}{\eta}\right|^8$$
$$+\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar{q}^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2}\frac{1}{2}\left|\frac{\theta_2}{\eta}\right|^{16} \tag{3.71}$$

$w \in 4\mathbb{Z}+2$

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{R,NS})} = Z_w^{(\mathrm{R,R})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar{\tau})$$
$$= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar{q}^{\ \frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}\frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^8-\left|\frac{\theta_4}{\eta}\right|^8\right\}\left|\frac{\theta_2}{\eta}\right|^8$$
$$+\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar{q}^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2}\frac{1}{2}\left|\frac{\theta_2}{\eta}\right|^{16} \tag{3.72}$$

$w \in 2\mathbb{Z}+1$

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar{\tau}) = \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{an}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}$$
$$\times\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4}\frac{1}{2}\left\{(-1)^a\left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4+\left(\frac{\theta_2(\tau)}{\eta}\right)^4\right\}\left\{\left(\frac{\theta_3}{\eta}\right)^4+\left(\frac{\theta_4}{\eta}\right)^4\right\} \tag{3.73}$$

$$Z_w^{(\mathrm{R,R})}(\tau,\bar{\tau}) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar{\tau}) = \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{an}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}$$
$$\times\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4}\frac{1}{2}\left\{(-1)^a\left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4+\left(\frac{\theta_2(\tau)}{\eta}\right)^4\right\}\left(\frac{\theta_2}{\eta}\right)^4 \tag{3.74}$$

The action of $(-1)^{F_R}|_{T^4[SO(8)]}$ impinges on the massive states in bosonic sector, so the massless states in untwisted sector are the same as (a) though the massive are not. In twisted sector, the lightest excitation is same as (a), that is, massive except at a massless point $R = 1/\sqrt{2}$.

Orbifold model (d)

The partition function vanishes for the same reasons as (b). The explicit form of the partition function (3.57d) is as follows.
$w \in 4\mathbb{Z}$

$$
\begin{aligned}
&Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) \\
&= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n}{2R}\right)^2} \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8 \\
&+ \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n+\frac12}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n+\frac12}{2R}\right)^2} \frac12 \left|\frac{\theta_2}{\eta}\right|^{16}
\end{aligned}
\tag{3.75}
$$

$$
\begin{aligned}
&Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau) \\
&= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n}{2R}\right)^2} \frac12 \left|\frac{\theta_2}{\eta}\right|^{16}
\end{aligned}
\tag{3.76}
$$

$$
+ \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n+\frac12}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n+\frac12}{2R}\right)^2} \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8 \tag{3.77}
$$

$w \in 4\mathbb{Z}+2$

$$
\begin{aligned}
&Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) \\
&= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\ \frac14\left(Rw-\frac{n}{2R}\right)^2} \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 - \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 \right\} \left\{ \overline{\left(\frac{\theta_3}{\eta}\right)}^4 + \overline{\left(\frac{\theta_4}{\eta}\right)}^4 \right\} \\
&+ \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n+\frac12}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n+\frac12}{2R}\right)^2} \frac12 \left|\frac{\theta_2}{\eta}\right|^8 \left\{ \left(\frac{\theta_3}{\eta}\right)^4 - \left(\frac{\theta_4}{\eta}\right)^4 \right\} \left\{ \overline{\left(\frac{\theta_3}{\eta}\right)}^4 + \overline{\left(\frac{\theta_4}{\eta}\right)}^4 \right\}
\end{aligned}
\tag{3.78}
$$

$$
\begin{aligned}
&Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau) \\
&= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n}{2R}\right)^2} \bar q^{\ \frac14\left(Rw-\frac{n}{2R}\right)^2} \frac12 \left|\frac{\theta_2}{\eta}\right|^{16} \\
&+ \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}} \frac{1}{2} q^{\frac14\left(Rw+\frac{n+\frac12}{2R}\right)^2} \bar q^{\frac14\left(Rw-\frac{n+\frac12}{2R}\right)^2} \frac12 \left\{ \left|\frac{\theta_3}{\eta}\right|^8 - \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8
\end{aligned}
\tag{3.79}
$$

$w \in 2\mathbb{Z}+1$

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau) = \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{an}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}$$
$$\times\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4}\frac{1}{2}\left\{(-1)^a\left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4+\left(\frac{\theta_2(\tau)}{\eta}\right)^4\right\}\left\{\left(\frac{\theta_3}{\eta}\right)^4+\left(\frac{\theta_4}{\eta}\right)^4\right\} \quad (3.80)$$

$$Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) = \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{an}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}$$
$$\times\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4}\frac{1}{2}\left\{(-1)^a\left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4+\left(\frac{\theta_2(\tau)}{\eta}\right)^4\right\}\left(\frac{\theta_2}{\eta}\right)^4 \quad (3.81)$$

In the twisted sectors, the lightest excitation is the same as (a). On the other hand, further massless point in $w=\pm 2$ does not exist, which is different from (b) due to $(-1)^{F_R}|_{T^4[SO(8)]}$.

Orbifold model on $\mathcal{M} = S^1\times\mathbb{R}\times T^2[SO(4)]\times T^2[SO(4)]$

Replacing $T^4[SO(8)]$ with $T^2[SO(4)]\times T^2[SO(4)]$ in (3.54), we consider the orbifold model defined by (3.53) but acting on the background $\mathcal{M} = S^1\times\mathbb{R}\times T^2[SO(4)]\times T^2[SO(4)]$. This corresponds to the replacement of building blocks for bosonic sector as follows.

$$Z^{T^4[SO(8)]}_{[0,4](w,m)}(\tau,\bar\tau) \text{ or } \mathbf{Z}^{T^4[SO(8)]}_{[0,4](w,m)}(\tau,\bar\tau) \to \begin{cases} Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) \\ \mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) \\ \mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)\mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) \end{cases} \quad (3.82)$$

These building blocks are calculated in subsection 3.1.

Here, for simplicity, we focus on the orbifold such that $\{(-\mathbf{1}_R)^{\otimes 4}\}^2 \equiv (-1)^{F_R}|_{\psi_R}$ compared with model (b). Instead of (3.57b), we consider the following partition fucntion.

$$Z(\tau,\bar\tau) = \frac{1}{4}Z^{\mathrm{tr}}_{M^4\times S^1}(\tau,\bar\tau)\sum_{w,m\in\mathbb{Z}}Z_{R(w,m)}(\tau,\bar\tau)\left(Z^{T^2[SO(4)]}_{[0,2](w,m)}(\tau,\bar\tau)\right)^2\mathbf{J}_{(w,m)}(\tau)\overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \quad (3.57\text{b'})$$

As seen in (3.27), $\left(Z^{T^2[SO(4)]}_{[0,2](w,m)}(\tau,\bar\tau)\right)^2$ is divided into $Z^{T^4[SO(8)]}_{[0,4](w,m)}(\tau,\bar\tau)$ and another part

$Z^{T^4[SO(8)]}_{[4,4](w,m)}(\tau,\bar\tau)$. The former has been calculated, and we do only the latter part, namely,

$$
\begin{aligned}
Z(\tau,\bar\tau) =& \frac{1}{4} Z^{\rm tr}_{M^4\times S^1}(\tau,\bar\tau) \\
&\times \sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar\tau)\left(\frac{1}{2}Z^{T^4[SO(8)]}_{[4,4](w,m)}(\tau,\bar\tau)\right)\mathbf{J}_{(w,m)}(\tau)\overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)}.
\end{aligned} \tag{3.83}
$$

The explicit form of this part (3.83) is as follows.
$w\in 4\mathbb{Z}$

$$
\begin{aligned}
&Z_w^{(\rm NS,NS)}(\tau,\bar\tau) = -Z_w^{(\rm R,NS)}(\tau,\bar\tau) \\
&= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}\left\{\left|\frac{\theta_3\theta_4}{\eta^2}\right|^4+\left|\frac{\theta_4\theta_2}{\eta^2}\right|^4+\left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}\left|\left(\frac{\theta_3}{\eta}\right)^4-\left(\frac{\theta_4}{\eta}\right)^4\right|^2
\end{aligned} \tag{3.84}
$$

$$
\begin{aligned}
&Z_w^{(\rm R,R)}(\tau,\bar\tau) = -Z_w^{(\rm NS,R)}(\tau,\bar\tau) \\
&= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2}\left\{\left|\frac{\theta_3\theta_4}{\eta^2}\right|^4+\left|\frac{\theta_4\theta_2}{\eta^2}\right|^4+\left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}\left|\frac{\theta_2}{\eta}\right|^8
\end{aligned} \tag{3.85}
$$

$w\in 4\mathbb{Z}+1$

$$
\begin{aligned}
&Z_w^{(\rm NS,NS)}(\tau,\bar\tau) = -Z_w^{(\rm NS,R)}(\tau,\bar\tau) \\
&= \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{an}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2} \\
&\times\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^4\left\{\left(\frac{\theta_3}{\eta}\right)^4+\left(\frac{\theta_4}{\eta}\right)^4\right\}\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^2}
\end{aligned} \tag{3.86}
$$

$$
\begin{aligned}
&Z_w^{(\rm R,R)}(\tau,\bar\tau) = -Z_w^{(\rm R,NS)}(\tau,\bar\tau) \\
&= \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{a(n+1)}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2} \\
&\times\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^4\left(\frac{\theta_2}{\eta}\right)^4\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^2}
\end{aligned} \tag{3.87}
$$

$w\in 4\mathbb{Z}+2$

$$
\begin{aligned}
&Z_w^{(\rm NS,NS)}(\tau,\bar\tau) = -Z_w^{(\rm R,NS)}(\tau,\bar\tau) = \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2} \\
&\times\left\{\left|\frac{\theta_3\theta_4}{\eta^2}\right|^4+\left|\frac{\theta_4\theta_2}{\eta^2}\right|^4+\left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}\left\{\left(\frac{\theta_3}{\eta}\right)^4-\left(\frac{\theta_4}{\eta}\right)^4\right\}\left\{\overline{\left(\frac{\theta_3}{\eta}\right)^4}+\overline{\left(\frac{\theta_4}{\eta}\right)^4}\right\}
\end{aligned} \tag{3.88}
$$

$$
\begin{aligned}
&Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau) \\
&= \frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}\left\{\left|\frac{\theta_3\theta_4}{\eta^2}\right|^4+\left|\frac{\theta_4\theta_2}{\eta^2}\right|^4+\left|\frac{\theta_2\theta_3}{\eta^2}\right|^4\right\}\left|\frac{\theta_2}{\eta}\right|^8
\end{aligned}
\tag{3.89}
$$

$w \in 4\mathbb{Z}+3$

$$
\begin{aligned}
&Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau) \\
&= \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{a(n+1)}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2} \\
&\times\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^4\left\{\left(\frac{\theta_3}{\eta}\right)^4+\left(\frac{\theta_4}{\eta}\right)^4\right\}\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^2}
\end{aligned}
\tag{3.90}
$$

$$
\begin{aligned}
&Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) \\
&= \sum_{a=0,1}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}(-1)^{an}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2} \\
&\times\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^4\left(\frac{\theta_2}{\eta}\right)^4\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^2}
\end{aligned}
\tag{3.91}
$$

The difference between $T^4[SO(8)]$ and $T^2[SO(4)]\times T^2[SO(4)]$ arises in massive spectrum of original background before orbifolding, so the massless states in untwisted sector are the same as model (b). In twisted sector, further massless point exists in addition to ones in model (b). As seen in (3.86) and (3.90), the massless states arise in $w=\pm1$ at special radius $R=1/2$.

3.3 Supersymmetry on asymmetric orbifold

We examine how to construct a non-supersymmetric orbifold with vanishing cosmological constant based on the result of the orbifold models (a)-(d). These all are asymmetric orbifolds where the partition function is zero in each w-twisted sector. The models (a) and (c) are supersymmetric orbifolds, and the others are not as follows.

In (b) and (d), the partition functions for the winding sectors have the relations summarized in table 3. These relations mean breaking supersymmetry in all sectors.

On the other hand, in (a) and (c), for each sector,

$$
\begin{aligned}
Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) &= -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau), \\
Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) &= -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau).
\end{aligned}
\tag{3.92}
$$

The relations (3.92) indicate the existence of the right-moving supersymmetry in the twisted sectors.

Table 3: Relations among the winding sectors in (b) and (d). ($^{\forall}w' \in \mathbb{Z}$)

$w \in 2\mathbb{Z}+1$		
	$Z_w^{(\mathrm{NS,NS})} \neq -Z_{w'}^{(\mathrm{R,NS})}$	$Z_w^{(\mathrm{NS,NS})} = -Z_w^{(\mathrm{NS,R})}$
	$Z_w^{(\mathrm{NS,R})} \neq -Z_{w'}^{(\mathrm{R,R})}$	$Z_w^{(\mathrm{R,NS})} = -Z_w^{(\mathrm{R,R})}$
$w \in 2\mathbb{Z}$		
	$Z_w^{(\mathrm{NS,NS})} = -Z_w^{(\mathrm{R,NS})}$	$Z_w^{(\mathrm{NS,NS})} \neq -Z_{w'}^{(\mathrm{NS,R})}$
	$Z_w^{(\mathrm{NS,R})} = -Z_w^{(\mathrm{R,R})}$	$Z_w^{(\mathrm{R,NS})} \neq -Z_{w'}^{(\mathrm{R,R})}$

For (b) and (d), the left-moving supercharges of the original background are removed by $(-1)^{F_L}|_{\psi_L}$ in the orbifold twist, and the right-moving ones are removed by $(-1)^{F_R}|_{\psi_R}$ in the square of the orbifold twist. If $(-1)^{F_L}|_{\psi_L}$ and $(-1)^{F_R}|_{\psi_R}$ are included in the same power of a twist operator, the total partition function does not vanish.

It is expected that $(-1)^{F_L}|_{\psi_L}$ is not only the operator breaking the left-moving supersymmetry to construct such orbifolds. Then, (i) we would like to examine if any other operator serves a similar role as $(-1)^{F_L}|_{\psi_L}$. For example, the chiral reflection $(-\mathbf{1}_L)^{\otimes 2}|_{T^4}$ acting on two coordinates of four-dimensional torus breaks at least the left-moving supersymmetry of the original background.

Moreover, in order to understand the supersymmmetry in such orbifolds, we should pay attention to the other supercharges potentially arising in twisted sectors as well as untwisted sector. For models (a)-(d), such possibility is related to $\mathcal{T}_{2\pi R}|_{\mathbb{R}}$ and $(-1)^{F_R}|_{T^4}$ in (3.53). Then, (ii) we consider the similar orbifolds to (b) and (d) but without $\mathcal{T}_{2\pi R}|_{\mathbb{R}}$.

Therefore, we study the following orbifold models in this subsection.
(i) The orbifolds defined by the twist operator schematically written as

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-\mathbf{1}_L)^{\otimes 2}|_{T^4} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4} \equiv \mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes \sigma_{[2,4]}|_{T^4} \tag{3.93}$$

(ii)The orbifolds defined by the twist operator schematically written as

$$(-1)^{F_L}|_{\psi_L} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4} \tag{3.94}$$

3.3 (i) Orbifolds defined by $\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes \sigma^2_{[2,4]}$

We study the construction of the string vacua defined by the twist operator [5]

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes \sigma_{[2,4]}|_{T^4}, \tag{3.93}$$

acting on the same background as (a)-(d), namely, $\mathcal{M} = S^1 \times \mathbb{R} \times T^4[SO(8)]$. The form of (3.93) also still does not specify a twist operator. We completely determine the twist

[5] This type might be less general than the models defined by the twist $(-\mathbf{1})^{\otimes 2}|_{T^2} \otimes (-\mathbf{1}_R)^{\otimes 2}|_{T^2}$ acting on $\mathcal{M} = S^1 \times \mathbb{R} \times T^2 \times T^2[SO(4)]$, where $-\mathbf{1}$ is ordinary reflection $X \to -X$ as discussed later. However, we examine the former here in order to explicitly calculate the partition function and to compare with (a)-(d).

operators by the distinction of their square $\sigma_{[2,4]}^2$ as folows.

$$\sigma_{[2,4]}^2 = \mathbf{1} \tag{3.93a'}$$
$$\sigma_{[2,4]}^2 = (-1)^{F_L}|_{\psi_L} \tag{3.93b'}$$
$$\sigma_{[2,4]}^2 = (-1)^{F_R}|_{\psi_R} \tag{3.93c'}$$
$$\sigma_{[2,4]}^2 = (-1)^{F_L}|_{\psi_L} \otimes (-1)^{F_R}|_{\psi_R} \tag{3.93d'}$$
$$\sigma_{[2,4]}^2 = (-1)^{F_L}|_{T^4} \tag{3.93e'}$$
$$\sigma_{[2,4]}^2 = (-1)^{F_L}|_{T^4} \otimes (-1)^{F_L}|_{\psi_L} \tag{3.93f'}$$
$$\sigma_{[2,4]}^2 = (-1)^{F_L}|_{T^4} \otimes (-1)^{F_R}|_{\psi_R} \tag{3.93g'}$$
$$\sigma_{[2,4]}^2 = (-1)^{F_L}|_{T^4} \otimes (-1)^{F_L}|_{\psi_L} \otimes (-1)^{F_R}|_{\psi_R} \tag{3.93h'}$$

These twists construct the different orbifolds respectively. We name them model (a')-(h') sorted by (3.93a')-(3.93h').

Supersymmetric type (a'), (b'), (e'), (f')

The partition function of (a') is as follows.

$$Z(\tau,\bar{\tau}) = \frac{1}{4} Z^{\mathrm{tr}}_{M^4\times S^1}(\tau,\bar{\tau}) \sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^4[SO(8)]}_{[2,4](w,m)}(\tau,\bar{\tau}) Z^{\psi}_{[2](w,m)}(\tau) \overline{Z^{\psi}_{[4](w,m)}(\tau)}, \tag{3.95}$$

This partition function vanishes because of $\overline{Z^{\psi}_{[4](w,m)}(\tau)}$, and 1/4 spacetime supersymmetry remain in untwisted sector. The same can be said of (b'), (e'), (f').

Non-supersymmetric type but non-zero vacuum energy (d'), (h')

The orbifold projection of this type removes all the (R, NS) and (NS, R)-sectors at least in untwisted sector. The partition function of (d') is as follows.

$$Z(\tau,\bar{\tau}) = \frac{1}{4} Z^{\mathrm{tr}}_{M^4\times S^1}(\tau,\bar{\tau}) \sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^4[SO(8)]}_{[2,4](w,m)}(\tau,\bar{\tau}) \mathbf{Z}^{\psi}_{[2](w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.96}$$

This partition function does not vanish since $\mathbf{Z}^{\psi}_{[2](w,m)} \neq 0$ and $\overline{\mathbf{Z}^{\psi}_{[4](w,m)}} \neq 0$ for $^{\forall}w \in 4\mathbb{Z}+2$ or $^{\forall}m \in 4\mathbb{Z}+2$. Thus we immediately find that this is non-supersymmetric orbifold model because of non-zero vacuum energy density perturbatively. The same can be said of (h').

Non-supersymmetric type and zero vacuum energy (c'), (g')

We focus on (c'). The partition function of (c') is as follows.

$$Z(\tau,\bar{\tau}) = \frac{1}{4} Z^{\rm tr}_{M^4\times S^1}(\tau,\bar{\tau}) \sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^4[SO(8)}_{[2,4](w,m)}(\tau,\bar{\tau}) Z^{\psi}_{[2](w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)}. \tag{3.97}$$

As in the case with (b) or (d), $\overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} = 0$ for ${}^{\forall}w \in 2\mathbb{Z}+1$ or ${}^{\forall}m \in 2\mathbb{Z}+1$, while $Z^{\psi}_{[2](w,m)}(\tau) = 0$ for ${}^{\forall}w, m \in 2\mathbb{Z}$. Thus this is the non-supersymmetric orbifold model with vanishing cosmological constant.

Table 4: massless spectrum in untwisted sector of (c')

spin structure	left		right	4D fields (d.o.f)
(NS, NS)	$\psi^{\mu}_{L,-1/2}\|0\rangle$ $(\mu=2,...,7)$	$\otimes$	$\psi^{\mu}_{R,-1/2}\|0\rangle$ $(\mu=2,...,5)$	graviton (2), 6 vectors (12), 10 (pseudo) scalars (10)
	$\psi^{\mu}_{L,-1/2}\|0\rangle$ $(\mu=8,9)$	$\otimes$	$\psi^{\mu}_{R,-1/2}\|0\rangle$ $(\mu=6,...,9)$	8 scalars (8)
(R , NS)	$\left[1+(-\mathbf{1}_L)^{\otimes 2}\right]\|\mathbf{s}\rangle_L$	$\otimes$	$\psi^{\mu}_{R,-1/2}\|0\rangle$ $(\mu=2,...,5)$	2 Rarita-Schwinger (4), 6 Weyl fermions (12)
	$\left[1-(-\mathbf{1}_L)^{\otimes 2}\right]\|\mathbf{s}\rangle_L$	$\otimes$	$\psi^{\mu}_{R,-1/2}\|0\rangle$ $(\mu=6,...,9)$	8 Weyl fermions (16)

Let us check the massless states. The massless spectrum in untwisted sector of (c') is displayed on table 4. Of course, these are different from (b). The left moving R sectors are slightly complicated. Although we cannot construct any supercharge in the left mover, there are the linear combinations of R sector in the right mover that are not removed. We consider the two distinguishable R sectors $|\mathbf{s}_1\rangle$, $|\mathbf{s}_2\rangle$ such that $|\mathbf{s}_1\rangle \to |\mathbf{s}_2\rangle$ and $|\mathbf{s}_2\rangle \to |\mathbf{s}_1\rangle$ under $(-\mathbf{1}_L)^{\otimes 2}$. In fact, there are such pairs $|\mathbf{s}_1\rangle$, $|\mathbf{s}_2\rangle$, for example,

$$\mathbf{s}_1 = \frac{1}{2}(+,+,+,+) \quad \text{and} \quad \mathbf{s}_2 = \frac{1}{2}(+,+,-,-). \tag{3.98}$$

Thus, we can find $|\mathbf{s}_1\rangle + |\mathbf{s}_2\rangle$ and $|\mathbf{s}_1\rangle - |\mathbf{s}_2\rangle$ where the phase ± 1 arise under the twist $(-\mathbf{1}_L)^{\otimes 2}$ respectively, and they compose the left moving R sector in (R, NS).

Next, we find the explicit form of the partition function of (c').

$w \in 4\mathbb{Z}+1$:

$$
\begin{aligned}
&Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau)\\
&= \sum_{a\in\mathbb{Z}_2}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2}\\
&\quad\times(-1)^{an}\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{m}{2})}{\eta^2}\right|^4\frac{1}{2}\left(\frac{\theta_3(\tau,\frac{m}{2})}{\eta}\right)^4\overline{\left(\frac{\theta_3(\tau,\frac{m}{2})\theta_2}{\eta^2}\right)}^2\\
&\quad+\sum_{a\in\mathbb{Z}_2}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}(-1)^{an}\frac{1}{2}\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{m}{2})}{\eta^2}\right|^8. \qquad (3.99)
\end{aligned}
$$

$$
\begin{aligned}
&Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau)\\
&= \sum_{a\in\mathbb{Z}_2}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{2}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{2}}{2R}\right)^2}\\
&\times(-1)^{a(n+1)}\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{m}{2})}{\eta^2}\right|^4\frac{1}{2}\left(\frac{\theta_2}{\eta}\right)^4\overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{m}{2})}{\eta^2}\right)}^2\\
&+\sum_{a\in\mathbb{Z}_2}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{2R}\right)^2}(-1)^{an}\frac{1}{2}\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{m}{2})}{\eta^2}\right|^8. \qquad (3.100)
\end{aligned}
$$

In $w=4\mathbb{Z}+3$, we obtain the result by replacing $(-1)^{an}$ in the first term of (3.99) with $(-1)^{a(n+1)}$, and by replacing $(-1)^{a(n+1)}$ in the first term of (3.100) with $(-1)^{an}$. Actually, the partition function in $w\in 2\mathbb{Z}+1$ is common in (a')-(h'). The partition function of (c') in $w\in 2\mathbb{Z}$ is the same as (b). Similarly, the partition function of (g') in $w\in 2\mathbb{Z}$ corresponds to (d). As seen in above results, any tachyonic state does not arise, but the place where the lightest excitation live depends on the radius of translation operator $\mathcal{T}_{2\pi R}$. In $w=\pm 2$, the massless points of (c') are same as (b). The other massless states appear at the special radius $R=1/2$ in $w=\pm 1$ (see (3.99)).

In addition, we mention the case of the twist $(-\mathbf{1}_L)^{\otimes 2}|_{T^4}\otimes(-\mathbf{1}_R)^{\otimes 4}|_{T^4}$ acting on $T^2[SO(4)]\times T^2[SO(4)]$ instead of $T^4[SO(8)]$. In this case, the twist acts in several ways related to (C.11)-(C.17). We need the following calculation like (a)-(d).

$$\sum_{\sigma_s,\tilde\sigma_s}\sum_{w\in\mathbb{Z}}Z_w^{(\sigma_s,\tilde\sigma_s)}(\tau,\bar\tau)=\sum_{w,m\in\mathbb{Z}}Z_{R(w,m)}\frac{1}{2}Z^{T^2[SO(8)]}_{[4,4](w,m)}Z^{\psi}_{[2](w,m)}\overline{\mathbf{Z}^{\psi}_{[4](w,m)}}. \qquad (3.101)$$

In $w=2\mathbb{Z}+1$,

$$
\begin{aligned}
&Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau)\\
&= \sum_{a\in\mathbb{Z}_2}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{8}q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2}(-1)^{\frac{1}{4}a(n-w)}\left\{1+(-1)^{\frac{1}{2}(n-w)}\right\}\{1-(-1)^n\}\\
&\qquad\times\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^6\left|\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right|^2\left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)\overline{\left(\frac{\theta_2}{\eta}\right)} \qquad (3.102)
\end{aligned}
$$

$$Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau)$$
$$= \sum_{a\in\mathbb{Z}_2}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{8}q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2}(-1)^{\frac{1}{4}a(n-3w)}\left\{1+(-1)^{\frac{1}{2}(n-3w)}\right\}\{1-(-1)^n\}$$
$$\times\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^6\left|\frac{\theta_2(\tau)}{\eta}\right|^2\left(\frac{\theta_2}{\eta}\right)\overline{\left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)} \tag{3.103}$$

Especially, in $w\in 8\mathbb{Z}+1$,

$$Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau)$$
$$= \sum_{a\in\mathbb{Z}_2}\frac{1}{|\eta|^2}\sum_{n\in\mathbb{Z}}\frac{1}{2}q^{\frac{1}{4}\left(Rw+\frac{n+\frac{1}{4}}{2R}\right)^2}\bar q^{\frac{1}{4}\left(Rw-\frac{n+\frac{1}{4}}{2R}\right)^2}$$
$$\times(-1)^{an}\left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^6\left|\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right|^2\left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)\overline{\left(\frac{\theta_2}{\eta}\right)}. \tag{3.104}$$

We can find the additional massless point at the radius $R=1/2$ in $w=\pm1$ (see (3.104)).

Eventually, we can find the non-supersymmetric asymmetric orbifold with vanishing cosmological constant by the twist (3.93) instead of (3.53) as expected. Of course, the spectra of these models of (3.93) are different from (3.53). Especially in (c') and (g'), massless Rarita-Schwinger fields arise even though they are not superpartners of the graviton.

3.3 (ii) Orbifolds defined by $(-1)^{F_L}|_{\psi_L}\otimes(-\mathbf{1}_R)^{\otimes4}|_{T^4}$

We study the construction of the string vacua defined by the twist operator

$$(-1)^{F_L}|_{\psi_L}\otimes(-\mathbf{1}_R)^{\otimes4}|_{T^4} \tag{3.94}$$

acting on $\mathcal{M}=T^2\times T^4[SO(8)]$ instead of $\mathcal{M}=S^1\times\mathbb{R}\times T^4[SO(8)]$, where T^2 is an arbitrary two-dimensional torus. This twist can be defined by four ways by distinction (3.53a) - (3.53d). It's not hard to anticipate the appearance of the supersymmetry in the models related to (3.53a) or (3.53c). Then, we discuss the cases of (3.53b) and (3.53d).

Orbifolds defined by (3.94) with $\{(-\mathbf{1}_R)^{\otimes4}\}^2\equiv(-1)^{F_R}|_{\psi_R}$

At first, we investigate the orbifold defined by the twist (3.94) with (3.53b). The order of this twist is four. The partition function of this orbifold is written as

$$Z(\tau,\bar\tau)=\frac{1}{4}Z^{\mathrm{tr}}_{M^4\times T^2}(\tau,\bar\tau)\sum_{a,b\in\mathbb{Z}_4}Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)\mathbf{J}_{(a,b)}(\tau)\overline{\mathbf{Z}^{\psi}_{[4](a,b)}(\tau)}. \tag{3.105}$$

Here, we reset the notation such that

$$Z(\tau,\bar\tau)=\frac{1}{16}Z^{\mathrm{tr}}_{M^4\times T^2(\tau,\bar\tau)}\sum_{a\in\mathbb{Z}_4}\sum_{\sigma_s,\tilde\sigma_s}Z_a^{(\sigma_s,\tilde\sigma_s)}(\tau,\bar\tau), \tag{3.106}$$

where the overall factor $\frac{1}{16} = \frac{1}{4} \times \frac{1}{4}$ is due to the $\mathbb{Z}_4$-orbifolding as well as the chiral GSO projection. We can find the following results by calculating (3.112) directly.

$$\begin{aligned} &Z_0^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_0^{(\mathrm{R,NS})}(\tau,\bar\tau) = Z_2^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_2^{(\mathrm{NS,R})}(\tau,\bar\tau) \\ &= Z_{1,3}^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_{1,3}^{(\mathrm{R,NS})}(\tau,\bar\tau) = \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8 \end{aligned} \tag{3.107}$$

$$\begin{aligned} Z_{1,3}^{(\mathrm{NS,NS})}(\tau,\bar\tau) &= -Z_{1,3}^{(\mathrm{NS,R})}(\tau,\bar\tau) \\ &= \overline{\left(\frac{\theta_2}{\eta}\right)^4} \left\{ \left|\frac{\theta_3}{\eta}\right|^8 - \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 \right\} \\ &\quad + \left|\frac{\theta_2}{\eta}\right|^8 \left\{ \overline{\left(\frac{\theta_3}{\eta}\right)^4} + \overline{\left(\frac{\theta_4}{\eta}\right)^4} \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 \right\} \end{aligned} \tag{3.108}$$

$$Z_2^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_2^{(\mathrm{R,NS})}(\tau,\bar\tau) = Z_0^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_0^{(\mathrm{NS,R})}(\tau,\bar\tau) = 0 \tag{3.109}$$

As seen above, no tachyonic state arises in this orbifold. The orbifold projection removes the left-moving supercharges in all sectors, which reflects the following relation.

$$Z_{1,3}^{(\mathrm{NS,NS})}(\tau,\bar\tau) \neq -Z_a^{(\mathrm{R,NS})}(\tau,\bar\tau) \quad ({}^\forall a \in \mathbb{Z}_4) \tag{3.110}$$

On the other hand, the orbifold projection does not remove the right-moving massless Ramond states in $a = 2$ twisted sector though the orbifold projection removes ones in untwisted sector. Thus the right-moving supercharges potentially exist in the twisted sector. In the current case, we actually confirm that some right-moving supercharges exist in a = 2 twisted sector by realizing the equalities

$$Z_a^{(*,\mathrm{NS})}(\tau,\bar\tau) = -Z_{a+2 \bmod 4}^{(*,\mathrm{R})}(\tau,\bar\tau) \quad ({}^\forall a \in \mathbb{Z}_4). \tag{3.111}$$

If the original background belongs to the type IIA (IIB) string theory, we find 8 supercharges that possess the reverse chirality to an original as in type IIB (IIA) theory in $a = 2$ sector, as mentioned *e.g.* in [33, 34].

The massless spectrum in the untwisted sector is the same as displayed in table 2. In addition, the massless states in the $a = 2$ sector is summarized in table 5. These states are combined into the super-multiplets in an $N = 2$ supersymmetric theory in 4-dimension.

This phenomenon is not seen in model (b). Compared with model (b), (NS, NS) and (R, NS) sector in $a = 2$ and (R, R) and (NS, R) sector in $a = 0$ cannot exist due to the level matching condition without translation twist. On the other hand, the winding excitations make (NS, NS) and (R, NS) sector in $w = 4\mathbb{Z} + 2$ and (R, R) and (NS, R) sector in $w = 4\mathbb{Z}$ level matching in model (b) as seen in (3.66) and (3.65), and they cause breaking supersymmetry.

Table 5: Massless spectrum in the $a = 2$ sector for (b) without translation twist.

spin structure	4D fields
(NS, R)	2 gravitini, 14 Weyl fermions
(R , R)	8 vectors, 16 (pseudo)scalars

Orbifolds defined by (3.94) **with** $\{(-\mathbf{1}_R)^{\otimes 4}\}^2 \equiv (-1)^{F_R}|_{T^4[SO(8)]} \otimes (-1)^{F_R}|_{\psi_R}$

Next, we investigate the orbifold defined by the twist (3.94) with (3.53d). The partition function of this orbifold is written as

$$Z(\tau,\bar{\tau}) = \frac{1}{4} Z^{\rm tr}_{M^4\times T^2}(\tau,\bar{\tau}) \sum_{a,b\in\mathbb{Z}_4} \mathbf{Z}^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar{\tau})\mathbf{J}_{(a,b)}(\tau)\overline{\mathbf{Z}^{\psi}_{[4](a,b)}(\tau)}. \tag{3.112}$$

The explicit partition functions are

$$\begin{aligned} Z_0^{\rm (NS,NS)}(\tau,\bar{\tau}) &= -Z_0^{\rm (R,NS)}(\tau,\bar{\tau}) = \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8, \\ Z_2^{\rm (NS,NS)}(\tau,\bar{\tau}) &= -Z_2^{\rm (R,NS)}(\tau,\bar{\tau}) \\ &= \left\{ \left|\frac{\theta_3}{\eta}\right|^8 - \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left(\frac{\theta_2}{\eta}\right)^4 \left\{ \overline{\left(\frac{\theta_3}{\eta}\right)^4} + \overline{\left(\frac{\theta_4}{\eta}\right)^4} \right\}, \\ Z_{0,2}^{\rm (R,R)}(\tau,\bar{\tau}) &= -Z_{0,2}^{\rm (NS,R)}(\tau,\bar{\tau}) = \left|\frac{\theta_2}{\eta}\right|^{16}, \end{aligned} \tag{3.113}$$

for the even sectors, and

$$\begin{aligned} Z_{1,3}^{\rm (NS,NS)}(\tau,\bar{\tau}) &= -Z_{1,3}^{\rm (NS,R)}(\tau,\bar{\tau}) = \overline{\left(\frac{\theta_2}{\eta}\right)^4} \left\{ \left|\frac{\theta_3}{\eta}\right|^8 - \left|\frac{\theta_4}{\eta}\right|^8 \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 \right\}, \\ &\quad + \left|\frac{\theta_2}{\eta}\right|^8 \left\{ \overline{\left(\frac{\theta_3}{\eta}\right)^4} + \overline{\left(\frac{\theta_4}{\eta}\right)^4} \right\} \left\{ \left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4 \right\}, \\ Z_{1,3}^{\rm (R,R)}(\tau,\bar{\tau}) &= -Z_{1,3}^{\rm (R,NS)}(\tau,\bar{\tau}) = \left\{ \left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8 \right\} \left|\frac{\theta_2}{\eta}\right|^8, \end{aligned} \tag{3.114}$$

for the odd sectors.

These relations are different from the supersymmetric case of the previous model. We never find the equalities of cancellation such as (3.111). Instead, we find the relations as displayed in table 3. Thus we can obtain the non-supersymmetric orbifold with vanishing cosmological constant without translation twist.

The massless spectrum in the untwisted sector is the same as model (b). In the twisted sectors, additional massless states also appear, but there are not the massless Ramond states which potentially compose the right-moving supercharges. This is because $(-1)^{F_R}|_{T^4[SO(8)]}$ in the square of the twist causes level mismatching to such massless Ramond states in the twisted sectors, alternatively, the other states appear with level matching.

3.4 Further torus asymmetric orbifolds

We have seen some non-supersymmetric orbifold models with vanishing cosmological constant compared with supersymmetric ones in previous subsection. In those models, the right-chiral reflection $(-\mathbf{1}_R)^{\otimes 4}|_{T^4}$ acts on the four coordinates of the four-dimensional torus of the background with symmetry enhancement which can be written in fermion description. However, fixing moduli of "four"-dimensional torus is not absolutely necessary to construct such non-supersymmetric orbifold.

For example, the discussion of the orbifold models (a')-(h') in subsection 3.3 (i) is almost the same as the orbifolds defined by the twist

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-\mathbf{1})^{\otimes 2}|_{T^3} \otimes (-\mathbf{1}_R)^{\otimes 2}|_{T^2[SO(4)]} \tag{3.115}$$

acting on

$$\mathcal{M} = \mathbb{R} \times T^3 \times T^2[SO(4)], \tag{3.116}$$

as displayed on table 6, where $-\mathbf{1}$ is ordinary reflection $X \to -X$.

Table 6: Action of (3.115)

space	$\mathbb{R}$	T^3			$T^2[SO(4)]$	
direction	X^4	X^5	X^6	X^7	X^8	X^9
Left	translation	$-\mathbf{1}$	$-\mathbf{1}$			
Right	$\mathcal{T}_{2\pi R}$	$-\mathbf{1}$	$-\mathbf{1}$		$-\mathbf{1}$	$-\mathbf{1}$

T^3 in (3.116) is an arbitrary three-dimensional torus, while two directions of $T^2[SO(4)]$ must be fixed due to the definition of chiral reflection. Of course, we may choose $S^1[SO(2)] \times S^1[SO(2)]$ instead of $T^2[SO(4)]$.

In addition, we can consider the similar orbifolds defined by the twist

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-\mathbf{1})|_{S^1} \otimes \sigma_{[1,3]}|_{T^4[SO(8)]} \tag{3.117}$$

acting on

$$\mathcal{M} = \mathbb{R} \times S^1 \times T^4[SO(8)], \tag{3.118}$$

Table 7: Action of (3.117)

space	$\mathbb{R}$	S^1	$T^4[SO(8)]$			
direction	X^4	X^5	X^6	X^7	X^8	X^9
Left	translation	$\mathbf{-1}$	$\mathbf{-1}$			
Right	$\mathcal{T}_{2\pi R}$	$\mathbf{-1}$		$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$

as displayed on table 7. If $\sigma^2_{[2,4]} = (-1)^{F_R}|_{\psi_R}$, the partition function of this orbifold can be written as

$$\begin{aligned} Z(\tau,\bar\tau) &= \frac{1}{4} Z^{\mathrm{tr}}_{M^4}(\tau,\bar\tau) \\ &\times \sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar\tau) Z^{S^1/\mathbb{Z}_2}_{R(w,m)} Z^{T^4[SO(8)]}_{[1,3](w,m)}(\tau,\bar\tau) Z^{\psi}_{[2](w,m)}(\tau)\overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)}, \end{aligned} \qquad (3.119)$$

where $Z^{S^1/\mathbb{Z}_2}_{R(w,m)}$ is defined in (2.51) or (3.25). In calculation of the partition function (3.119), non-obvious terms are in $w \in 2\mathbb{Z}+1$, or $m \in 2\mathbb{Z}+1$, especially for the building block $Z^{S^1/\mathbb{Z}_2}_{R(w,m)} Z^{T^4[SO(8)]}_{[1,3](w,m)}(\tau,\bar\tau)$. Then, we describe this part as

$$\begin{aligned} & Z^{S^1/\mathbb{Z}_2}_{(a,b)}(\tau,\bar\tau) Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar\tau) \\ &= \begin{cases} Z^{S^1}_R(\tau,\bar\tau) Z^{T^4[SO(8)]}(\tau,\bar\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ Z^{T^5[SO(10)]}_{[2,4](a,b)}(\tau,\bar\tau) & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1). \end{cases} \end{aligned} \qquad (3.120)$$

The phases arising from the building block $Z^{T^5[SO(10)]}_{[2,4](w,m)}(\tau,\bar\tau)$ under the modular transformation is similar to $Z^{T^2[SO(4)]}_{[0,2](w,m)}(\tau,\bar\tau)$ in (3.18), so we can calculate the explicit form of this building blocks immediately as follows.

$$Z^{T^5[SO(10)]}_{[2,4](a,b)}(\tau,\bar\tau)$$
$$\equiv \begin{cases} Z^{T^5[SO(10)]}(\tau,\bar\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ e^{\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)} \left|\frac{\theta_3\theta_4}{\eta^2}\right|^2 \frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2 \left|\frac{\theta_3}{\eta}\right|^2 + (-1)^{\frac{a}{2}}\left(\frac{\theta_4}{\eta}\right)^2\left|\frac{\theta_4}{\eta}\right|^2\right\} & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}+1) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_3}{\eta^2}\right)} \left|\frac{\theta_2\theta_3}{\eta^2}\right|^2 \frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2 \left|\frac{\theta_3}{\eta}\right|^2 + (-1)^{\frac{b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\left|\frac{\theta_2}{\eta}\right|^2\right\} & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_4}{\eta^2}\right)} \left|\frac{\theta_2\theta_4}{\eta^2}\right|^2 \frac{1}{2}\left\{\left(\frac{\theta_4}{\eta}\right)^2 \left|\frac{\theta_4}{\eta}\right|^2 - i(-1)^{\frac{a+b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\left|\frac{\theta_2}{\eta}\right|^2\right\} & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}+1) \end{cases}$$
$$(3.121)$$

More generally, the building block $Z^{T^N[SO(2N)]}_{[\alpha,\alpha+2](w,m)}(\tau,\bar\tau)$ can be explicitly written as (C.20). Therefore, we can easily obtain the explicit form of the partition function (3.119) from the partition function (3.97) for the model (c').

Furthermore, there are several similar orbifolds to replace $T^4[SO(8)]$ with the products of tori with $SO(N)$-symmetry enhancement in (3.118), which are displayed on table 8.

Table 8: Variety of the backgrounds T^4 where $(-\mathbf{1}_L)^{\otimes 1} \otimes (-\mathbf{1}_R)^{\otimes 3}$ can act

<table>
<tr><td colspan="5">T^4</td><td>building block</td></tr>
<tr><td>direction</td><td>X^6</td><td>X^7</td><td>X^8</td><td>X^9</td><td rowspan="3"></td></tr>
<tr><td>Left</td><td>$-\mathbf{1}$</td><td></td><td></td><td></td></tr>
<tr><td>Right</td><td></td><td>$-\mathbf{1}$</td><td>$-\mathbf{1}$</td><td>$-\mathbf{1}$</td></tr>
<tr><td rowspan="7">space</td><td>$S^1[SO(2)]$</td><td>$S^1[SO(2)]$</td><td>$S^1[SO(2)]$</td><td>$S^1[SO(2)]$</td><td>(C.22)</td></tr>
<tr><td>$S^1[SO(2)]$</td><td>$S^1[SO(2)]$</td><td colspan="2">$T^2[SO(4)]$</td><td>(C.23)</td></tr>
<tr><td>$S^1[SO(2)]$</td><td colspan="3">$T^3[SO(6)]$</td><td>(C.24)</td></tr>
<tr><td colspan="2">$T^2[SO(4)]$</td><td>$S^1[SO(2)]$</td><td>$S^1[SO(2)]$</td><td>(C.25)</td></tr>
<tr><td colspan="2">$T^2[SO(4)]$</td><td colspan="2">$T^2[SO(4)]$</td><td>(C.26)</td></tr>
<tr><td colspan="3">$T^3[SO(6)]$</td><td>$S^1[SO(2)]$</td><td>(C.27)</td></tr>
<tr><td colspan="4">$T^4[SO(8)]$</td><td>$Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar{\tau})$</td></tr>
</table>

They all are non-supersymmetric orbifolds with vanishing cosmological constant, but the partition functions of these orbifolds are different. Then, we introduce the further building blocks (C.22)-(C.27) to efficiently calculate those partition functions. It is too hard to directly calculate some simplex building blocks such as $Z^{S^1[SO(2)]}_{[1,0](a,b)}(\tau,\bar{\tau})$ or $Z^{T^3[SO(6)]}_{[0,3](a,b)}(\tau,\bar{\tau})$ because of the small phases, but these products may become easy to calculate. Indeed, the building blocks (C.22)-(C.27) arrive at the sum of the combination of (C.19), (C.20) and (C.21) for a or $b \in 2\mathbb{Z}+1$. We combine them with $Z^{S^1/\mathbb{Z}_2}_{(a,b)}(\tau,\bar{\tau})$ as (C.28). Eventually, we find the terms proportional to the following forms.

$$\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^5}_{[2,6](w,m)}(\tau,\bar{\tau}) Z^{\psi}_{[2](w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.122}$$

$$\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^5}_{[\alpha,\alpha+2](w,m)}(\tau,\bar{\tau}) Z^{\psi}_{[2](w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.123}$$

$$\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^5}_{[4,4](w,m)}(\tau,\bar{\tau}) Z^{\psi}_{[2](w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.124}$$

The new type of calculation arises in (3.122) for $w \in 2\mathbb{Z}+1$. The others are similar to ones in previous subsection. The explicit form are as follows in $w \in 2\mathbb{Z}+1$.

$$
\begin{aligned}
&Z_w^{(\mathrm{NS,NS})}(\tau,\bar\tau) = -Z_w^{(\mathrm{NS,R})}(\tau,\bar\tau) \\
\propto &\sum_{a\in\mathbb{Z}_2} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{8} q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2} \bar q^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2} (-1)^{\frac{1}{4}a(n+w)} \left\{1+(-1)^{\frac{1}{2}(n+w)}\right\} \{1-(-1)^n\} \\
&\times \frac{1}{2} \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4} \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^3 \left(\frac{\theta_2}{\eta}\right) \left\{ \left(\frac{\theta_2}{\eta}\right)^4 \left|\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right|^2 + (-1)^a \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4 \left|\frac{\theta_2}{\eta}\right|^2 \right\}
\end{aligned}
\tag{3.125}
$$

$$
\begin{aligned}
&Z_w^{(\mathrm{R,R})}(\tau,\bar\tau) = -Z_w^{(\mathrm{R,NS})}(\tau,\bar\tau) \\
\propto &\sum_{a\in\mathbb{Z}_2} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{8} q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2} \bar q^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2} (-1)^{\frac{1}{4}a(n-w)} \left\{1+(-1)^{\frac{1}{2}(n-w)}\right\} \{1-(-1)^n\} \\
&\times \frac{1}{2} \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^4} \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right) \left(\frac{\theta_2}{\eta}\right)^3 \left\{ \left(\frac{\theta_2}{\eta}\right)^4 \left|\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right|^2 + (-1)^a \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^4 \left|\frac{\theta_2}{\eta}\right|^2 \right\}
\end{aligned}
\tag{3.126}
$$

The new massless states do not appear.

Next, we also consider the similar orbifolds to model (b), defined by the twist

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-1)^{F_L}|_{\psi_L} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4} \quad \text{with} \quad \left\{(-\mathbf{1}_R)^{\otimes 4}\right\}^2 = (-1)^{F_R}|_{\psi_R}.$$

Again, $(-\mathbf{1}_R)^{\otimes 4}$ can act on several T^4 as displayed on table 9. We have already inves-

Table 9: Variety of the backgrounds T^4 where $(-\mathbf{1}_R)^{\otimes 4}$ can act

T^4					building block
direction	X^6	X^7	X^8	X^9	
Left					
Right	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{1}$	
space	$S^1[SO(2)]$	$S^1[SO(2)]$	$S^1[SO(2)]$	$S^1[SO(2)]$	(C.29)
	$S^1[SO(2)]$	$S^1[SO(2)]$	$T^2[SO(4)]$		(C.30)
	$S^1[SO(2)]$	$T^3[SO(6)]$			(C.31)
	$T^2[SO(4)]$		$T^2[SO(4)]$		(3.27)
	$T^4[SO(8)]$				$Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)$

tigated the case that T^4 is $T^4[SO(8)]$ or $T^2[SO(4)] \times T^2[SO(4)]$ in previous subsection. The building blocks (C.29)-(C.31) can also be written as the sum of the combination of $Z^{T^4}_{[0,4](a,b)}(\tau,\bar{\tau})$, $Z^{T^4}_{[2,4](a,b)}(\tau,\bar{\tau})$ and $Z^{T^4}_{[4,4](a,b)}(\tau,\bar{\tau})$ for a or $b \in 2\mathbb{Z}+1$. Then, we find the terms proportional to the following forms.

$$\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^4}_{[0,4](w,m)}(\tau,\bar{\tau}) \mathbf{J}_{(w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.127}$$

$$\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^4}_{[2,4](w,m)}(\tau,\bar{\tau}) \mathbf{J}_{(w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.128}$$

$$\sum_{w,m\in\mathbb{Z}} Z_{R(w,m)}(\tau,\bar{\tau}) Z^{T^4}_{[4,4](w,m)}(\tau,\bar{\tau}) \mathbf{J}_{(w,m)}(\tau) \overline{\mathbf{Z}^{\psi}_{[4](w,m)}(\tau)} \tag{3.129}$$

The new type of calculation arises in (3.128) for $w \in 2\mathbb{Z}+1$. The others are similar to ones in previous subsection. The explicit form are as follows in $w \in 2\mathbb{Z}+1$.

$$\begin{aligned}
&Z^{(\mathrm{NS,NS})}_{w}(\tau,\bar{\tau}) = -Z^{(\mathrm{NS,R})}_{w}(\tau,\bar{\tau}) \\
&\propto \sum_{a\in\mathbb{Z}_2} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{8} q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2} (-1)^{\frac{1}{4}a(n-3w)} \left\{1+(-1)^{\frac{1}{2}(n-3w)}\right\} \{1-(-1)^n\} \\
&\qquad \times \frac{1}{2} \left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^2 \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^3} \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^2 \left\{\left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4\right\} \\
&+ \sum_{a\in\mathbb{Z}_2} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{8} q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2} (-1)^{\frac{1}{4}a(n-w)} \left\{1+(-1)^{\frac{1}{2}(n-w)}\right\} \{1-(-1)^n\} \\
&\qquad \times \frac{1}{2} \left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^2 \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^3} \left(\frac{\theta_2(\tau)}{\eta}\right)^2 \left\{\left(\frac{\theta_3}{\eta}\right)^4 + \left(\frac{\theta_4}{\eta}\right)^4\right\}
\end{aligned} \tag{3.130}$$

$$\begin{aligned}
&Z^{(\mathrm{R,R})}_{w}(\tau,\bar{\tau}) = -Z^{(\mathrm{R,NS})}_{w}(\tau,\bar{\tau}) \\
&\propto \sum_{a\in\mathbb{Z}_2} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{8} q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2} (-1)^{\frac{1}{4}a(n-3w)} \left\{1+(-1)^{\frac{1}{2}(n-3w)}\right\} \{1-(-1)^n\} \\
&\qquad \times \frac{1}{2} \left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^2 \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^3} \left(\frac{\theta_3(\tau,\frac{a}{2})}{\eta}\right)^2 \left(\frac{\theta_2}{\eta}\right)^4 \\
&+ \sum_{a\in\mathbb{Z}_2} \frac{1}{|\eta|^2} \sum_{n\in\mathbb{Z}} \frac{1}{8} q^{\frac{1}{4}\left(Rw+\frac{n}{8R}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw-\frac{n}{8R}\right)^2} (-1)^{\frac{1}{4}a(n-w)} \left\{1+(-1)^{\frac{1}{2}(n-w)}\right\} \{1-(-1)^n\} \\
&\qquad \times \frac{1}{2} \left|\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right|^2 \overline{\left(\frac{\theta_2(\tau)\theta_3(\tau,\frac{a}{2})}{\eta^2}\right)^3} \left(\frac{\theta_2(\tau)}{\eta}\right)^6
\end{aligned} \tag{3.131}$$

In the current case, tachyonic states do not arise, but some massless points appear. From the first term in (3.130), the massless states exist at $R = \frac{\sqrt{6}}{4}$ in $w = \pm 1$, and $R = 2\sqrt{6}$ in $w = \pm 3$. From the second term in (3.130), they exist at $R = \frac{\sqrt{2}}{4}$ in $w = \pm 1$.

3.5 Generality

Unitarity

The unitarity is not obvious for asymmetric orbifolds. For example, the building blocks defined by asymmetric twists have non-trivial phases as seen in subsection 3.1. The orbifolds we have studied are consistent with unitarity. We have directly checked that by writing down the partition functions explicitly in previous subsection.

Here, we would like to find the explanation of the unitarity, namely, the absence of the negative norm states of such asymmetric orbifolds without writing down all the terms of the partition function. We have calculated the partition functions of the orbifolds using the conception of twisting introduced in section 2. Let us organize this procedure again. At the beginning, we have inserted the projection operator of the orbifold $\mathbf{P}_{\rm orb}$ in the trace corresponding to the original space naming "untwisted",

$$\mathrm{Tr}_{\rm untwisted}\left[(\mathbf{P}_{\rm orb})q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right], \tag{3.132}$$

and have decomposed it into the terms as follows.

$$\frac{1}{M}\sum_{b\in\mathbb{Z}_M} Z_{(0,b)}(\tau,\bar{\tau}), \tag{3.133}$$

where M is the order of the orbifold action. We have regarded $(0,b)$-terms as the trace where the twists are inserted respectively, and have described the partition function in untwisted sector as the sum over them with the coefficient $1/M$. We assume that M is finite temporarily for the discussion. We have found the partition function in a-twisted sectors by the modular transformations. Then, expressing each terms of the partition function by q-expansions, the partition function of a-twisted sector can be interpreted as the trace of $(a,0)$ where a projection operator $\mathbf{P}_a$ is inserted as follows.

$$\frac{1}{M}\sum_{b\in\mathbb{Z}_{M_a}} Z_{(a,b)}(\tau,\bar{\tau}) = \frac{1}{M}\sum_{b\in\mathbb{Z}_M} Z'_{(a,b)}(\tau,\bar{\tau}) = \mathrm{Tr}_{a-\rm twisted}\left[(\mathbf{P}_a)q^{L_0-\frac{c}{24}}\bar{q}^{\tilde{L}_0-\frac{c}{24}}\right], \tag{3.134}$$

where we use the relation of (2.41). Therefore, we can confirm the unitarity by checking the coefficients of $Z_{(a,0)}(\tau,\bar{\tau})$ before the projection.

In the models we have studied, we should firstly look at the building blocks (C.19)-(C.21), and (3.12). For all $b=0$, these building blocks do not have the phase, so there are no negative norm states in the orbifold whose partition function is composed of these combinations.

This can be extended to the case that the twists include the translation $\mathcal{T}_{2\pi R}$. In this case, it is useful to separate out the translation of infinite order and the other parts of order M. The building block of translation is written as $Z_{R(a,b)}$ seen in (2.10). We assume that the phases from the other parts do not arise when $b=0$, and these phases can be written as

$$\exp\left(2\pi i b L\right), \tag{3.135}$$

where $L \in \frac{1}{M}\mathbb{Z}$. The Poisson resummation of this partition function relates to the following part.

$$\sum_{m\in\mathbb{Z}} Z_{R(w,m)} \exp\left(2\pi i m L\right) \tag{3.136}$$

Separating this infinite sum such that

$$\sum_{b\in\mathbb{Z}_M} \sum_{m\in M\mathbb{Z}+b} Z_{R(w,m)} \exp\left(2\pi i m L\right), \tag{3.137}$$

we obtain

$$\sum_{b\in\mathbb{Z}_M} \sum_{n\in\mathbb{Z}} \frac{1}{M} \exp\left[2\pi i b\left(\frac{n}{M}+L\right)\right] q^{\frac{1}{4}\left(Rw-\frac{n}{MR}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw+\frac{n}{MR}\right)^2}. \tag{3.138}$$

In this way, the phases are something to the power of b again. Eventually, negative norm states do not arise in the asymmetric orbifolds defined by the twist of the translation of one direction in addition to asymmetric twists as we have confirmed directly.

Stability for asymmetric orbifolds with vanishing cosmological constant

The tachyonic states do not arise in the orbifolds with vanishing cosmological constant we have studied regardless of the presence or absence of supersymmetry of spacetime. We would like to explain that clearly. Obviously, no tachyonic states appear in untwisted sector, which is inherited from the original coordinates, so we focus on twisted sectors. In order to observe the physical states on the asymetric orbifolds, it is convenient to check the level-matching condition as follows.

We begin to consider the conformal weights from each building blocks. When the partition function vanishes in our models, the partition function for each a-twisted sector includes the building blocks $\mathcal{J}(\tau)$ or $\overline{Z^{\psi}_{[4](a,b)}(\tau)}$ or these conjugates. We can read the confomal weight 1/2 from the leading of $\mathcal{J}(\tau)$, and 1/4 from that of $\overline{Z^{\psi}_{[4](a,b)}(\tau)}$. When the right-moving fermionic sector's building block is $\overline{Z^{\psi}_{[4](a,b)}(\tau)}$, the building block of the bosonic sector includes $Z^{T^N}_{[\alpha,\beta](a,b)}(\tau,\bar{\tau}), (\beta \geq 2)$, where the minimum conformal weight is more than 1/4. Therefore, the conformal weight is more than 1/2 in the left or right-moving sector at least when the partition function is zero by these building blocks.

In the case that the twist does not include the translation, we can immediately find that all the states in the orbifolds with these building blocks are not tachyonic by level matching condition

$$h_L - h_R \in \mathbb{Z}, \tag{3.139}$$

where h_L, h_R denote the conformal weights in left and right movers.

Next, we consider the case that the twist includes the translation to use the result of (3.138) again. We assume that the phase $\exp\left(2\pi i b L\right)$ in (3.135) arises from the building blocks of the directions other than the translation in w-twisted sector. This means that the phase $\exp(2\pi i w L)$ arises under the modular transformation $\tau \to \tau + 1$, so wL is the difference between the contributions of the zero point every from the left and right-mover of these building blocks. Therefore, zero point energy from these building blocks is more than $|w|L$, where $0 \leq L \leq 1$. Furthermore, when the total partition function vanishes, and $|w|L \leq \frac{1}{2}$, the contribution of this zero point vacuum energy to mass squared is more than

$$\frac{1}{2} + \left(\frac{1}{2} - |w|L\right), \tag{3.140}$$

by the discussion about the conformal weight above.

On the other hand, this phase influences the excitations from the direction where the translation acts. As seen in (3.138), the Kaluza-Klein momentum in this direction is shifted by L such that

$$\frac{n}{M} \in \mathbb{Z} - L. \tag{3.141}$$

The contribution of the mass squared from this direction is

$$\frac{1}{2}\left\{(Rw)^2 + \left(\frac{n}{MR}\right)^2\right\} \geq \left|w\frac{n}{M}\right|. \tag{3.142}$$

Let us consider the mass squared of the lightest excitation in twisted sectors.

- If $\frac{1}{2} \leq L \leq 1$, the lightest contribution from this direction is $|w|(1-L)$. Then, the mass squared is more than

  $$|w|(1-L) + |w|L - 1 = |w| - 1 \geq 0, \tag{3.143}$$

  in twisted sectors.

- If $0 \leq L \leq \frac{1}{2}$, the lightest contribution from the direction where the translation acts is $|w|L$. Then, the mass squared is more than

  $$|w|L + |w|L - 1 = 2|w|L - 1. \tag{3.144}$$

- In addition to $0 \leq L \leq \frac{1}{2}$, if the partition function vanishes due to certain building blocks such as $\mathcal{J}(\tau)$, $\overline{Z^{\psi}_{[4](a,b)}}(\tau)$, the mass squared is more than

  $$|w|L + \left\{\frac{1}{2} + \left(\frac{1}{2} - |w|L\right)\right\} - 1 = 0. \tag{3.145}$$

As a result, the tachyonic states do not appear in the asymmetric orbifolds where the partition function vanishes in each w-twisted sector. Otherwise, the tachyonic states potentially exist in the case of (3.144) when the partition function does not vanish.

4 Summary

We have discussed the spectra of the strings on various asymmetric orbifolds by means of twisting construction mentioned in section 2. At the beginning, we have investigated the building blocks of the bosonic and fermionic sectors in order to efficiently calculate the partition functions of variety of the orbifolds in section 3. The orders of periodicity of some building blocks in twisted sectors are larger than those of the corresponding twists, which is related to the restriction on the level of the spectra to match them on right and left-movers.

We have studied the non-supersymmetric orbifolds with vanishing vacuum energy density at one-loop by using these building blocks. In subsection 3.2, we actually investigate the orbifold defined by the twist

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-1)^{F_L}|_{\psi_L} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4}, \tag{3.53}$$

acting on $\mathcal{M} = S^1 \times \mathbb{R} \times T^4[SO(8)]$ as a first example. We find the desired non-supersymmetric orbifold models when the square of $(-\mathbf{1}_R)^{\otimes 4}|_{T^4}$ in this twist is $(-1)^{F_R}|_{\psi_R}$ or $(-1)^{F_R}|_{T^4[SO(8)]} \otimes (-1)^{F_R}|_{\psi_R}$, which we call "model (b)", "model (d)" respectively. The $\mathbb{Z}_4$-chiral reflection $(-\mathbf{1}_R)|_{\psi_R}$ in twists plays an important part in constructing such orbifolds, keeping bose-fermi cancellation but breaking supersymmetry of the spacetime in untwisted sector. However, the supercharges are potentially constructed by the spin fields in the twisted sectors even if the twist includes the $\mathbb{Z}_4$-chiral reflection. Thus, we have considered the twists with the translation $\mathcal{T}_{2\pi R}|_{\mathbb{R}}$ as well as the $\mathbb{Z}_4$-chiral reflection.

In order to study how to break the supersymmetry of orbifold in spite of bose-fermi cancellation in detail, in subsection 3.3, we have studied two modifications as follows.

(i) We consider the orbifolds defined by the twist operator schematically written as

$$\mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes (-\mathbf{1}_L)^{\otimes 2}|_{T^4} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4} \equiv \mathcal{T}_{2\pi R}|_{\mathbb{R}} \otimes \sigma_{[2,4]}|_{T^4}. \tag{3.93}$$

Namely, the twist includes $(-\mathbf{1}_L)^{\otimes 2}|_{T^4}$ instead of $(-1)^{F_L}|_{\psi_L}$ in (3.53).

(ii)We consider the orbifolds defined by the twist operator schematically written as

$$(-1)^{F_L}|_{\psi_L} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4}. \tag{3.94}$$

As a result, the models obtained by these modifications include the non-supersymmetric orbifolds with vanishing cosmological constant. For the modification (i), we obtain the similar results to the models of (3.53), but the parts of the left-moving spin fields remain, which do not construct the supercharges of the spacetime. For the modification (ii), supercharges are not constructed in the twisted sectors when the square of (3.94) includes $(-1)^{F_R}|_{T^4}$ even if the translation $\mathcal{T}_{2\pi R}$ is not used. This is because level matching condition removes the states that construct the supercurrents.

In subsection 3.4, we have investigated the various models, changing the coordinates which the twists act on. In calculating these partition functions, we can use the same building blocks for fermionic sector. For bosonic sector, we can reduce the calculation of the building blocks to some results.

In subsection 3.5, we check that these orbifold models are consistent with unitarity. In addition, we confirm that the tachyonic states do not appear if the vacuum energy density vanishes the bose-fermi cancellation in each a-twisted sector regardless of the presence or absence of supersymmetry.

There are some interesting subjects left unsolved. Firstly, we analyze the string vacua on asymmetric orbifolds at one-loop level, but it is important to reveal whether we obtain the similar results of the vacuum energy density at more than two-loop to at one-loop. However, it become very hard to calculate the more than two-loop partition function. There are attempts to calculate the two-loop partition function on non-supersymmetric asymmetric orbifolds, but the conclusion has not been reached this matter [35].

Secondly, we do not study what the non-supersymmetric toroidal types of the asymmetric orbifolds are constructed with consistency. For example, it is simple problem whether the orbifolds defined by the twist with the chiral reflection $\sigma_{[\alpha,\beta]}$ for $\alpha-\beta \in 2\mathbb{Z}+1$ can be constructed consistently or technically. This seems to be similar to the case of $\sigma_{[\alpha,\beta]}$ for $\alpha - \beta \in 2\mathbb{Z}$, but, for example, we cannot easily calculate the building blocks related to $\sigma_{[1,4]}$ in the same way of $\sigma_{[2,4]}$. We have not confirmed the presence of the modular invariant partition function yet though the partition function may be easily calculated by combining the building blocks of bosonic and fermionic sectors practically. Another example is the extension of the chiral reflection. There may be the discrete rotations which assume the same role of breaking supersymmetry as $\mathbb{Z}_2$ or $\mathbb{Z}_4$-chiral reflection.

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A Notational Convention

Theta functions

$$\theta_1(\tau,z) := i\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2}y^{n-\frac{1}{2}} \equiv 2\sin(\pi z)q^{\frac{1}{8}}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m)$$
$$\theta_2(\tau,z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2}y^{n-\frac{1}{2}} \equiv 2\cos(\pi z)q^{\frac{1}{8}}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m)$$
$$\theta_3(\tau,z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-\frac{1}{2}})(1+y^{-1}q^{m-\frac{1}{2}})$$
$$\theta_4(\tau,z) := \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n^2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-\frac{1}{2}})(1-y^{-1}q^{m-\frac{1}{2}})$$

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m)$$

We have set $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$ ($^\forall\tau\in\mathbb{H}^+, ^\forall z\in\mathbb{C}$), and used abbreviations, $\theta_i(\tau)\equiv\theta_i(\tau,0)$.

In this paper, we use Jacobi's abstruse identity,

$$\theta_3(\tau,z)^4-\theta_4(\tau,z)^4-\theta_2(\tau,z)^4+\theta_1(\tau,z)^4=0, \tag{A.1}$$

and other identities as follows;

$$\theta_2(2\tau)=\sqrt{\frac{1}{2}\{\theta_3(\tau)^2-\theta_4(\tau)^2\}}, \tag{A.2}$$

$$\theta_3(2\tau)=\sqrt{\frac{1}{2}\{\theta_3(\tau)^2+\theta_4(\tau)^2\}}, \tag{A.3}$$

$$\theta_4(2\tau)=\sqrt{\theta_3(\tau)\theta_4(\tau)}. \tag{A.4}$$

B Poisson Resummation Formula

For a partition function f on $\mathbb{R}$, one can use the relation

$$\sum_{n\in\mathbb{Z}}f(n)=\sum_{k\in\mathbb{Z}}\hat{f}(k), \tag{B.1}$$

where the Fourier transform $\hat{f}(k)$ is defined as

$$\hat{f}(k)=\int_{\mathbb{R}}dxe^{-2\pi kx}f(x). \tag{B.2}$$

Proof:

Let $F(x)$ be the periodic function given by

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n).$$

Then, the Fourier coefficients of F are

$$c_k = \int_0^1 dx F(x) e^{-2\pi ikx} = \int_0^1 dx \sum_{n\in\mathbb{Z}} f(x+n) e^{-2\pi ikx} \tag{B.3}$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} dx f(x) e^{-2\pi ikx} = \int_{\mathbb{R}} dx e^{-2\pi kx} f(x) = \hat{f}(k). \tag{B.4}$$

Evaluating at $x = 0$, we obtain (B.1).

Useful Form:

Practically, the following relation is often used, derived from (B.1),

$$\sum_{n\in\mathbb{Z}} \exp\left(-\pi\alpha(n+a)^2 + 2\pi ib(n+a)\right) = \frac{1}{\sqrt{\alpha}} \sum_{m\in\mathbb{Z}} \exp\left(-\frac{\pi\alpha(n-b)^2}{\alpha} + 2\pi ima\right), \tag{B.5}$$

$$(\alpha > 0, a, b \in \mathbb{R}),$$

but in this paper, it is more useful to introduce the explicit description as follows,

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} Z_{R(w,m)} \exp(2\pi imL) \\ &= \sum_{b\in\mathbb{Z}_M} \sum_{n\in\mathbb{Z}} \frac{1}{M} (-1)^{2n\frac{b}{M}} q^{\frac{1}{4}\left(Rw - \frac{n}{MR}\right)^2} \bar{q}^{\frac{1}{4}\left(Rw + \frac{n}{MR}\right)^2} \exp(2\pi ibL),\end{aligned} \tag{B.6}$$

$$(M, b \in \mathbb{Z}, \quad L \in \frac{1}{M}\mathbb{Z}),$$

where $Z_{R(w,m)}$ is defined in (2.10).

C Building blocks

We summarize the building blocks of the partition function in this paper.

C.1 Bosonic Sector

At first, we introduce the building blocks studied in subsection 3.1. It is useful to introduce $T^N[SO(2N)]$ building blocks which are defined by orbifolding by $(-1)^{F_L}$ or

$(-1)^{F_R}$;

$$\mathbf{Z}^{T^N[SO(2N)]}_{(a,b)}(\tau,\bar\tau)$$
$$\equiv \begin{cases} \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N}+\left|\frac{\theta_4}{\eta}\right|^{2N}+\left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N}+\left|\frac{\theta_4}{\eta}\right|^{2N}-\left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\ \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N}-\left|\frac{\theta_4}{\eta}\right|^{2N}+\left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left\{-\left|\frac{\theta_3}{\eta}\right|^{2N}+\left|\frac{\theta_4}{\eta}\right|^{2N}+\left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1). \end{cases} \tag{C.1}$$

Followings are asymmetric building blocks which are defined by applying twist operator

$$\sigma_{\text{boson1}} \equiv (-\mathbf{1}_R)^{\otimes 4}|_{T^4} \tag{C.2}$$

to the partition function (2.80).

(i)$(\sigma_{\text{boson1}})^2 = \mathbf{1}$

$$Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau) \tag{3.20}$$

(ii)$(\sigma_{\text{boson1}})^2 = (-1)^{F_R}|_{T^4}$

$$\mathbf{Z}^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau) \tag{3.21}$$

Similarly, the building blocks which are obtained by asymmetric twist operator

$$\sigma_{\text{boson1}'} \equiv (-\mathbf{1}_R)^{\otimes 2}|_{T^2_1} \times (-\mathbf{1}_R)^{\otimes 2}|_{T^2_2} \tag{C.3}$$

acting on the partition function (2.78) are following blocks;

(i) $(\sigma_{\text{boson1}'})^2 = \mathbf{1}$

$$\left(Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)\right)^2 = \begin{cases} \left(Z^{T^2[SO(4)]}(\tau,\bar\tau)\right)^2 & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left(Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[4,4](a,b)}(\tau,\bar\tau)\right) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases} \tag{C.4}$$

(ii) $(\sigma_{\text{boson1}'})^2 = (-1)^{F_R}|_{T^2_1}\otimes \mathbf{1}|_{T^2_2}$ or $(\sigma_{\text{boson1}'})^2 = \mathbf{1}|_{T^2_1}\otimes(-1)^{F_R}|_{T^2_2}$

$$\mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) \tag{C.5}$$

(iii) $(\sigma_{\text{boson}1'})^2 = (-1)^{F_R}|_{T_1^2} \otimes (-1)^{F_R}|_{T_2^2}$

$$\left(\mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar{\tau})\right)^2 \tag{C.6}$$

Next, other types of building block are obtained by replacing the twist operator (C.2) with

$$\sigma_{\text{boson}2} \equiv (-\mathbf{1}_L)^{\otimes 2} \otimes (-\mathbf{1}_R)^{\otimes 4}|_{T^4}, \tag{C.7}$$

and apply it to the partition function (2.80).

(i) $(\sigma_{\text{boson}2})^2 = 1_{\text{R}}$

$$Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar{\tau}) \equiv \begin{cases} \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^8 + \left|\frac{\theta_4}{\eta}\right|^8 + \left|\frac{\theta_2}{\eta}\right|^8\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ e^{\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)}\left|\frac{\theta_3\theta_4}{\eta^2}\right|^2 \frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2 + (-1)^{\frac{a}{2}}\left(\frac{\theta_4}{\eta}\right)^2\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_3}{\eta^2}\right)}\left|\frac{\theta_2\theta_3}{\eta^2}\right|^2 \frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2 + (-1)^{\frac{b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_4}{\eta^2}\right)}\left|\frac{\theta_2\theta_4}{\eta^2}\right|^2 \frac{1}{2}\left\{\left(\frac{\theta_4}{\eta}\right)^2 - i(-1)^{\frac{a+b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1) \end{cases} \tag{C.8}$$

(ii) $(\sigma_{\text{boson}2})^2 = (-1)^{F_R}|_{T^4}$

$$\mathbf{Z}^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar{\tau}) \equiv \begin{cases} E^{T^4[SO(8)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar{\tau}) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar{\tau}) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases} \tag{C.9}$$

Similarly, the building blocks corresponding to the orbifolding twist operator,

$$\sigma_{\text{boson}2'} \equiv (-\mathbf{1}_L)^{\otimes 2} \otimes (-\mathbf{1}_R)^{\otimes 2}|_{T_1^2} \otimes (-\mathbf{1}_R)^{\otimes 2}|_{T_2^2}, \tag{C.10}$$

acting on the partition function (2.78) are as follows.

(i) $(\sigma_{\text{boson}2'})^2 = 1$

$$Z^{T^2[SO(4)]}_{[2,2](a,b)}(\tau,\bar{\tau}) Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar{\tau}) = \begin{cases} \left\{Z^{T^2[SO(4)]}(\tau,\bar{\tau})\right\}^2 & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar{\tau}) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1) \end{cases} \tag{C.11}$$

(ii) $(\sigma_{\text{boson}2'})^2 = (-1)^{F_R}|_{T_1^2} \otimes \mathbf{1}|_{T_2^2}$ or $(\sigma_{\text{boson}2'})^2 = \mathbf{1}|_{T_1^2} \otimes (-1)^{F_R}|_{T_2^2}$

$$\mathbf{Z}^{T^2[SO(4)]}_{[2,2](a,b)}(\tau,\bar\tau) Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) = \begin{cases} E^{T^2[SO(4)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau) \times Z^{T^2[SO(4)]}(\tau,\bar\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar\tau) & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1). \end{cases} \tag{C.12}$$

(iii) $(\sigma_{\text{boson}2'})^2 = (-1)^{F_R}|_{T_1^2} \otimes (-1)^{F_R}|_{T_2^2}$

$$\mathbf{Z}^{T^2[SO(4)]}_{[2,2](a,b)}(\tau,\bar\tau) \mathbf{Z}^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) = \begin{cases} \left\{ E^{T^2[SO(4)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau) \right\}^2 & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar\tau) & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1) \end{cases} \tag{C.13}$$

Replacing the twist operator (C.10) with

$$\sigma_{\text{boson}2''} \equiv (-\mathbf{1}_L)^{\otimes 1} \otimes (-\mathbf{1}_R)^{\otimes 2}|_{T_1^2} \otimes (-\mathbf{1}_L)^{\otimes 1} \otimes (-\mathbf{1}_R)^{\otimes 2}|_{T_2^2}, \tag{C.14}$$

one can obtain following building blocks.

(i) $(\sigma_{\text{boson}2''})^2 = 1$

$$\begin{aligned} &\left\{ Z^{T^2[SO(4)]}_{[1,2](a,b)}(\tau,\bar\tau) \right\}^2 \\ &= \begin{cases} \frac{1}{2}\left[Z^{T^4[SO(8)]}(\tau,\bar\tau) + \left|\frac{\theta_3\theta_4}{\eta^2}\right|^4 + \left|\frac{\theta_4\theta_2}{\eta^2}\right|^4 + \left|\frac{\theta_2\theta_3}{\eta^2}\right|^4 \right] & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ \frac{1}{2}\left(Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar\tau) + Z^{T^4[SO(8)]}_{[4,4](a,b)}(\tau,\bar\tau) \right) & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1). \end{cases} \end{aligned} \tag{C.15}$$

(ii) $(\sigma_{\text{boson}2''})^2 = (-1)^{F_R}|_{T_1^2} \otimes \mathbf{1}|_{T_2^2}$ or $(\sigma_{\text{boson}2''})^2 = \mathbf{1}|_{T_1^2} \otimes (-1)^{F_R}|_{T_2^2}$

$$\begin{aligned} &\mathbf{Z}^{T^2[SO(4)]}_{[1,2](a,b)}(\tau,\bar\tau) Z^{T^2[SO(4)]}_{[1,2](a,b)}(\tau,\bar\tau) \\ &\equiv \begin{cases} E^{T^2[SO(4)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau) \times Z^{T^2[SO(4)]}(\tau,\bar\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ \left\{ Z^{T^2[SO(4)]}_{[1,2](a,b)}(\tau,\bar\tau) \right\}^2 & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1) \end{cases} \end{aligned} \tag{C.16}$$

(iii) $(\sigma_{\text{boson2}''})^2 = (-1)^{F_R}|_{T_1^2} \otimes (-1)^{F_R}|_{T_2^2}$

$$\left\{\mathbf{Z}^{T^2[SO(4)]}_{[1,2](a,b)}(\tau,\bar\tau)\right\}^2 \equiv \begin{cases} \left\{E^{T^2[SO(4)]}_{(\frac{a}{2},\frac{b}{2})}(\tau,\bar\tau)\right\}^2 & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \left\{Z^{T^2[SO(4)]}_{[1,2](a,b)}(\tau,\bar\tau)\right\}^2 & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1) \end{cases} \tag{C.17}$$

In general,

$$Z^{T^2[SO(4)]}_{[\alpha_1,\beta_1](a,b)}(\tau,\bar\tau)Z^{T^2[SO(4)]}_{[\alpha_2,\beta_2](a,b)}(\tau,\bar\tau) = \begin{cases} \left(Z^{T^2[SO(4)]}(\tau,\bar\tau)\right)^2 & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left(Z^{T^4[SO(8)]}_{[\alpha_1+\alpha_2,\beta_1+\beta_2](a,b)}(\tau,\bar\tau) + Z^{T^4[SO(8)]}_{[4-\alpha_1+\alpha_2,4-\beta_1+\beta_2](a,b)}(\tau,\bar\tau)\right) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases} \tag{C.18}$$

Similarly, we summarize the other building blocks we use in subsection 3.4.

$$Z^{T^N[SO(2N)]}_{[\alpha,\alpha+4](a,b)}(\tau,\bar\tau) \equiv \begin{cases} \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N} + \left|\frac{\theta_4}{\eta}\right|^{2N} + \left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ (-1)^{\frac{a}{2}}\overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)^2}\left|\frac{\theta_3\theta_4}{\eta^2}\right|^{\alpha}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^4\left|\frac{\theta_3}{\eta}\right|^{2N-8-2\alpha} + \left(\frac{\theta_4}{\eta}\right)^4\left|\frac{\theta_4}{\eta}\right|^{2N-8-2\alpha}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\ (-1)^{\frac{b}{2}}\overline{\left(\frac{\theta_2\theta_3}{\eta^2}\right)^2}\left|\frac{\theta_2\theta_3}{\eta^2}\right|^{\alpha}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^4\left|\frac{\theta_3}{\eta}\right|^{2N-8-2\alpha} + \left(\frac{\theta_2}{\eta}\right)^4\left|\frac{\theta_2}{\eta}\right|^{2N-8-2\alpha}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\ e^{-\frac{i\pi}{2}ab}\overline{\left(\frac{\theta_2\theta_4}{\eta^2}\right)^2}\left|\frac{\theta_2\theta_4}{\eta^2}\right|^{\alpha}\frac{1}{2}\left\{\left(\frac{\theta_4}{\eta}\right)^4\left|\frac{\theta_4}{\eta}\right|^{2N-8-2\alpha} - \left(\frac{\theta_2}{\eta}\right)^4\left|\frac{\theta_2}{\eta}\right|^{2N-8-2\alpha}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1) \end{cases} \tag{C.19}$$

$$Z^{T^N[SO(2N)]}_{[\alpha,\alpha+2](a,b)}(\tau,\bar\tau) \equiv \begin{cases} \frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{2N} + \left|\frac{\theta_4}{\eta}\right|^{2N} + \left|\frac{\theta_2}{\eta}\right|^{2N}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ e^{\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_3\theta_4}{\eta^2}\right)}\left|\frac{\theta_3\theta_4}{\eta^2}\right|^{\alpha}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2\left|\frac{\theta_3}{\eta}\right|^{2N-4-2\alpha} + (-1)^{\frac{a}{2}}\left(\frac{\theta_4}{\eta}\right)^2\left|\frac{\theta_4}{\eta}\right|^{2N-4-2\alpha}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_3}{\eta^2}\right)}\left|\frac{\theta_2\theta_3}{\eta^2}\right|^{\alpha}\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^2\left|\frac{\theta_3}{\eta}\right|^{2N-4-2\alpha} + (-1)^{\frac{b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\left|\frac{\theta_2}{\eta}\right|^{2N-4-2\alpha}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\ e^{-\frac{i\pi}{4}ab}\overline{\left(\frac{\theta_2\theta_4}{\eta^2}\right)}\left|\frac{\theta_2\theta_4}{\eta^2}\right|^{\alpha}\frac{1}{2}\left\{\left(\frac{\theta_4}{\eta}\right)^2\left|\frac{\theta_4}{\eta}\right|^{2N-4-2\alpha} - i(-1)^{\frac{a+b}{2}}\left(\frac{\theta_2}{\eta}\right)^2\left|\frac{\theta_2}{\eta}\right|^{2N-4-2\alpha}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1) \end{cases} \tag{C.20}$$

$$
Z^{T^{\frac{N}{2}}[SO(N)]}_{[\alpha,\alpha](a,b)}(\tau,\bar\tau)
\equiv \begin{cases}
\frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^N+\left|\frac{\theta_4}{\eta}\right|^N+\left|\frac{\theta_2}{\eta}\right|^N\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\
\frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{N-\alpha}\left|\frac{\theta_4}{\eta}\right|^{\alpha}+\left|\frac{\theta_4}{\eta}\right|^{N-\alpha}\left|\frac{\theta_3}{\eta}\right|^{\alpha}\right\} & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}+1) \\
\frac{1}{2}\left\{\left|\frac{\theta_3}{\eta}\right|^{N-\alpha}\left|\frac{\theta_2}{\eta}\right|^{\alpha}+\left|\frac{\theta_3}{\eta}\right|^{N-\alpha}\left|\frac{\theta_2}{\eta}\right|^{\alpha}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}) \\
\frac{1}{2}\left\{\left|\frac{\theta_4}{\eta}\right|^{N-\alpha}\left|\frac{\theta_2}{\eta}\right|^{\alpha}+\left|\frac{\theta_2}{\eta}\right|^{N-\alpha}\left|\frac{\theta_4}{\eta}\right|^{\alpha}\right\} & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1)
\end{cases}
\tag{C.21}
$$

$$
Z^{S^1[SO(2)]}_{[1,0](a,b)}(\tau,\bar\tau)\left(Z^{S^1[SO(2)]}_{[0,1](a,b)}(\tau,\bar\tau)\right)^3
= \begin{cases}
\left(Z^{S^1[SO(2)]}(\tau,\bar\tau)\right)^4 Z^{T^2[SO(4)]}(\tau,\bar\tau) \quad (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\
\frac{1}{8}\left(Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[1,5](a,b)}(\tau,\bar\tau) \quad +3Z^{T^4[SO(8)]}_{[3,3](a,b)}(\tau,\bar\tau)+3Z^{T^4[SO(8)]}_{[3,5](a,b)}(\tau,\bar\tau)\right) \\
\qquad (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1).
\end{cases}
\tag{C.22}
$$

$$
Z^{S^1[SO(2)]}_{[1,0](a,b)}(\tau,\bar\tau)Z^{S^1[SO(4)]}_{[0,1](a,b)}(\tau,\bar\tau)Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau)
= \begin{cases}
\left(Z^{S^1[SO(2)]}(\tau,\bar\tau)\right)^2 Z^{T^2[SO(4)]}(\tau,\bar\tau) \quad (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\
\frac{1}{4}\left(Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[1,5](a,b)}(\tau,\bar\tau) \quad +Z^{T^4[SO(8)]}_{[3,3](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[3,5](a,b)}(\tau,\bar\tau)\right) \\
\qquad (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1).
\end{cases}
\tag{C.23}
$$

$$
Z^{S^1[SO(2)]}_{[1,0](a,b)}(\tau,\bar\tau)Z^{T^3[SO(6)]}_{[0,3](a,b)}(\tau,\bar\tau)
= \begin{cases}
Z^{S^1[SO(2)]}(\tau,\bar\tau)Z^{T^3[SO(6)]}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\
\frac{1}{2}\left(Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[1,5](a,b)}(\tau,\bar\tau)\right) & (a\in 2\mathbb{Z}+1, b\in 2\mathbb{Z}+1).
\end{cases}
\tag{C.24}
$$

$$
Z^{T^2[SO(4)]}_{[1,1](a,b)}(\tau,\bar\tau)\left(Z^{T^1[SO(2)]}_{[0,1](a,b)}(\tau,\bar\tau)\right)^2
= \begin{cases}
Z^{T^2[SO(4)]}(\tau,\bar\tau)\left(Z^{T^1[SO(2)]}(\tau,\bar\tau)\right)^2 \quad (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\
\frac{1}{4}\left(Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar\tau)+2Z^{T^4[SO(8)]}_{[3,3](a,b)}(\tau,\bar\tau) \quad +Z^{T^4[SO(8)]}_{[3,5](a,b)}(\tau,\bar\tau)\right) \\
\qquad (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1).
\end{cases}
\tag{C.25}
$$

$$
\begin{aligned}
&Z^{T^2[SO(4)]}_{[1,1](a,b)}(\tau,\bar\tau)Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) \\
&= \begin{cases} \left(Z^{T^2[SO(4)]}(\tau,\bar\tau)\right)^2 & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left(Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[3,5](a,b)}(\tau,\bar\tau)\right) & \\ & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases}
\end{aligned} \tag{C.26}
$$

$$
\begin{aligned}
&Z^{T^3[SO(6)]}_{[1,2](a,b)}(\tau,\bar\tau)Z^{S^1[SO(2)]}_{[0,1](a,b)}(\tau,\bar\tau) \\
&= \begin{cases} Z^{T^3[SO(6)]}(\tau,\bar\tau)Z^{S^1[SO(2)]}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left(Z^{T^4[SO(8)]}_{[1,3](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[3,3](a,b)}(\tau,\bar\tau)\right) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases}
\end{aligned} \tag{C.27}
$$

$$
\begin{aligned}
&Z^{S^1/\mathbb{Z}_2}_{(a,b)}(\tau,\bar\tau)Z^{T^N[SO(2N)]}_{[\alpha,\beta](a,b)}(\tau,\bar\tau) \\
&= \begin{cases} Z^{S^1}_R(\tau,\bar\tau)Z^{T^N[SO(2N)]}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ Z^{T^N[SO(2N)]}_{[\alpha+1,\beta+1](a,b)}(\tau,\bar\tau) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases}
\end{aligned} \tag{C.28}
$$

$$
\begin{aligned}
&\left(Z^{S^1[SO(2)]}_{[0,1](a,b)}(\tau,\bar\tau)\right)^4 \\
&= \begin{cases} \left(Z^{S^1[SO(2)]}(\tau,\bar\tau)\right)^4 & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{8}\left(Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)+4Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar\tau)\right. & \left.+3Z^{T^4[SO(8)]}_{[4,4](a,b)}(\tau,\bar\tau)\right) \\ & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases}
\end{aligned} \tag{C.29}
$$

$$
\begin{aligned}
&\left(Z^{S^1[SO(2)]}_{[0,1](a,b)}(\tau,\bar\tau)\right)^2 Z^{T^2[SO(4)]}_{[0,2](a,b)}(\tau,\bar\tau) \\
&= \begin{cases} \left(Z^{S^1[SO(2)]}(\tau,\bar\tau)\right)^2 Z^{T^2[SO(4)]}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{4}\left(Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)+2Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar\tau)\right. & \left.+Z^{T^4[SO(8)]}_{[4,4](a,b)}(\tau,\bar\tau)\right) \\ & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases}
\end{aligned} \tag{C.30}
$$

$$
\begin{aligned}
&Z^{T^1[SO(2)]}_{[0,1](a,b)}(\tau,\bar\tau)Z^{T^3[SO(6)]}_{[0,3](a,b)}(\tau,\bar\tau) \\
&= \begin{cases} Z^{T^1[SO(2)]}(\tau,\bar\tau)Z^{T^3[SO(6)]}(\tau,\bar\tau) & (a\in 2\mathbb{Z}, b\in 2\mathbb{Z}) \\ \frac{1}{2}\left(Z^{T^4[SO(8)]}_{[0,4](a,b)}(\tau,\bar\tau)+Z^{T^4[SO(8)]}_{[2,4](a,b)}(\tau,\bar\tau)\right) & (a\in 2\mathbb{Z}+1 \text{ or } b\in 2\mathbb{Z}+1). \end{cases}
\end{aligned} \tag{C.31}
$$

C.2 Fermionic Sector

Fermionic part of the partition function of type II string on ten-dimensional Minkowski coordinate is

$$Z^{\psi,\tilde{\psi}}_{\text{typeII}}(\tau,\bar{\tau}) = \frac{1}{4}|\mathcal{J}(\tau)|^2. \tag{3.38}$$

We consider the fermion chiral blocks from $\mathcal{J}(\tau)$.

1. The non-supersymmetric chiral blocks corresponding to $(-1)^{F_L}$**;**

$$\mathbf{J}_{(a,b)}(\tau) \tag{3.45}$$

2. The chiral blocks corresponding to $(-1_L)^{\otimes 4}$**;**

(i) $\{(-\mathbf{1}_L)^{\otimes 4}\}^2 = 1$

$$Z^{\psi}_{[4](a,b)}(\tau) \tag{3.51}$$

(ii) $\{(-\mathbf{1}_L)^{\otimes 4}\}^2 = (-1)^{F_L}$

$$\mathbf{Z}^{\psi}_{[4](a,b)}(\tau) \equiv \begin{cases} \mathbf{J}_{(\frac{a}{2},\frac{b}{2})}(\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ \\ Z^{\psi}_{[4](a,b)}(\tau) & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1) \end{cases} \tag{C.32}$$

3. The non-supersymmetric chiral blocks corresponding to $(-1_L)^{\otimes 2}$**;**

(i) $\{(-\mathbf{1}_L)^{\otimes 2}\}^2 = 1$

$$Z^{\psi}_{[2](a,b)}(\tau) \equiv \begin{cases} \mathcal{J}(\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ \\ e^{-\frac{i\pi}{4}ab}\left\{\left(\frac{\theta_3}{\eta}\right)^3\left(\frac{\theta_4}{\eta}\right) - (-1)^{\frac{a}{2}}\left(\frac{\theta_4}{\eta}\right)^3\left(\frac{\theta_3}{\eta}\right) + 0\right\} & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}+1) \\ e^{\frac{i\pi}{4}ab}\left\{\left(\frac{\theta_3}{\eta}\right)^3\left(\frac{\theta_2}{\eta}\right) + 0 - (-1)^{\frac{b}{2}}\left(\frac{\theta_2}{\eta}\right)^3\left(\frac{\theta_3}{\eta}\right)\right\} & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}) \\ -e^{\frac{i\pi}{4}ab}\left\{0 + \left(\frac{\theta_4}{\eta}\right)^3\left(\frac{\theta_2}{\eta}\right) + i(-1)^{\frac{a+b}{2}}\left(\frac{\theta_2}{\eta}\right)^3\left(\frac{\theta_4}{\eta}\right)\right\} & (a \in 2\mathbb{Z}+1, b \in 2\mathbb{Z}+1) \end{cases} \tag{C.33}$$

(ii) $\{(-\mathbf{1}_L)^{\otimes 2}\}^2 = (-1)^{F_L}$

$$\mathbf{Z}^{\psi}_{[2](a,b)}(\tau) \equiv \begin{cases} \mathbf{J}_{(\frac{a}{2},\frac{b}{2})}(\tau) & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z}) \\ \\ Z^{\psi}_{[2](a,b)}(\tau) & (a \in 2\mathbb{Z}+1 \text{ or } b \in 2\mathbb{Z}+1) \end{cases} \tag{C.34}$$

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