Doctoral Thesis

Applications of Stochastic Calculus based on Hyperbolic Brownian Motion

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(双曲ブラウン運動に基づく確率解析の応用)

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Abstract

The present thesis consists of four parts. The first part, Chapter 2, introduces a framework of a discrete stochastic calculus based on *Parisian walk*, a special kind of symmetric random walk in the complex plane, listing some results analogue to those for complex Brownian motion. As an application to mathematical finance, a Parisian-walk analogue of Heston's stochastic volatility model is discussed. In the second part, Chapter 3, an expansion of the transition density of Hyperbolic Brownian motion with drift is given, which is potentially useful for pricing and hedging of options under stochastic volatility models. We work on a condition on the drift which dramatically simplifies the proof. In the third part, Chapter 4, in view of application to pricing of Barrier options under a stochastic volatility model, we study a reflection principle for the hyperbolic Brownian motion, and introduce a hyperbolic version of Imamura-Ishigaki-Okumura's symmetrization. Some results of numerical experiments, which imply the efficiency of the numerical scheme based on the symmetrization, are given. The last part, Chapter 5, is devoted to the study of the time-average of exponential of a fractional Brownian motion from the point of view of so-called PCOC.

Contents

Chapter 1

Introduction

1.1 Parisian Walk and Related Issues

In Chapter 2, the following symmetric random walk on $\mathbb{Z}[\zeta]$ is studied.

Let $\tau_1, \dots, \tau_t, \dots$ be an i.i.d. sequence with $\mathrm{P}(\tau_i = 1) = \mathrm{P}(\tau_i = \zeta) = \mathrm{P}(\tau_i = \zeta^2) = 1/3$ for $i \in \mathbb{N}$, where we denote $\zeta = (-1 + \sqrt{-3})/2$. The filtration generated by $\{\tau\}$ will be denoted by $\mathrm{F} \equiv \{\mathcal{F}_t\}$.

Definition 1.1. *An* F*-adapted complex valued process* $\{Z_t\}$ *is called* **Parisian** *(walk) if (i) it is a martingale starting from a point in* $\mathbb{Z}[\zeta] \equiv \{a + b\zeta : a, b \in \mathbb{Z}\}$*, and (ii)* $Z_{t+1} - Z_t =: \Delta Z_t \in \{1, \zeta, \zeta^2\}$ *for all* t.

Thus, a Parisian walk (PW for short) is a random walk on $\mathbb{Z}[\zeta]$. Note that there are a lot of Parisian walks as functions of $\{\tau\}$, but the law is unique up to the initial point. We claim that PW is a discrete analogue of the complex

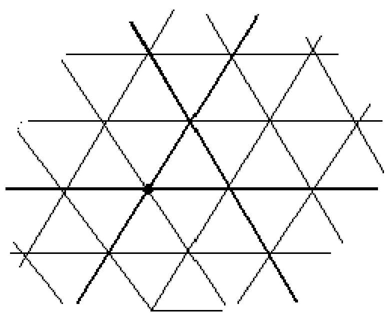

Figure 1.1: Parisian walkways $\mathbb{Z}[\zeta]$.

Brownian motion (cBM for short), just as the simple symmetric random walk on $\mathbb{Z}$ is to the real one dimensional Brownian motion.

The latter analogy, the real case, is already established since many similar properties are known (see e.g. [15, 16]). For our complex case, we have the following "evidences" so far;

(i) The scaling limit of PW converges to cBM.

(ii) The Itô's formula for PW (Proposition 2.2 below) looks very much like (symbolically the same as) the one for cBM.

(iii) We can define a discrete analogue of the conformal martingale, which is shown to be a time changed Parisian walk(Theorem 2.8). The result is analogue to the well-known fact with the conformal martingales.

The first fact (i) is due to the orthogonality between $\mathrm{Re}(\Delta Z)$ and $\mathrm{Im}(\Delta Z)$. The second one is coming from the fact that the martingale dimension of F is two. We give a proof for (ii) in section 2.1, and to discuss (iii) we first argue the conformalilty in $\mathbb{Z}[\zeta]$ in section 2.2.2, then give a proof of (ii) in section 2.3.

In section 2.4, we present a potential application of our Parisian discrete stochastic calculus to mathematical finance. We first remark that with a parametric restriction the Heston model, one of the most popular stochastic volatility model, has a representation in terms of the squared norm and the area of a two-dimensional Ornstein-Uhlenbeck process (Proposition 2.10). Then relying on the fact that the pair of the squared norm and the area of a two-dimensional process has a representation in terms of a complex martingale (Lemma 2.11), we construct a Parisian discrete analogue of the Heston model.

Remark 1.2. *A fully general discrete Itô formula is given in [1], where the convergence rate of a scaling limit is also discussed, but analogies with complex stochastic calculus was out of the scope. Discrete analogues of Malliavin calculus are also studied in [2], [7], and [39], etc, but, again, none of them is interested in analogies with complex stochastic calculus.*

The results presented in Chapter 2 is based on the paper [4] by Akahori and Ida published in *JSIAM Letters*.

1.2 Expansion of Heat Kernel of Hyperbolic Brownian Motion with Drift

As is well recognized, "local stochastic volatility models" can be reduced to Brownian motion with drift thanks to Lamperti's transform. This is not the case when one works with stochastic volatility (henceforth SV) models where the stock price S and its instantaneous volatility V are modeled by a two-dimensional diffusion process. One can not transform it into a two dimensional Brownian motion with drift in general.

As is pointed in [22], however, most of existing stochastic volatility models are "conformally equivalent" to *hyperbolic* Brownian motion (HBM for short) instead; or in other words, many SV diffusion processes (S, V) can be transformed to HBM with drift by a diffeomorphism.

In Chapter3, we shall give an asymptotic expansion formula of the transition density of HBM with drift with respect to the so-called *McKean kernel*; density kernel. That is, the HBM without drift. We claim that this formula can be used in numerical calculations for the under SV models, although in this paper we will not go in depth in this direction.

Our formula is in fact a parametrix one, so along the line of Bally-Kohatsu [9]'s idea, we give an exact simulation interpretation of the parametrix formula[1].

In section 3.1, we briefly recall some basic facts about HBM. In section 3.2, we introduce a drift to the HBM, and describe its transition density by using as parametrix a HBM (Theorem 3.2). In section 3.3, we give an interpretation of the formula given in Theorem 3.2 that it gives a description of an *exact simulation*.

In the present paper we restrict ourselves to working on a simple situation given by (3.4); no drift in the volatility, and (3.5), which reduce the computational complexity of the proof dramatically. Also, we omit the description of how SV models can be transformed to HBM in this paper since it is well-known (see e.g.[21]). The main aim of the present paper is then to show that the condition (3.4) simplifies the proof quite a lot.

The results presented in Chapter 3 is based on the paper [28] by Ida and Imamura published in *Japan Journal of Industrial and Applied Mathematics*.

1.3 Hyperbolic Reflection Principle and Associated Symmetrization

1.3.1 Reflection principle and the static hedge of barrier options

The reflection principle of standard Brownian motion relates the probability distribution of a first hitting time to a boundary to the 1-dimensional marginal distribution of the process. The formula has a direct application in continuous-time finance, that is, the *static hedging* of barrier options[2]. The idea is explained roughly as follows. Suppose that we sold a knock-out call option[3] (which is a typical barrier option). Its pay-off is described as

$$(S_T - K)_+ 1_{\{\tau > T\}},$$

where

- T is the expiry date of the option,
- K is the exercise price,

[1]Here the term "exact" is used because it is not an approximation, but the equality. It may be also referred to as "unbiased" since it is only simulate the expectation of a functional of (S_t, V_t).

[2]A barrier option is a financial derivative with an additional condition that is made active when the underlying price process goes beyond/below a certain level. For details, see e.g. [27].

[3]An option is called of "knock-out" type if the pay-off becomes zero if the underlying price process hits a certain value.

- S is the price process of a risky asset, with $S_0 > K$,

and

- $\tau := \inf\{s > 0 : S_s < K'\}$, the first hitting time of S to K', the knock-out boundary, with $K' < K$.

The static hedge of the knock-out option consists of two plain-vanilla (=without knock-out condition) options, long position of call option with pay-off $(S_T - K)_+$, and short position of "put option" whose value

- at τ equals the call, and
- is zero at T on $\tau > T$.

This simple portfolio hedges the knock-out option since it is

- zero at T on $\tau \le T$ since at τ it is liquidated, and
- $(S_T - K)$ at T on $\tau > T$.

This relation can be expressed as

$$E[(S_T - K)_+ 1_{\{\tau > T\}} | \mathcal{F}_{t\wedge\tau}] = E[(S_T - K)_+ | \mathcal{F}_{t\wedge\tau}] - E[\text{"put option"} | \mathcal{F}_{t\wedge\tau}], \tag{1.1}$$

for $0 \le t \le T$, where $\{\mathcal{F}_t\}$ is the filtration generated by S. The existence of such an option is ensured by the reflection principle. If S is geometric Brownian motion, it can be the option with pay-off $(K - S_T)_+$ since

$$(S_t - K)_{\tau \le t \le T} \overset{\text{(law)}}{=} (K - S_t)_{\tau \le t \le T} \tag{1.2}$$

by the reflection principle. In general, the property is referred to as (arithmetic) *put-call symmetry* at K [12], which is weaker than the reflection principle that ensures put-call symmetry for any K.

The interpretation is first proposed in [10], and there are vast literatures since then (see e.g.[3] and references therein). Among these, we just mention a multi-dimensional extension proposed in [31], where the reflection principle with respect to reflection groups is applied to the pricing of multi-asset barrier options, barrier being the boundary of a Weyl chamber. To the best of our knowledge, it is the first attempt to deal with the cases where the barrier(=knock-out/in boundary) is not a one point set.

1.3.2 Symmetrization and its application to numerical calculation of the price of barrier options

A new point of view in the literature, the *symmetrization*, was first introduced in [30], and further generalized in [5]. The symmetrization is a procedure to convert a given diffusion into the one with a weaker version of reflection principle, aiming at obtaining a precise numerical value of the price of barrier options in a reasonable computational time, rather than static-hedge in the market described in the previous section.

Let us briefly explain their idea. We work on 1-dimensional case for simplicity. Let S be a diffusion process satisfying the following stochastic differential equation:

$$dS_t = \sigma(S_t)\,dW_t + \mu(S_t)\,dt, \tag{1.3}$$

where, σ and μ are piece-wise continuous functions with linear growth and W_t is 1-dimensional Brownian motion. In general we cannot expect the formula (1.1) to hold. The symmetrization $\tilde{S}$ of S alternatively satisfies (1.1). It is defined as a (weak) solution to

$$d\tilde{S}_t = \tilde{\sigma}(\tilde{S}_t)\,dW_t + \tilde{\mu}(\tilde{S}_t)\,dt, \tag{1.4}$$

where

$$\tilde{\sigma}(x) = \begin{cases} \sigma(x) & x \geq K' \\ \sigma(2K - x) & x < K' \end{cases},$$

and

$$\tilde{\mu}(x) = \begin{cases} \mu(x) & x \geq K' \\ -\mu(2K - x) & x < K' \end{cases}, \tag{1.5}$$

The following is proven in [30].

Theorem 1.3 (Imamura-Ishigaki-Okumura[30])**.** *The law-unique solution $\tilde{S}$ of* (1.4) *satisfies the put-call symmetry at K, and $(\tilde{S}_t)_{0\leq t\leq\tau}$ has the same law as $(S_t)_{0\leq t\leq\tau}$.*

It then implies

$$\mathrm{E}[(S_T - K)_+ 1_{\{\tau > T\}}] = \mathrm{E}[(\tilde{S}_T - K)_+] - \mathrm{E}[(K - \tilde{S}_T)_+]. \tag{1.6}$$

The formula (1.6), however, is not anymore interpreted as static hedge relation, but it has another application. The equation (1.6) now reads that

- **An expectation with stopping time is converted to the one without it**.

A numerical calculation of an expectation with stopping time often is a tough challenge due to its path-dependent nature. On the other hand, an expectation with respect to one dimensional marginal of a diffusion process is in most cases numerically tractable. Thus the equation (1.6) gives a new insight to the numerical analysis of barrier options/stopping times.

1.3.3 Euler-Maruyama approximation of the price of barrier options

The most common technique to numerically approximate an expectation with respect to a diffusion process would be so-called "Euler-Maruyama" scheme. Here we briefly recall the scheme.

An Euler-Maruyama discretization of (1.3) with respect to a time partition $0 = t_0 < t_1 < \cdots < t_n = T$ is given by

$$\begin{aligned} S^n_{t_0} &= S_0, \\ S^n_{t_{k+1}} &= S^n_{t_k} + \sigma(S_{t_k})\Delta W_{t_k} \\ &\quad + \mu(S_{t_k})(t_{k+1} - t_k), \\ k &= 0, 1, \cdots, n-1, \end{aligned} \tag{1.7}$$

where $\Delta W^n_{t_k} \sim N(0, t_{k+1} - t_k)$, mutually independent for $k = 0, 1, \cdots, n-1$. The stopping time τ is also approximated by

$$\tau^n := \min\{j : S^n_{t_{j+1}} < K'\}.$$

The expectation in the left-hand-side of (1.6) is approximated by (Monte-Carlo simulation of)

$$\mathrm{E}[(S^n_T - K)_+ 1_{\{\tau^n > T\}}], \tag{1.8}$$

while the right-hand-side counterpart is

$$\mathrm{E}[(\tilde{S}^n_T - K)_+] - \mathrm{E}[(K - \tilde{S}^n_T)_+], \tag{1.9}$$

where $\tilde{S}^n$ is obtained by the same procedure as (1.4).

The discretization error, by which we mean the difference between the true value of the expectation and its Euler-Maruyama approximation like (1.8) or (1.9), is known to be of $O(n^{-1/2})$ in general when $t_k - t_{k-1} = T/n$ for all k. It is reported in [18] that the one with stopping time like (1.8) cannot be improved, while the one with one-dimensional marginal like (1.9) is, provided some continuity of the coefficients, known to be of $O(n^{-1})$.

The symmetrized drift coefficient (1.5) may not be continuous in general even if the original one is very smooth, and as far as we know, no existing result ensures the order is of $O(n^{-1})$ though recently there have been several papers ([36], [38], and [42]) to deal with discontinuous coefficients in line with the problem posed here. In [30], however, they conjecture that it is the case by performing numerical experiments.

References for more detailed and precise results of the order can be found in [30].

1.3.4 SABR model and hyperbolic Brownian motion

In Chapter 4, we study a *hyperbolic* version of the symmetrization, with a view to the application of the pricing of barrier options under SABR model, which is known to be transformed to hyperbolic Brownian motion with drift.

The SABR (stochastic alha-beta-rho) model was introduced in [20]. It is given by

$$\begin{aligned} dS_t &= v_t \sigma(S_t)\, dW_t^1 \\ dv_t &= \nu\, v_t(\sqrt{1-\rho^2} dW_t^2 + \rho dW_t^1), \end{aligned}$$

where (W^1, W^2) is a two dimensional Brownian motion, $\rho \in (-1, 1)$ and ν is a constant. We note that

- A driftless local volatility model is obtained by setting $\nu = 0$, and
- $Z_t := \psi(S_{t/\nu^2}, V_{t/\nu^2}) + \sqrt{-1} V_{t/\nu^2}$ with $\psi(x, y) = (\int_c^x \frac{dz}{\sigma(z)} - \rho y)/\sqrt{1-\rho^2}$ is a hyperbolic Brownian motion with drift, a solution to (4.4) in section 4.2 (for details see [21]).

The following is a “motto" widely accepted among researchers and practitioners in finance (see e.g. [22]): *as tractability of one dimensional diffusion processes is attributed to the reduction to the standard Brownian motion with drift by the Lamperti transform, so the analysis of SABR model will be converted to that of hyperbolic Brownian motion with drift, where we can still work on symmetries from linear fractional transformations.*

We shall observe a realization of this idea in Chapter 4

The results presented in Chapter 4 is based on the paper by Ida, Kinoshita and Matsumoto in *Pacific Journal of Mathematics for Industry*.

1.4 PCOC Property of Fractional Volatility of SABR type

PCOC (pronounced as peacock) is an acronym for French words *Processus Croissant pour l’Ordre Convexe*, words for an integrable process which is increasing in the convex order. The class is important because it characterizes the property that a martingale can “mimic" one-dimensional distribution of the process in the class.

An easy, but remarkable application of this equivalence to mathematical finance, is done by showing that the time-average of the geometric Brownian motion is a PCOC, using a Brownian sheet [8]. This fact implies that the price of Asian option is increasing in the volatility and the maturity, which was first proven in [11].

The technique for the proof using Gaussian sheet has been further developed in [25] and [26] (see [24, Chapter 2] for further references). In the present paper we trace the proof in [8] to extend it to the cases of fractional Brownian motion.

The time-average of exponential of fractional Brownian motion on the other hand can be understood as *integrated volatility*, or quadratic variation of log-price under *log-normal fractional SABR* model[4] introduced in [6].

[4]The fractional volatility is first suggested in [17] with several empirical evidences.

Chapter 2

Some results on Parisian walk

2.1 An Itô formula for Parisian walks

We begin with a lemma.

Lemma 2.1. *Let Z be a Parisian walk. Then the two dimensional process* $(Z, \bar{Z})$ *enjoys martingale representation property; every complex valued* F*-martingale is represented as a stochastic integral with respect to* $(Z, \bar{Z})$*.*

Proof. Denote $\Delta Z_t := Z_t - Z_{t-1}$ for $t \in \mathbb{Z}_{>0}$. Fix t and set

$$\Delta Z_S := \prod_{s_i=\zeta} \Delta Z_i \prod_{s_i=\zeta^2} \overline{\Delta Z_i}$$

for $S = (s_1, \ldots, s_t) \in \{1, \zeta, \zeta^2\}^t$. Then we have $\mathrm{E}[\Delta Z_S \overline{\Delta Z_{S'}}] = 1$ if $S = S'$ and $= 0$ otherwise because of the martingale property and of the fact that $(\Delta Z_t)^2 = \overline{\Delta Z_t}$. Therefore $\{\Delta Z_S \,|\, S \in \{1, \zeta, \zeta^2\}^t\}$ forms an orthonormal basis (ONB) of $L^2(\mathcal{F}_t)$ since $\sharp\{1, \zeta, \zeta^2\}^t = \dim L^2(\mathcal{F}_t) = 3^t$.

For an adapted (X_t), expanding $X_t - X_{t-1}$ with respect to this ONB and denoting

$$\mathrm{E}[(X_t - X_{t-1})\overline{\Delta Z_S}] = x_S,$$

we have

$$\begin{aligned} X_t - X_{t-1} &= \sum_{s_t=\zeta} x_S \Delta Z_S + \sum_{s_t=\zeta^2} x_S \Delta Z_S + \sum_{s_t=1} x_S \Delta Z_S \\ &= \left(\sum_{s_t=\zeta} x_S \Delta Z_{(s_1,\ldots,s_{t-1})} \right) \Delta Z_t + \left(\sum_{s_t=\zeta^2} x_S \Delta Z_{(s_1,\ldots,s_{t-1})} \right) \Delta \bar{Z}_t \\ &\quad + \sum_{s_t=1} x_S \Delta Z_{(s_1,\ldots,s_{t-1})}. \end{aligned} \tag{2.1}$$

By summing up the above equations, we obtain the Doob decomposition of X, and this completes the proof. □

The above lemma can be easily extended to general unit root cases. The point here is that Parisian walk is the right discrete analogue of planar Brownian motion as can be seen by the following proposition.

Proposition 2.2. *Let (Z_t) be a Parisian walk, and let f be a complex valued function on $\mathbb{Z}[\zeta]$. Then we have the following formula, which would correspond to an Itô's formula in* F. *For $t = 0, 1, 2, \ldots$, we have*

$$\begin{aligned} f(Z_{t+1}) - f(Z_t) = &\frac{1}{3}(Z_{t+1} - Z_t)\left\{f(Z_t+1) + \zeta^2 f(Z_t+\zeta) + \zeta f(Z_t+\zeta^2)\right\} \\ &+ \frac{1}{3}(\bar{Z}_{t+1} - \bar{Z}_t)\left\{f(Z_t+1) + \zeta f(Z_t+\zeta) + \zeta^2 f(Z_t+\zeta^2)\right\} \\ &+ \frac{1}{3}\left\{f(Z_t+1) + f(Z_t+\zeta) + f(Z_t+\zeta^2) - 3f(Z_t)\right\}. \end{aligned} \tag{2.2}$$

Proof. As in the expression (2.1),

$$f(Z_{t+1}) - f(Z_t) = \alpha \Delta Z_{t+1} + \beta \Delta \bar{Z}_{t+1} + \gamma$$

for some $\mathcal{F}_t$-measurable α, β and γ. On the set of $\Delta Z_{t+1} = 1$, $\Delta Z_{t+1} = \zeta$, and $\Delta Z_{t+1} = \zeta^2$ respectively, we have

$$\begin{aligned} f(Z_t+1) - f(Z_t) &= \alpha + \beta + \gamma, \\ f(Z_t+\zeta) - f(Z_t) &= \alpha\zeta + \beta\zeta^2 + \gamma, \\ \text{and } f(Z_t+\zeta^2) - f(Z_t) &= \alpha\zeta^2 + \beta\zeta + \gamma. \end{aligned} \tag{2.3}$$

Solving (2.3) in terms of (α, β, γ), we obtain (2.2). □

2.2 A discrete analogue of conformality

2.2.1 Analogy in Itô formulas

If we put

$$Df(z) = \frac{1}{3}\sum_{j=0,1,2} \zeta^{-j} f(z+\zeta^j), \bar{D}f(z) = \frac{1}{3}\sum_{j=0,1,2} \zeta^j f(z+\zeta^j),$$

and

$$Lf(z) = \frac{1}{3}\sum_{j=0,1,2}\left\{f(z+\zeta^j) - f(z)\right\},$$

the formula (2.2) becomes

$$\begin{aligned} \Delta f(Z_t) :&= f(Z_t) - f(Z_{t-1}) \\ &= Df(Z_{t-1})\Delta Z_t + \bar{D}f(Z_{t-1})\overline{\Delta Z_t} + Lf(Z_{t-1}). \end{aligned} \tag{2.4}$$

The discrete Itô's formula symbolically coincides with the one of cBM Z_t; for $f(x+iy) = f_1(x,y) + i f_2(x,y)$ with $f_1, f_2 \in C^2(\mathbb{R})$, we have that

$$df(Z_t) = \partial_z f(Z_t)dZ + \partial_{\bar{z}} f(Z_t)d\overline{Z} + \mathcal{L}f(Z_t)dt,$$

where

$$\partial_z = \frac{1}{2}(\partial_x + i\partial_y), \tag{2.5}$$

and

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y), \tag{2.6}$$

the right-hand-side of (2.5) and (2.6) acting on f_1 and f_2,

$$\mathcal{L} = \frac{1}{2}\partial_z \partial_{\bar{z}},$$

is the Laplacian (see e.g. [29]).

2.2.2 Conformality in $\mathbb{Z}[\zeta]$

If f is analytic, or equivalently $\partial_{\bar{z}} = 0$, we have that

$$df(Z_t) = f'(Z_t)dZ.$$

With this in mind, we define the conformality in $\mathbb{Z}[\zeta]$ as follows:

Definition 2.3. *We say a map $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is* **Parisian conformal** *or* **p-conformal** *if $\bar{D}f(z) = 0$ and $Lf(z) = 0$ for all $z \in \mathbb{Z}[\zeta]$.*

We give the following basic result, which insists that our definition of conformality is proper in a geometric sense.

Proposition 2.4. *A map $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is p-conformal if and only if it has the following property; any triangle of the form $\{z+1, z+\zeta, z+\zeta^2\}$ is mapped to a triangle $\{f(z)+c, f(z)+c\zeta, f(z)+c\zeta^2\}$ for some $c \in \mathbb{Z}[\zeta]$ with $f(z+\zeta^j) = f(z) + c\zeta^j$, $j = 0, 1, 2$.*

Proof. The "if" part is straightforward since we can calculate directly $\bar{D}f$ and Lf by the assumed property. The converse is also easy to see; since $\bar{D}f(z) = Lf(z) = 0$, we obtain by the Itô's formula (2.4),

$$f(z+\zeta^j) = f(z) + Df(z)\zeta^j, j = 0, 1, 2.$$

Here we notice in particular that $Df(z) \in \mathbb{Z}[\zeta]$, which we state separately as a corollary. □

Corollary 2.5. *If $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is p-conformal, then the image of the map Df is in $\mathbb{Z}[\zeta]$.*

Example 2.6. *Except the trivial ones like constant or linear functions in z, the simplest example of p-conformal map would be $z^2 - |z|$. Note that z^2 is not p-conformal. We conjecture that, in general, among monic polynomials of a fixed degree, there is only one p-conformal map.*

2.3 Discrete analogue of conformal martingales

In this section, we give a probabilistic "credit" that ours is a discrete analogue of the conformality. First we give a definition of a "Parisian" conformal martingale.

Definition 2.7. *We say an* F *martingale* (M_t) *is p-conformal if it is* $\mathbb{Z}[\zeta]$*-valued and is represented by a martingale transform with respect to* (Z_t)*.*

From the definitions and the Itô's formula (2.4), it is straightforward that $(f(Z_t))$ is a p-conformal martingale if and only if $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is p-conformal.

We yet have the following theorem, which also implies that the definition is proper since it is also a discrete analogue of the fact that "*a conformal martingale is a time changed cBM*" (see e.g. [29, Proposition 3.6.2]).

Theorem 2.8. *For a p-conformal martingale* (M_k) *with respect to a Parisian walk* (Z_k)*, there exists another Parisian Walk* $(\tilde{Z}_k)$ *and a sequence of stopping times* $T_1 < T_2 < \cdots < \infty$ *such that* (M_k) *is identically distributed as* $(\tilde{Z}_{T_t})$ *as a stochastic process.*

Proof. We first note that a Parisian walk is recurrent, which can be proven in a similar way as the case with the simple random walk on $\mathbb{Z}^2$. We choose a sequence of stopping times $T_1 < T_2 < \cdots < \infty$ recursively as $T_0 = 0$,

$$T_k := \inf\{t > T_{k-1} : Z_t \in \{Z_{T_{k-1}} + M_{k-1}\zeta^j, j = 0, 1, 2\}\}, \quad k = 1, 2, \cdots,$$

and put $\tilde{Z}_k = Z_{T_k}$ for $k = 0, 1, \cdots$. Then it is easy to see that the sequence satisfies the desired property. □

Theorem 2.9. *For* $(k_1, \cdots, k_m) \in \{1, 2\}^m$ *define*

$$M_t^{(m;k_1,\cdots,k_m)} := \sum_{\sigma \in \mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)} \Delta Z^{k_2}_{\sigma(2)} \cdots \Delta Z^{k_m}_{\sigma(m)}, \quad t \geq m.$$

There exist $F_{k_1,\ldots,k_m} : \mathbb{Z}[\zeta] \times \mathbb{Z} \to \mathbb{Z}[\zeta]$ *such that*

$$F_{k_1,\ldots,k_m}(Z_t, t) = M_t^{(m;k_1,\cdots,k_m)} \tag{2.7}$$

for any $t \geq m$ *and* $\omega \in \Omega$*.*

Proof. If $m = 1$, then the assertion is true since

$$\sum_{\sigma \in \mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)} = (t-1)! {Z_t}^{k_1}.$$

Let $m = 2$. If $k_1 \neq k_2$, we may assume without loss of generality $k_1 = 1, k_2 = 2$ since $\sum_{\sigma \in \mathfrak{S}_t} \overline{\Delta Z}_{\sigma(1)} \Delta Z_{\sigma(2)} = \sum_{\sigma \in \mathfrak{S}_t} \Delta Z_{\sigma(1)} \overline{\Delta Z}_{\sigma(2)}$. In such a case we have

$$\sum_{\sigma \in \mathfrak{S}_t} \Delta Z_{\sigma(1)} \overline{\Delta Z}_{\sigma(2)} = \frac{2(t-2)!}{2} \left(|Z_t|^2 - t \right)$$

If $k_1 = k_2$, we may assume $k_1 = 1$ since the other case is just its complex conjugate. In such a case we have

$$\sum_{\sigma\in\mathfrak{S}_t} \Delta Z_{\sigma(1)}\Delta Z_{\sigma(2)} = \frac{2(t-2)!}{2}\left({Z_t}^2 - \bar{Z}_t\right).$$

Thus, we have the assertion when $m = 2$. Suppose that the assertion holds when $m = l-1$ and $m = l$ for $l \geq 3$;

$$\sum_{\sigma\in\mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_{l-1}}_{\sigma(l-1)} = F_{k_1,\cdots,k_l}(Z_t, t)$$

and

$$\sum_{\sigma\in\mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)} = F_{k_1,\cdots,k_l}(Z_t, t).$$

Fix $\sigma \in \mathfrak{S}_t$ and let $k_{l+1} = 1$. Then

$$\begin{aligned}
&\Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\Delta Z_{\sigma(l+1)}\\
&= \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\left(\Delta Z_{\sigma(1)} + \cdots + \Delta Z_{\sigma(t)}\right)\\
&- \Delta Z^{k_1+1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)} - \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2+1}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\\
&- \cdots - \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l+1}_{\sigma(l)} - \sum_{i=l+2}^{t}\Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\Delta Z_{\sigma(i)},
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_{i=l+1}^{t}\Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\Delta Z_{\sigma(i)}\\
&= \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\left(\Delta Z_{\sigma(1)} + \cdots + \Delta Z_{\sigma(t)}\right) - \Delta Z^{k_1+1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\\
&- \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2+1}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)} - \cdots - \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l+1}_{\sigma(l)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_t}\sum_{i=l+1}^{t}\Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)}\Delta Z_{\sigma(i)}\\
&= \sum_{\sigma\in\mathfrak{S}_t}\Big\{\Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_m}_{\sigma(m)}\left(\Delta Z_{\sigma(1)} + \cdots + \Delta Z_{\sigma(t)}\right)\\
&- \Delta Z^{k_1+1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)} - \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2+1}_{\sigma(2)}\cdots\Delta Z^{k_l}_{\sigma(l)} - \cdots - \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\cdots\Delta Z^{k_l+1}_{\sigma(l)}\Big\}.
\end{aligned} \tag{2.8}$$

The left-hand-side of (2.8) is equal to (const) ×

$$\sum_{\sigma\in\mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\dots\Delta Z^{k_m}_{\sigma(m)}\Delta Z_{\sigma(m+1)}.$$

On the other hand, the first term of the right-hand-side of (2.8) equals

$$Z_t \sum_{\sigma\in\mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\dots\Delta Z^{k_m}_{\sigma(m)}.$$

The other terms are equal to either $\sum_{\sigma\in\mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\dots\overline{\Delta Z}_{\sigma(i)}\dots\Delta Z^{k_m}_{\sigma(m)}$ or $\sum_{\sigma\in\mathfrak{S}_t} \Delta Z^{k_1}_{\sigma(1)}\Delta Z^{k_2}_{\sigma(2)}\dots\Delta Z^{k_{i-1}}_{\sigma(i-1)}\Delta Z^{k_{i+1}}_{i+1}\dots\Delta Z^{k_m}_{\sigma(m)}$, since $k_i+1\in\{2,3\}$. By the inductive assumption we have the assertion when $k_{m+1}=1$. The case $k_{m+1}=2$ is handled in the same way. □

2.4 Parisian analogue of Heston's stochastic volatility model

In this section, we discuss a potential application of our Parisian stochastic calculus to mathematical finance.

2.4.1 Heston's stochastic volatility model

In Heston's model [23], the stock price at time t is given by

$$S_t = S_0 \exp\left\{\int_0^t \sqrt{\nu_s}dB_s + rt - \frac{1}{2}\int_0^t \nu_s ds\right\},$$

where $r>0$ stands for the risk-free rate, B is a 1-dimensional standard Brownian motion (under an equivalent martingale measure), and (the square root) of the volatility ν_t is a Cox-Ingersol-Ross process;

$$d\nu_t = \xi\sqrt{\nu_t}dB'_t + \kappa(\theta-\nu_t)dt, \tag{2.9}$$

where ξ is the *volatility of volatility* that is assumed to be constant, κ is the rate at which ν_t reverts to θ, the *long variance*. Here B' is another standard Brownian motion such that

$$d\langle B, B'\rangle_t = \rho\, dt$$

for some $\rho\in[-1,1]$.

It is known that when $\xi^2\le 2\kappa\theta$, the process stays strictly positive (see e.g. [33, Chapter 6, Section 3.1]).

2.4.2 A decomposition of a Heston process

We work on the special case $\xi^2 = 2\kappa\theta$. It is also well known that in this case the unique strong solution to (2.9) is given by

$$\nu_t = |O_t|^2,$$

where $O_t = (O_t^1, O_t^2)$ is a two-dimensional Ornstein-Uhlenbeck process; $O^j, j = 1, 2$ solve

$$dO_t^j = \frac{\xi}{2} dW_t^j - \frac{\kappa}{2} O_t^j dt. \tag{2.10}$$

Here (W^1, W^2) is a two-dimensional Brownian motion.

In fact, since

$$\begin{aligned} d(|O_t|^2) &= 2O_t^1 dO_t^1 + 2O_t^2 dO_t^2 + \frac{\xi^2}{2} dt \\ &= \xi \left(O_t^1 dW_t^1 + O_t^2 dW_t^2 \right) - \kappa\{(O^1)^2 + (O^2)^2\} dt + \frac{\xi^2}{2} dt \\ &= \xi |O_t| \frac{O_t^1 dW_t^1 + O_t^2 dW_t^2}{|O_t|} + \kappa \left(\theta - |O_t|^2 \right) \, dt, \end{aligned}$$

and since

$$\left\langle \frac{O^1 dW^1 + O^2 dW^2}{|O_t|} \right\rangle_t = dt,$$

it is a Brownian motion, we see that $|O_t|^2$ solves (2.9).

Further, we have the following fact, which may not be new.

Proposition 2.10. *When $\xi^2 = 2\kappa\theta$, $X_t := \log S_t$ has the following identity in law as a stochastic process:*

$$\begin{aligned} X_t - X_0 = &\sqrt{1-\rho^2} \left(\int_0^t O_s^1 dO_s^2 - \int_0^t O_s^2 dO_s^1 \right) \\ &+ \frac{\rho}{\xi} (|O_t|^2 - |O_0|^2) + (\frac{\rho\xi}{\theta} - \frac{1}{2}) \int_0^t |O_t|^2 ds + (r - \frac{\rho\xi}{2}) t. \end{aligned} \tag{2.11}$$

Proof. Observe that

$$\begin{aligned} O_t^1 dO_t^2 - O_t^2 dO_t^1 &= O_t^1 dW_t^2 - O_t^2 dW_t^1 \\ &= |O_t| \frac{O_t^1 dW_t^2 - O_t^2 dW_t^1}{|O_t|} =: |O_t| dB_t'', \end{aligned}$$

by (2.10). Here B'' is a Brownian motion independent of B' since it is a martingale, $d\langle B'' \rangle_t = dt$, and $d\langle B', B'' \rangle_t = 0$.

On the other hand, since $d\langle B, B'\rangle_t = \rho\, dt$, we have

$$\begin{aligned}
&X_t - X_0 \\
&\stackrel{d}{=} \int_0^t \sqrt{\nu_s}(\rho dB'_s + \sqrt{1-\rho^2} dB''_s) + rt - \frac{1}{2}\int_0^t \nu_s ds \\
&= \frac{\rho}{\xi}(\nu_t - \nu_0) + \sqrt{1-\rho^2}\int_0^t \sqrt{\nu_s} dB''_s + \left(r - \frac{\rho\xi}{2}\right)t + \left(\frac{\rho\xi}{\theta} - \frac{1}{2}\right)\int_0^t \nu_s ds,
\end{aligned}$$

which leads to (2.11). □

2.4.3 Discrete analogue of the stochastic area and the squared Bessel processes

For a discrete deterministic process $X : \mathbb{Z}_0 \to \mathbb{C}$, define $H(X) : \mathbb{Z}_0 \to \mathbb{C}$ by

$$H_0(X) = |X_0|^2, \quad \Delta H_t(X) = \bar{X}_{t-1}\Delta X_t, \quad t = 1, 2, \cdots .$$

Then, we have the following.

Lemma 2.11. *(i) The squared distance from* 0 *of* X_t *is represented by the real part of* $H_t(X)$ *for each* $t > 0$*;*

$$|X_t|^2 = 2\mathrm{Re} H_t(X) + \sum_{j=1}^t |\Delta X_j|^2,$$

and (ii) the total sum of the (oriented) areas of the triangle drawn by X_{s-1}, X_s *and* 0*,* $s = 1, \cdots, t$ *is represented by the imaginary part of* $H_t(X)$ *for each* $t > 0$*;*

$$\begin{aligned}
A_t(X) :&= \frac{1}{2}\sum_{s=1}^t \{(\mathrm{Re}X_{s-1})(\mathrm{Im}X_s) - (\mathrm{Re}X_s)(\mathrm{Im}X_{s-1})\} \\
&= \frac{1}{2}\mathrm{Im}H_t(X).
\end{aligned}$$

Proof. For (i),

$$\begin{aligned}
&(X_0 + \Delta X_1 + \cdots + \Delta X_t)\overline{(X_0 + \Delta X_1 + \cdots + \Delta X_t)} \\
&= |X_0|^2 + \sum_{j=1}^t |\Delta X_j|^2 + 2\mathrm{Re}(X_0 \sum_{j=1}^t \Delta X_j) + 2\mathrm{Re}\sum_{j=1}^t \sum_{i=1}^{j-1} \overline{\Delta X_i}\Delta X_j \\
&= 2\mathrm{Re}H_t(X) + \sum_{j=1}^t |\Delta X_j|^2.
\end{aligned}$$

The relation (ii) is obvious. □

Remark 2.12. *Since the Ornstein-Uhlenbeck process (2.10) can be approximated by our Parisian walk by taking a scaling limit with a Girsanov-Maruyama type measure-change, we may claim that*

$$
\begin{aligned}
S_t^P := \sqrt{1-\rho^2}\mathrm{Im}H_t(Z) + \frac{2\rho}{\xi}(2\mathrm{Re}H_t(Z) - |Z_0|^2) \\
+ (\frac{2\rho\xi}{\theta} - 1)\sum \mathrm{Re}H_t(Z)\Delta t + (r - \frac{\rho\xi}{2})t,
\end{aligned}
\tag{2.12}
$$

where Z is a Parisian walk, is a discrete analogue of Heston's model, with a proper change of measures. Precise discussions and proofs will be given in another paper.

Chapter 3

Towards the Exact Simulation Using Hyperbolic Brownian motion

3.1 Hyperbolic Brownian Motions

In this section, we recall basic facts about hyperbolic Brownian motions.

Let $n \geq 2$ and

$$\mathbb{H}^n := \{z = (x, y) = (x^1, \cdots, x^{n-1}, y); x \in \mathbb{R}^{n-1}, y > 0\},$$

the upper half space in $\mathbb{R}^n$, endowed with the Poincaré metric [1]

$$ds^2 = y^{-2}(\sum_{i=1}^{n-1}(dx^i)^2 + (dy)^2).$$

The Riemannian volume element is given by $dv = y^{-n}dx^1 \cdots dx^{n-1}dy$ and the distance $d_{\mathbb{H}^n}(z, z')$ for $z = (x, y)$, $z' = (x', y') \in \mathbb{H}^n$ is given by

$$\cosh(d_{\mathbb{H}^n}(z, z')) = \frac{d_{\mathbb{R}^{n-1}}(x, x')^2 + y^2 + (y')^2}{2yy'}. \tag{3.1}$$

The Laplace-Beltrami operator is

$$\Delta_n := y^2 \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + y^2 \frac{\partial^2}{\partial y^2} - (n-2)y\frac{\partial}{\partial y}.$$

We denote by $q_n(t, z, z')$ the heat kernel with respect to the volume element dv of the semigroup generated by $\Delta_n/2$; that is to say,

$$\partial_t q_n = \frac{1}{2}\Delta_n q_n,$$

[1]A metric, at each point, is a bi-linear form on the tangent space, or equivalently, an element of the tensor product of the cotangent space. The convention $(dx)^2$ should then be understood as $dx \otimes dx$, and so on.

and

$$\lim_{t\to 0}\int_{\mathbb{H}^n} q_n(t,z',(x,y))f(x,y)y^{-n}dxdy = f(z')$$

for any bounded continuous function f. In other words,

$$\mathbb{P}((X_t,Y_t)\in dxdy|(X_0,Y_0)=z') = q_n(t,z',(x,y))y^{-n}dxdy, \tag{3.2}$$

where (X_t,Y_t) is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^i &= Y_t dW_t^i,\ i=1,\cdots,n-1,\\ dY_t &= Y_t dW_t^n - (n-2)Y_t dt, \end{aligned} \tag{3.3}$$

where $W^1,\cdots,W^n$ are mutually independent Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The diffusion (X,Y) is the one associated with the semigroup $\Delta_n/2$.

The following formulas for q_n are known (see e.g.[13] and [37]):

Theorem 3.1. *The heat kernel with respect to the volume form has the following explicit expressions. Here we put $r=r(z,z')=d_{\mathbb{H}^2}(z,z')$.*
i) (McKean's kernel) In the case of $n=2$;

$$q_2(t,z,z') =: p_2(t,r) = \frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}}\int_r^\infty \frac{be^{-b^2/2t}}{(\cosh(b)-\cosh(r))^{1/2}}db.$$

ii) (Milson's formula) For $n\geq 2$, we have the following recursive relation;

$$q_{n+2}(t,z,z') =: p_{n+2}(t,r) = -\frac{e^{-nt/2}}{2\pi\sinh(r)}\frac{\partial}{\partial r}p_n(t,r).$$

(iii) (Gruet's formula [19]) For every $n\geq 2, t>0, z,z'\in\mathbb{H}^n$, it holds that

$$\begin{aligned} q_n(t,z,z') &= p_n(t,r)\\ &= \frac{e^{-(n-1)^2t/8}}{\pi(2\pi)^{n/2}t^{1/2}}\Gamma\left(\frac{n+1}{2}\right)\int_0^\infty \frac{e^{(\pi^2-b^2)/2t}\sinh(b)\sin(\pi b/t)}{(\cosh(b)+\cosh(r))^{(n+1)/2}}db. \end{aligned}$$

3.2 HBM with drift, and its parametrix

We consider the following stochastic differential equation:

$$\begin{aligned} dX_t &= Y_t dW_t^1 + \mu(X_t,Y_t)\,dt\\ dY_t &= Y_t dW_t^2,\\ (X_0,Y_0) &= (x,y) = z, \end{aligned} \tag{3.4}$$

where $(x,y)=z\in\mathbb{H}^2$, $\mu:\mathbb{H}^2\to\mathbb{R}$ be a Lipschitz function, bounded in x and

$$|\mu(x,y)|\leq K_0|y|,\quad (x,y)\in\mathbb{H}^2 \tag{3.5}$$

with some positive constant K_0. The unique strong solution to (3.4) exists, and will be denoted by $(X^\mu, Y^\mu) =: Z^\mu$, while the 2-dimensional HBM given by (3.3) with $n = 2$ will be denoted by $(X^0, Y^0) =: Z^0$.

Put

$$\begin{aligned}\theta(t, z, z') :&= \mu(x, y)\frac{\partial}{\partial x}\log q_2(t, (x, y), (x', y')) \\ &= \mu(x, y)\frac{\frac{\partial}{\partial x}q_2(t, (x, y), (x', y'))}{q_2(t, (x, y), (x', y'))}, \\ &\qquad t > 0, z, z' \in \mathbb{H}^2.\end{aligned}$$

For $t > 0$ and each n, let

$$\Delta_n(t) := \{(u_1, u_2, \cdots, u_n) \in [0, t]^n : u_1 < \cdots < u_n\}.$$

The following is the main theorem of the present paper:

Theorem 3.2. *(i) We have that*

$$|\theta(t, z, z')| \le \frac{3K_0}{2} \tag{3.6}$$

and therefore for each $n \ge 2$, $t > 0$ and $(s_1, \cdots, s_{n-1}) \in \Delta_{n-1}(t)$, the random variable $\prod_{i=1}^n \theta(s_i - s_{i-1}, Z^0_{s_{i-1}}, Z^0_{s_i})$, where $s_0 = 0$ and $s_n = t$, is in $L^\infty(\mathbb{P})$ and

$$\mathbb{E}[\prod_{i=1}^n \theta(s_i - s_{i-1}, Z^0_{s_{i-1}}, Z^0_{s_i})|Z^0_t = z'] \in L^\infty(\Delta_{n-1}(t))$$

for each $t > 0$ and $z, z' \in \mathbb{H}^2$.

(ii) Set

$$h_1(t, z, z') = \mu(z)\frac{\partial}{\partial x}q_2(t, z, z')(y')^{-2}. \tag{3.7}$$

and

$$h_n(t, z, z') := \int_{\Delta_{n-1}(t)} \mathbb{E}[\prod_{i=1}^n \theta(s_i - s_{i-1}, Z^0_{s_{i-1}}, Z^0_{s_i})|Z^0_0 = z, Z^0_t = z']q_2(t, z, z')/(y')^2 ds_1 \cdots ds_{n-1}$$

for $n \ge 2$. Then, the series $\sum_{n=1}^N h_n(t, z, z')$ is absolutely convergent as $N \to \infty$ uniformly in (t, z, z') on every compact set.

(iii) The transition density of Z^μ, i.e. the Radon-Nikodym derivative of the measure $P(Z^\mu_t \in dz'|Z^\mu_0 = z)$, is given by

$$s(t, z, z') := \frac{q_2(t, z, z')}{(y')^2} + \int_{\mathbb{H}^2}\int_0^t \frac{q_2(t - s, z, z'')}{(y'')^2}\Phi(s, z'', z')ds dz'',$$

where $\Phi(t, z, z') = \sum_{n=1}^\infty h_n(t, z, z')$.

Proof. Since $q_n(t,z,z') = p_n(t, r(z,z'))$, we have that

$$\begin{aligned}
\frac{\partial}{\partial x} q_2(t,(x,y),(x',y')) &= \frac{\partial}{\partial x} p_2(t, r((x,y),(x',y'))) \\
&= \frac{\partial r}{\partial x}\frac{\partial p_2}{\partial r}(t, r((x,y),(x',y'))) \\
&= \frac{x-x'}{yy'\sinh(r)}\left(-e^t 2\pi \sinh(r) p_4(t, r((x,y),(x',y')))\right) \\
&= \frac{x-x'}{yy'}\left(-e^t (2\pi) p_4(t, r((x,y),(x',y')))\right)
\end{aligned}$$

by (ii) of Theorem 3.1. Also, (iii) of Theorem 3.1 tells us that

$$\begin{aligned}
e^t(2\pi)p_4(t,r) &= \frac{e^{-t/8}}{\pi(2\pi)t^{1/2}}\Gamma\left(\frac{5}{2}\right)\int_0^\infty \frac{e^{(\pi^2-b^2)/2t}\sinh(b)\sin(\pi b/t)}{(\cosh(b)+\cosh(r))^{5/2}}db \\
&= \frac{3}{2}\frac{e^{-t/8}}{\pi(2\pi)t^{1/2}}\Gamma\left(\frac{3}{2}\right)\int_0^\infty \frac{e^{(\pi^2-b^2)/2t}\sinh(b)\sin(\pi b/t)}{(\cosh(b)+\cosh(r))(\cosh(b)+\cosh(r))^{3/2}}db \\
&\le \frac{3}{2}\frac{1}{1+\cosh(r)}p_2(t,r)
\end{aligned}$$

since $\cosh(x) \ge 1$ for all x. Therefore, we see that

$$\begin{aligned}
|\theta(t,(x,y),(x',y'))| &\le |\mu(z)|\frac{|\frac{\partial}{\partial x}q_2(t,(x,y),(x',y'))|}{q_2(t,(x,y),(x',y'))} \\
&\le \frac{3K_0}{2}\frac{|y||x-x'|}{yy'(1+\cosh(r(z,z')))}.
\end{aligned}$$

Here, we have used (3.5) in the last inequality. By (3.1),

$$\begin{aligned}
&\frac{|y||x-x'|}{yy'(1+\cosh(r(z,z')))} \\
&= \frac{|y||x-x'|}{yy'(1+\frac{|x-x'|^2+y^2+(y')^2}{2yy'})} = \frac{2|y||x-x'|}{|x-x'|^2+|y+y'|^2} \\
&\le \frac{|y|}{|y+y'|} \le 1.
\end{aligned}$$

Thus we obtained (3.6). Here in the last line we have used the following elementary inequality:

$$|x-x'|^2 + |y+y'|^2 \ge 2|x-x'||y+y'|.$$

Let us consider (ii). By (3.6), we have that for n bigger than 2,

$$\begin{aligned} |h_n(t,z,z')| &\le \frac{q_2(t,z,z')}{(y')^2} \int_{\Delta_{n-1}(t)} \mathbb{E}[\Big(\frac{3}{2}K_0\Big)^n \, |Z_t^0 = z'] ds_1 \cdots ds_{n-1} \\ &= \Big(\frac{3}{2}K_0\Big)^n \frac{q_2(t,z,z')}{(y')^2} \int_{\Delta_{n-1}(t)} ds_1 \cdots ds_{n-1} \\ &= \Big(\frac{3}{2}K_0\Big)^n \frac{q_2(t,z,z')}{(y')^2} \frac{t^{n-1}}{(n-1)!}. \end{aligned} \tag{3.8}$$

Hence we have

$$\begin{aligned} \sum_{n=1}^{\infty} |h_n(t,z,z')| &\le \frac{q_2(t,z,z')}{(y')^2} \sum_{n=1}^{\infty} \Big(\frac{3}{2}K_0\Big)^n \frac{t^{n-1}}{(n-1)!} \\ &= \frac{3}{2}K_0 \frac{q_2(t,z,z')}{(y')^2} \sum_{n=0}^{\infty} \Big(\frac{3}{2}K_0 t\Big)^n \frac{1}{n!} \\ &= \frac{3}{2}K_0 \frac{q_2(t,z,z')}{(y')^2} e^{\frac{3}{2}K_0 t}, \end{aligned}$$

which complete the proof of (ii).

Finally, we shall prove (iii). Since

$$h_n(t,z,z') = \int_{\mathbb{H}^2} \int_0^t h_1(t-s,z,z'') h_{n-1}(s,z'',z') ds dz'',$$

we see that the sum $\sum_{n=1}^{\infty} h_n(t,z,z') =: \Phi(t,z,z')$ satisfies

$$\Phi(t,z,z') = h_1(t,z,z') + \int_{\mathbb{H}^2} \int_0^t h_1(t-s,z,z'') \Phi(s,z'',z') ds dz''. \tag{3.9}$$

Note that since we have, by (3.8),

$$\begin{aligned} |\Phi(t,z,z')| &= |\sum_{n=1}^{\infty} h_n(t,z,z')| \\ &\le \sum_{n=1}^{\infty} |h_n(t,z,z')| \le \frac{3}{2}K_0 \frac{q_2(t,z,z')}{(y')^2} e^{\frac{3}{2}K_0 t}, \end{aligned}$$

we see that Φ is integrable:

$$\begin{aligned} \int_0^T \int_{\mathbb{H}^2} |\Phi(t,z,z')| dz' dt &\le \frac{3}{2}K_0 \int_0^T \int_{\mathbb{H}^2} \frac{q_2(t,z,z')}{(y')^2} e^{\frac{3}{2}K_0 t} dz' dt \\ &\le \frac{3}{2}K_0 e^{\frac{3}{2}K_0 T} \int_0^T \int_{\mathbb{H}^2} \frac{q_2(t,z,z')}{(y')^2} dz' dt = \frac{3}{2}K_0 T e^{\frac{3}{2}K_0 T} < \infty. \end{aligned}$$

We know that

$$\left(\frac{1}{2}\Delta_2 - \partial_t\right) q_2(t,z,z') = 0,$$

and

$$\left(\frac{1}{2}\Delta_2 - \partial_t\right) \int_{\mathbb{H}^2} \int_0^t \frac{q_2(t-s,z,z'')}{(y'')^2} \Phi(s,z'',z') ds dz'' = -\Phi(t,z,z')$$

by Feynman-Kac formula (see e.g. [34, Theorem 7.6]). Therefore, we have that

$$\begin{aligned}
&\left(\frac{1}{2}\Delta_2 + \mu \frac{\partial}{\partial x} - \partial_t\right) s(t,z,z') \\
&= \left(\frac{1}{2}\Delta_2 + \mu \frac{\partial}{\partial x} - \partial_t\right) \left(\frac{q_2(t,z,z')}{(y')^2} + \int_{\mathbb{H}^2} \int_0^t \frac{q_2(t-s,z,z'')}{(y'')^2} \Phi(s,z'',z') ds dz''\right) \\
&= \mu \frac{\partial q_2}{\partial x_1} \frac{1}{(y')^2} + \int_{\mathbb{H}^2} \int_0^t \frac{\mu}{(y'')^2} \frac{\partial q_2}{\partial x}(t-s,z,z'') \Phi(s,z'',z') ds dz'' - \Phi(t,z,z'),
\end{aligned}$$

which is seen to be zero by (3.7) and (3.9).

Clearly, the property that $s(t,z,z')dz$ converges to $\delta_{z'}(dz)$ is inherited from q_2. □

3.3 Exact Simulation Interpretation

In the spirit of Bally-Kohatsu [9], we give the following "exact simulation interpretation" to Theorem 3.2.

Theorem 3.3. *Let S_i, $i = 1 \cdots$, are independent copies of an exponentially distributed random variable with mean 1, which are also independent of the Brownian motion (W^1, W^2). Let $T_i := S_1 + \cdots + S_i$ and $N_t := \sum_i 1_{\{T_i \le t\}}$, $t > 0$. Then, for any bounded measurable f, we have that*

$$\mathbb{E}[f(Z_t^\mu)] = e^t \mathbb{E}[\prod_{i=1}^{N_t} \theta(T_i - T_{i-1}, Z^0_{T_{i-1}}, Z^0_{T_i}) f(Z^0_t)].$$

Even though this is an almost direct corollary of Theorem 3.2 and Bally-Kohatsu's general theory, we give a self-contained proof below.

Proof. First we claim that for a positive measurable function

$$G \equiv G(s_1, \cdots s_{k+1}, z_1, \cdots, z_{k+1}),$$

we have that

$$\begin{aligned}
&\mathbb{E}\left[1_{\{N_t = k\}} G(T_1, \cdots, T_{k+1}, Z^0_{T_1}, \cdots, Z^0_{T_k}, Z^0_t)\right] \\
&= \mathbb{E}\left[\int_{\Delta_k(t) \times [t,\infty)} G(s_1, \cdots, s_{k+1}, Z^0_{s_1}, \cdots, Z^0_{s_k}, Z^0_t) ds_1 \cdots ds_k e^{-s_{k+1}} ds_{k+1}\right].
\end{aligned} \tag{3.10}$$

In fact, since

$$
\begin{aligned}
&\mathbb{E}\left[1_{\{N_t=k\}}G(T_1,\cdots,T_{k+1},Z^0_{T_1},\cdots,Z^0_{T_k},Z^0_t)\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[1_{\{T_1\le t,\cdots,T_k\le t,T_{k+1}>t\}}G(T_1,\cdots T_{k+1},Z^0_{T_1},\cdots,Z^0_{T_k},Z^0_t)|\mathcal{F}^Z\right]\right]\\
&=\mathbb{E}\left[\int_{[0,t]^k\times(t,\infty)}G(s_1,\cdots,s_{k+1},Z^0_{s_1},\cdots,Z^0_{s_k},Z^0_t)\mathbb{P}(T_1\in ds_1,\cdots,T_{k+1}\in ds_{k+1})\right],
\end{aligned}
$$

and since the joint density of $T_1,\cdots,T_k$ is given by

$$
\begin{aligned}
&\mathbb{P}(T_1\in ds_1,\cdots,T_{k+1}\in ds_{k+1})\\
&=1_{\{s_{k+1}>s_k>s_{k-1}>\cdots>s_1>0\}}e^{-s_{k+1}}ds_1\cdots ds_{k+1},
\end{aligned}
$$

we have (3.10).

In particular, if G is independent to s_{k+1}, we have the following reduction:

$$
\begin{aligned}
&\mathbb{E}\left[1_{\{N_t=k\}}G(T_1,\cdots T_k,Z^0_{T_1},\cdots,Z^0_{T_k},Z^0_t)\right]\\
&=e^{-t}\mathbb{E}\left[\int_{\Delta_k(t)}G(s_1,\cdots s_k,Z^0_{s_1},\cdots,Z^0_{s_k},Z^0_t)ds_1\cdots ds_k\right].
\end{aligned}
\tag{3.11}
$$

We note that we can substitute the following representations to (3.11):

$$
\begin{aligned}
G_+(s_1,\cdots\ s_k,z_1,\cdots,z_{k+1})&=\left(\prod_{i=1}^k\theta(s_i-s_{i-1},z_{i-1},z_i)f(z_{k+1})\right)_+\\
G_-(s_1,\cdots\ s_k,z_1,\cdots,z_{k+1})&=\left(\prod_{i=1}^k\theta(s_i-s_{i-1},z_{i-1},z_i)f(z_{k+1})\right)_-,
\end{aligned}
$$

and so we have

$$
\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^k 1_{\{N_t=k\}}\theta(T_i-T_{i-1},Z^0_{T_{i-1}},Z^0_{T_i})f(Z^0_t)\right]\\
&=e^{-t}\mathbb{E}\left[\int_{\Delta_k(t)}\prod_{i=1}^k\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})f(Z_t)ds_1\cdots ds_k\right].
\end{aligned}
\tag{3.12}
$$

Since we know from (i) of Theorem 3.2 that $\prod_{i=1}^k\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})\in L^\infty(\mathbb{P})$, we see that $\prod_{i=1}^k\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})f(Z^0_t)$ is in $L^1(\mathbb{P})$ by the requirement that $f(Z^0_t)\in L^1(\mathbb{P})$. Therefore, the right-hand-side of (3.12) is equal to

$$
e^{-t}\int_{\Delta_k(t)}\mathbb{E}\left[\prod_{i=1}^k\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})f(Z^0_t)\right]ds_1\cdots ds_k.
$$

Noting that

$$\mathbb{E}\left[\prod_{i=1}^{k}\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})f(Z^0_t)\right]$$
$$=\int_{(\mathbb{H}^2)^{k+1}}\prod_{i=1}^{k}h_1(s_i-s_{i-1},z_{i-1},z_i)f(z')\frac{q_2(t-s_k,z_k,z')}{(y')^2}\,dz_1\cdots dz_k dz',$$

we obtain that

$$\begin{aligned}&\int_{\Delta_k(t)}\mathbb{E}\left[\prod_{i=1}^{k}\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})f(Z^0_t)\right]ds_1\cdots ds_k\\&=\int_{\mathbb{H}^2}\left(\int_{(\mathbb{H}^2\times[0,t])^k}\prod_{i=1}^{k}h_1(s_i-s_{i-1},z_{i-1},z_i)\frac{q_2(t-s_k,z_k,z')}{(y')^2}ds_idz_i\right)f(z')\,dz'\\&=\int_{\mathbb{H}^2}\left(\int_{\mathbb{H}^2\times[0,t]}h_k(s_k,z'',z')\frac{q_2(t-s_k,z,z'')}{(y'')^2}ds_kdz''\right)f(z')\,dz',\end{aligned}\tag{3.13}$$

which is bounded by

$$\left(\frac{3}{2}K_0\right)^k\frac{t^{k-1}}{(k-1)!}\int_{\mathbb{H}^2}\frac{q_2(t,z,z')}{(y')^2}|f(z')|\,dz'$$
$$=\left(\frac{3}{2}K_0\right)^k\frac{t^{k-1}}{(k-1)!}\mathbb{E}[|f(Z^0_t)|],$$

as we see from (3.8). Therefore, we can change the order between the summation and the expectation in

$$\mathbb{E}\left[\prod_{i=1}^{N_t}\theta(T_i-T_{i-1},Z^0_{T_{i-1}},Z^0_{T_i})f(Z^0_t)\right]$$
$$=\mathbb{E}\left[f(Z^0_t)1_{\{N_t=0\}}+\sum_{k=1}^{\infty}\prod_{i=1}^{k}1_{\{N_t=k\}}\theta(T_i-T_{i-1},Z^0_{T_{i-1}},Z^0_{T_i})f(Z^0_t)\right].$$

On the other hand, by (3.13),

$$
\begin{aligned}
&\mathbb{E}\left[f(Z_t^0)1_{\{N_t=0\}}\right] + \sum_{k=1}^{\infty}\mathbb{E}\left[\prod_{i=1}^{k} 1_{\{N_t=k\}}\theta(T_i - T_{i-1}, Z_{T_{i-1}}^0, Z_{T_i}^0) f(Z_t^0)\right] \\
&= e^{-t}\int_{\mathbb{H}^2} \frac{q_2(t,z,z')}{(y')^2} f(z')\,dz' \\
&+ e^{-t}\sum_{k=1}^{\infty}\int_{\mathbb{H}^2} f(z')dz' \int_{\mathbb{H}^2}\int_0^t \frac{q_2(t-s,z,z'')}{(y'')^2} h_k(s,z'',z')ds dz'' \\
&= e^{-t}\int_{\mathbb{H}^2} \frac{q_2(t,z,z')}{(y')^2} f(z')\,dz' \\
&+ e^{-t}\int_{\mathbb{H}^2} f(z')dz' \int_{\mathbb{H}^2}\int_0^t \frac{q_2(t-s,z,z'')}{(y'')^2} \Phi(s,z'',z')ds dz'' \\
&= e^{-t}\mathbb{E}[f(Z_t^{\mu})],
\end{aligned}
$$

where the last equality is valid by (iii) of Theorem 3.2.

□

Chapter 4

Symmetrization Associated with Hyperbolic Reflection Principle

In this Chapter, we start with introducing a hyperbolic version of the reflection principle that parallels the one with the standard Brownian motion in Section 4.1. We introduce in Section 4.2 a weak version of the reflection principle, which also parallels with the classical put-call symmetry. Associated symmetrization is then introduced. Section 4.3 is devoted to numerical studies. As in the case of the Imamura-Ishigaki-Okumura's scheme using classical symmetrization, the error is not proven to be $O(n^{-1})$ mathematically but the numerical results support the conjecture of the hyperbolic case as well.

4.1 Hyperbolic Reflection Principle

4.1.1 Invariant Property of Hyperbolic Brownian Motion

A Hyperbolic Brownian motion is the unique solution to

$$\begin{cases} dX_t = Y_t dW_t^1 \\ dY_t = Y_t dW_t^2, \end{cases}$$

where W^1 and W^2 are independent Brownian motions. It is defined on the upper-half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and we may and sometimes will embed it to $\mathbb{C}$ by $Z_t = X_t + iY_t$, where $i = \sqrt{-1}$.

Proposition 4.1. *Let $f : \mathbb{H} \to \mathbb{H}$ be such that $f(z) := \frac{az+b}{cz+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. Then $(f(Z_t))_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ are equivalent in law provided that $f(Z_0) = Z_0$.*

Proof. Since $Z_t = X_t + iY_t$, using Ito's formula for Z_t,

$$\begin{aligned} dZ_t &= dX_t + idY_t = Y_t(dW_t^1 + idW_t^2) \\ &= \mathrm{Im}(Z_t) dW_t^{\mathbb{C}}, \end{aligned}$$

where $dW_t^{\mathbb{C}} = dW_t^1 + idW_t^2$, which we define to be a complex Brownian motion. On the other hand, since Z_t is a conformal martingale and f is a holomorphic function, we can use Ito's formula for conformal martingales to get

$$\begin{aligned} df(Z_t) &= \partial_z f(Z_t) dZ_t \\ &= \frac{1}{(cZ_t + d)^2} \mathrm{Im}(Z_t) dW_t^{\mathbb{C}} \\ &= \frac{|cZ_t + d|^2}{(cZ_t + d)^2} \mathrm{Im}(f(Z_t)) dW_t^{\mathbb{C}} \\ &= \mathrm{Im}(f(Z_t)) d\widetilde{W}_t^{\mathbb{C}}, \end{aligned}$$

where $d\widetilde{W}_t^{\mathbb{C}} = \frac{|cZ_t+d|^2}{(cZ_t+d)^2} dW_t^{\mathbb{C}}$, which is another complex Brownian motion. Hence Z_t and $f(Z_t)$ are equivalent in law if they start from the same point, as they are defined by the same SDE. □

4.1.2 Hyperbolic Reflections

Let $\mathscr{C}$ be the totality of such isometries π on the upper-half plane $\mathbb{H}$ that $\pi^2 = \mathrm{Id}$ and that the invariant set $\mathrm{Inv}_\pi := \{z \in \mathbb{H} : \pi(z) = z\}$ is a geodesic on $\mathbb{H}$.

Proposition 4.2. *We have that*

$$\mathscr{C} = \left\{ \Phi_A \circ \Phi_0 \in \mathrm{Isom}(\mathbb{H}) : A = \begin{pmatrix} a & b \\ \frac{a^2-1}{b} & a \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix} a, c \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\} \right\},$$

where $\Phi_A(z) = \dfrac{az+b}{cz+d}$ *for* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ *and* $\Phi_0(z) := -\overline{z}$.

Proof. It is well-known that an isometry on $\mathbb{H}$ is either Φ_A or $\Phi_A \circ \Phi_0$ for some $A \in \mathrm{SL}(2, \mathbb{R})$. By the fundamental theorem of algebra, we know that the equation $\Phi_A(z) = z$ has at most two solutions of complex for $A \in \mathrm{SL}(2, \mathbb{R})$. So $\Phi_A(z) \notin \mathscr{C}$.

For $\Phi_A \circ \Phi_0 \in \mathrm{Isom}(\mathbb{H})$ and for $z = x + iy$,

$$(\Phi_A \circ \Phi_0)^2(z) = \frac{(a^2 - bc)z - b(a-d)}{c(a-d)z - (bc - d^2)}.$$

By a simple calculation,

$$\begin{aligned} &\frac{(a^2 - bc)z - b(a-d)}{c(a-d)z - (bc - d^2)} = z \\ \Longleftrightarrow\ &(a^2 - bc)z - b(a-d) = c(a-d)z^2 - (bc - d^2)z \end{aligned} \tag{4.1}$$

- If $a = d$, for any b and c, (4.1) is satisfied.

Since $a^2 - bc = 1$, we have $c = \frac{a^2-1}{b}$ if $b \neq 0$, that is;

$$A = \begin{pmatrix} a & b \\ \frac{a^2-1}{b} & a \end{pmatrix}.$$

For $b = 0$, c is an arbitrary real number and $a = \pm 1$ from $a^2 = 1$;

$$A = \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}.$$

- If $a \neq d$, the equation(4.1) is

$$cz^2 - (a+d)z + b = 0.$$

We get $a = -d$ and $b = c = 0$.

Finally, we should find that the invariant set is geodesic. A geodesics of upper half plane is a line perpendicular to the real line, or a half-circle orthogonal to the real line.

- If

$$A = \begin{pmatrix} a & b \\ \frac{a^2-1}{b} & a \end{pmatrix},$$

and if $a \neq \pm 1$,

$$\frac{-a\bar{z} + b}{-\dfrac{a^2-1}{b}\bar{z} + a} = z,$$
$$\iff (a^2-1)|z|^2 - ab(z+\bar{z}) + b^2 = 0.$$

The last equation means that it is a half circle, with center $\left(\frac{ab}{a^2-1}, 0\right)$ and radius of $\left|\frac{b}{a^2-1}\right|$.

If $a = \pm 1$,

$$-\bar{z} + b = z.$$

The equation means that the invariant set is lines perpendicular to the real line.

- If

$$A = \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix},$$

and if $c \neq 0$, without loss of generality, we may set $a = 1$;

$$\frac{-\bar{z}}{-c\bar{z} + 1} = z$$
$$\iff -c|z|^2 + (z+\bar{z}) = 0.$$

The invariant set is a circle with center $\left(\frac{1}{c}, 0\right)$ and the radius $\frac{1}{|c|}$.

If $c = 0$, the invariant set is the lines perpendicular to the real line. □

4.1.3 Hyperbolic Reflection Principle

Let $\pi \in \mathscr{C}$. Then, $\mathbb{H} = D_+ \cup \mathrm{Inv}_\pi \cup D_-$, where $D_\pm$ are the connected components of $\mathbb{H} \setminus \mathrm{Inv}_\pi$.

Proposition 4.3. *[Hyperbolic Reflection Principle] Let $Z_0 \in D_+$ and $\tau = \inf\{t \geq 0 : Z_t \notin D_+\} = \inf\{t \geq 0 : Z_t \in \mathrm{Inv}_\pi\}$. If we put $\widetilde{Z_t} = Z_t 1_{\{t<\tau\}} + \pi(Z_t) 1_{\{t\geq\tau\}}$, then we have $(Z_t) = (\widetilde{Z_t})$ in law.*

Proof. It suffices to show that if π is a reflection of $\mathbb{H}$, then $(\pi(Z_t))_{t\geq 0} = (Z_t)_{t\geq 0}$ in law if $Z_0 \in \mathrm{Inv}_\pi$ since Z is a strong Markov process and $Z_\tau \in \mathrm{Inv}_\pi$. As we have seen that $\pi = \Phi_A \circ \Phi_0$ for some specific $A \in \mathrm{SL}(2, \mathbb{R})$, and by Proposition 4.1, we only need to check that $(-\overline{Z}_t)$ is identically distributed as (Z_t) as a stochastic process, but this is obvious since (X_t) is identically distributed as $(-X_t)$. □

4.2 Hyperbolic Symmetrization

4.2.1 Hyperbolic Put-Call Symmetry

Let $\pi \in \mathscr{C}$. Then, by Proposition 4.2, we know that

$$\pi = \Phi_A \circ \Phi_0$$

for

$$A = \begin{pmatrix} a & b \\ \frac{a^2-1}{b} & a \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix} \quad a, c \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}. \tag{4.2}$$

A Hyperbolic Brownian motion with drift is a unique solution in $\mathbb{H}$ (if it exists) to

$$\begin{cases} dX_t = Y_t dW_t^1 + \mu_1(X_t, Y_t)dt \\ dY_t = Y_t dW_t^2 + \mu_2(X_t, Y_t)dt, \end{cases} \tag{4.3}$$

where W^1 and W^2 are independent Brownian motions and μ_1 and μ_2 are measurable functions. If we use complex coordinate, the SDE (4.3) is rewritten as

$$dZ_t = \mathrm{Im}(Z_t) dW_t^{\mathbb{C}} + \mu(Z_t)\, dt, \tag{4.4}$$

where $W^{\mathbb{C}} := W^1 + iW^2$ and $\mu(Z) = \mu_1(\mathrm{Re}(Z), \mathrm{Im}(Z)) + i\mu_2(\mathrm{Re}(Z), \mathrm{Im}(Z))$.

Theorem 4.4. *Let $\pi = \Phi_A \circ \Phi_0 \in \mathscr{C}$ and we write*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

to unify the two classes in the expression (4.2)*. Suppose that* μ *satisfies*

$$\mu(z) = \frac{\Phi_0 \circ \mu \circ \pi(z)}{(c\Phi_0 \circ \pi(z) + d)^2}, \tag{4.5}$$

and (4.4) *has a unique weak solution. Then* $(\pi(Z_t))$ *and* (Z_t) *have the same law as a stochastic process, provided that* $Z_0 \in \mathrm{Inv}_\pi$

Proof. Using Itô formula for $\pi(Z_t)$, we have

$$\begin{aligned}
d\pi(Z_t) &= d(\Phi_A \circ \Phi_0(Z_t)) \\
&= \partial_{\bar{z}}(\Phi_A \circ \Phi_0)(Z_t) d\overline{Z}_t \\
&= -\frac{1}{(c\overline{Z_t} - d)^2} \mathrm{Im}(Z_t) d\overline{W_t^{\mathbb{C}}} - \frac{\overline{\mu(Z_t)}}{(-c\overline{Z_t} + d)^2} dt \\
&= -\frac{|c\overline{Z_t} - d|^2}{(c\overline{Z_t} - d)^2} \mathrm{Im}(\pi(Z_t)) d\overline{W_t^{\mathbb{C}}} + \frac{\Phi_0 \circ \mu \circ \pi^2(Z_t)}{(c\Phi_0 \circ \pi^2(Z_t) + d)^2} dt \\
&= \mathrm{Im}(\pi(Z_t)) d\widetilde{W}_t^{\mathbb{C}} + \mu(\pi(Z_t)) dt,
\end{aligned}$$

where we have used assumption (4.5) in the last line and

$$d\widetilde{W}_t^{\mathbb{C}} = -\frac{|c\overline{Z_t} - d|^2}{(c\overline{Z_t} - d)^2} d\overline{W_t^{\mathbb{C}}},$$

which is another complex Brownian motion. Now Theorem follows by the law-uniqueness of the SDE (4.4). □

4.2.2 Symmetrization

Here we present a hyperbolic version of the symmetrization introduced in [5] and [30].

Theorem 4.5. *We keep the setting of Section 4.1 and Theorem 4.4 except for the drift function* μ*. We let*

$$\tilde{\mu}(z) = \begin{cases} \mu(z) & z \in D_+ \\ \dfrac{\Phi_0 \circ \mu \circ \pi(z)}{(c\Phi_0 \circ \pi(z) + d)^2}. & z \in \mathbb{H} \setminus D_+. \end{cases}$$

Then,

(i) the law unique solution of the SDE, if it exists,

$$dZ_t = \mathrm{Im}(Z_t) dW^{\mathbb{C}} + \tilde{\mu}(Z_t)\, dt$$

satisfies $(\pi(Z_t)) = (Z_t)$ *in law, provided that* $Z_0 \in \mathrm{Inv}_\pi$*.*

(ii) *Let* $Z_0 \in D_+$ *and* $\tau = \inf\{t \geq 0 : Z_t \notin D_+\} = \inf\{t \geq 0 : Z_t \in \mathrm{Inv}_\pi\}$. *If we put* $\widetilde{Z_t} = Z_t 1_{\{t<\tau\}} + \pi(Z_t) 1_{\{t \geq \tau\}}$, *then we have* $(Z_t) = (\widetilde{Z_t})$ *in law.*

(iii) *[Conversion Formula] Suppose that* F *is a bounded measurable function on* $\mathbb{H}$ *with support in* D_+. *Then,*

$$E[F(Z_t)1_{\{\tau>t\}}] = E[F(Z_t)] - E[F(\pi(Z_t))].$$

Proof. (i) and (ii) are direct consequences of Theorem 4.4 and Proposition 4.3. (iii) can be proven in the same manner as in [30]. □

Example 4.6. *Let* Z *be the unique solution to* (4.4), $\pi(z) = \frac{1}{\bar{z}}$ *and* $\mathrm{Inv}_\pi = \{z \in \mathbb{H} : |z| = 1\}$. *We let* $D_+ := \{z \in \mathbb{H} : |z| > 1\}$ *and*

$$\mu(z) = c\,\mathrm{Im}(z),$$

where c *is a constant, then the symmetrization* $\tilde{\mu}$ *in Theorem 4.5 is*

$$\tilde{\mu}(z) = \begin{cases} c\,\mathrm{Im}(z) & z \in D_+ \\ -cz^2\mathrm{Im}\left(\frac{1}{\bar{z}}\right) & \textit{otherwise.} \end{cases}$$

4.3 Numerical Experiments

In the hyperbolic symmetrization proposed in the present paper the symmetrized drift may not be continuous in general, as in the case of the symmetrization in [30]. This means that no rigorous mathematical result guarantees the efficiency— (high) order of convergence— in Euler-Maruyama approximation. In [30], it is claimed, however, that numerical experiments show the efficiency. In this section we present some simulation results of the example 4.6 with $c = 1$, $t = 1$, and $F(z) = (|z| - 1)_+ \wedge N$ with $N = 10^4$, which suggest that in the hyperbolic case the conjecture is still likely to be true.

We work on Euler-Maruyama discretization scheme with Monte-Carlo simulation, described below.

1. Let n be the number of discretization; we put $t_k = k/n$, $k = 0, 1, \cdots, n$.

2. Let Z be the original process and $\widetilde{Z}$ be the symmetrized one. We approximate Z and $\widetilde{Z}$ by $Z^n = (X^n, Y^n)$ and $\widetilde{Z}^n = (\widetilde{X}^n, \widetilde{Y}^n)$, defined as

$$X^n_{t_k} - X^n_{t_{k-1}} = Y^n_{t_{k-1}} \Delta W^n_{t_k} + \mu(Y^n_{t_{k-1}}) n^{-1},$$
$$Y^n_{t_k} = Y^n_{t_{k-1}} \exp(\Delta W^n_{t_k} - (2n)^{-1}), \quad k = 1, 2, \cdots, n$$

and

$$\widetilde{X}^n_{t_k} - \widetilde{X}^n_{t_{k-1}} = \widetilde{Y}^n_{t_{k-1}} \Delta W^n_{t_k} + \tilde{\mu}_1(\widetilde{X}^n_{t_{k-1}}, \widetilde{Y}^n_{t_{k-1}}) n^{-1},$$
$$\widetilde{Y}^n_{t_k} - \widetilde{Y}^n_{t_{k-1}} = \widetilde{Y}^n_{t_{k-1}} \Delta W^n_{t_k} + \tilde{\mu}_2(\widetilde{X}^n_{t_{k-1}}, \widetilde{Y}^n_{t_{k-1}}) n^{-1},$$
$$k = 1, 2, \cdots, n,$$

where $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are such that $\tilde{\mu} = \tilde{\mu}_1 + i\tilde{\mu}_2$. Here $\{\Delta W^n_{t_k} : k = 1, 2, \cdots, n\}$ simulates, by pseudo random numbers, independent copies of centered Gaussian random variables with variance n^{-1}.

3. The Monte-Carlo simulation of Path-Wise Euler-Maruyama approximation of $E[F(Z_1)1_{\{\tau>1\}}]$ is obtained by

$$\text{PW-EM}(n) := \frac{1}{M}\sum_{m=1}^{M} F(Z_1^{n,m})1_{\{\tau^{n,m}>1\}},$$

where $Z^{n,m}$ stands for the m-th simulation of Z^n, and

$$\tau^{n,m} = \min\left\{t_k : |Z^{n,m}_{t_k}| \le 1\right\}$$

4. The Monte-Carlo simulation of $E[F(\widetilde{Z}_1)] - E[F(\pi(\widetilde{Z}_1))]$ is given by

$$\text{Symmetrization}(n) := \frac{1}{M}\sum_{m=1}^{M}\left(F(\widetilde{Z}_1^{n,m}) - F(\pi(\widetilde{Z}_1^{n,m}))\right),$$

5. The "true" value $\text{Tr}(n)$ is set to be $\text{Symmetrization}(n)$ for some large n.

6. The errors are calculated accordingly as

$$\text{PW EM Error}(n) := \log|\text{Tr}(n) - \text{PW EM}(n)|$$

and

$$\text{Sym Error}(n) := \log|\text{Tr}(n) - \text{Symmetrization}(n)|.$$

The results are visualized as follows. The figures 4.1 and 4.2 show the results when $(X_0, Y_0) = (0.75, 0.7)$ and $(X_0, Y_0) = (1.0, 1.0)$, respectively, and the "true" value is calculate for $n = 1000$. The tables 4.1 and 4.2 are the values of the dotted points in the figures 4.1 and 4.2, respectively. The tangent of the regression line corresponds to the order of the convergence, which may suggest that it is of order 1 in the case of symmetrization.

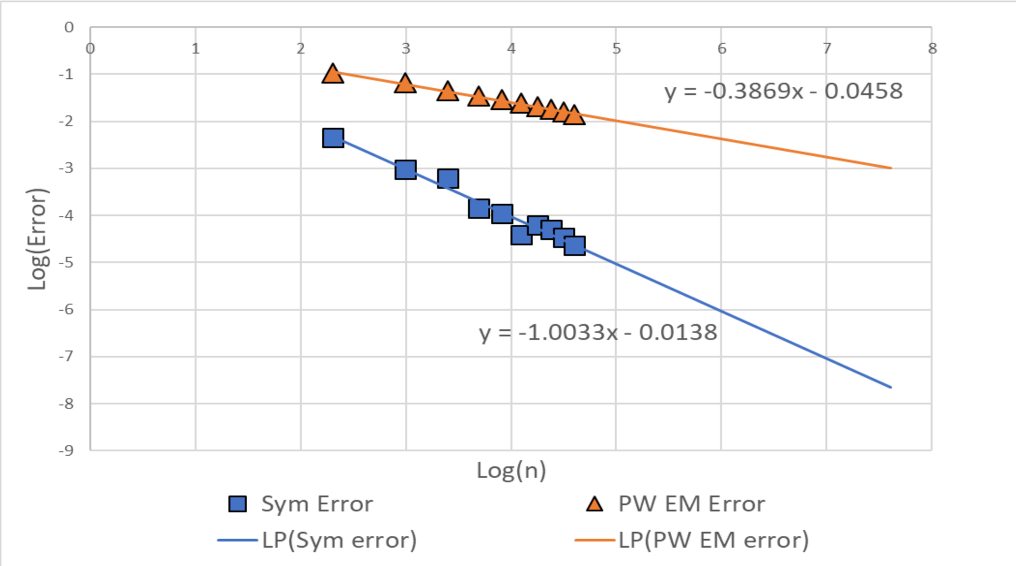

Figure 4.1: $(X_0, Y_0) = (0.75, 0.7)$, Tr 1000=0.116674

M:No. of simmulation trials	n:No. of time steps	Symmetrization	PW EM	Sym Error	PW EM Error
11000	10	0.494071	0.212777	-2.342335	-0.974458
18000	20	0.424634	0.165305	-3.023494	-1.177785
27000	30	0.374365	0.15732	-3.202855	-1.355994
64000	40	0.350142	0.138187	-3.839098	-1.454710
125000	50	0.33173	0.135724	-3.960688	-1.536857
216000	60	0.316112	0.128837	-4.409357	-1.612252
343000	70	0.301227	0.13156	-4.207334	-1.689819
512000	80	0.290315	0.130349	-4.292186	-1.750765
729000	90	0.281466	0.128077	-4.473879	-1.803071
1000000	100	0.273353	0.126315	-4.641730	-1.853556

Table 4.1: $(X_0, Y_0) = (0.75, 0.7)$, Tr 1000=0.116674

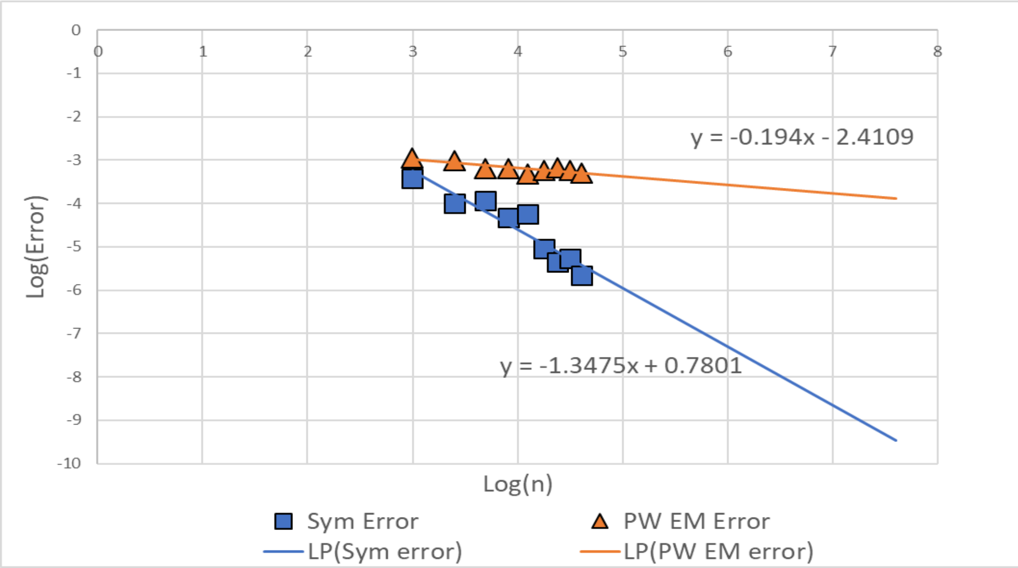

Figure 4.2: $(X_0, Y_0) = (1.0, 1.0)$, Tr 1000=1.253903

M:No. of simmulation trials	n:No. of time steps	Symmetrization	PW EM	Sym Error	PW EM Error
18000	20	1.305908	1.22153	-3.430431	-2.956415
27000	30	1.302775	1.235505	-3.995513	-3.018551
64000	40	1.294995	1.234476	-3.941091	-3.191942
125000	50	1.294358	1.2407	-4.327311	-3.207565
216000	60	1.290095	1.23953	-4.242404	-3.318917
343000	70	1.293495	1.247477	-5.047403	-3.229128
512000	80	1.295325	1.249126	-5.343942	-3.183943
729000	90	1.293081	1.248791	-5.276165	-3.239640
1000000	100	1.290695	1.250402	-5.654707	-3.302475

Table 4.2: $(X_0, Y_0) = (1.0, 1.0)$, Tr 1000=1.253903

Chapter 5

PCOCs with Fractional Brownian Motion and Pricing of Options on Fractional Volatility

The present Chapter is organized as follows. After the preliminary section where we give a precise definition of PCOC, we present in section 5.2, the main result of the PCOC property of the time average of space-time harmonic function of a fractional Brownian motion.

5.1 Preliminary

Let $(B_t^H)_{t\geq 0}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$, which is a Gaussian process with

$$E[B_t^H B_s^H] = \frac{|t|^{2H} + |s|^{2H} - |t-s|^{2H}}{2}.$$

Definition 5.1. *A function $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R})$ is called a space-time harmonic function if*

$$(h(s, W_s), s \geq 0) \text{ is a } (\mathcal{F}_s)_{s\geq 0} - \text{martingale},$$

where $(W_s)_{s\geq 0}$ is a standard Brownian motion and $(\mathcal{F}_s)_{s\geq 0}$ denotes its filtration.

Remark 5.2. *The function* $\exp(x - s/2)$ *is a space-time harmonic function.*

Lemma 5.3. *$h(\sigma^2 u, \sigma W_u)$ is a $(\mathcal{F}_u)_{u\geq 0}$–martingale for $\sigma \in \mathbb{R}$.*

Proof. Since h is a space-time harmonic function, for $t > s$ we have

$$\begin{aligned} h(s, W_s) &= \mathbb{E}[h(t, W_t)|\mathcal{F}_s] \\ &= \mathbb{E}[h(t, W_t - W_s + W_s)|\mathcal{F}_s] \\ &= \mathbb{E}[h(t, b + \sqrt{t-s}X)]|_{b=W_s}, \end{aligned}$$

where X is a centered Gaussian random variable. This equation means

$$h(s,b) = \mathbb{E}[h(t, b + \sqrt{t-s}X)].$$

We have

$$\begin{aligned}
&\mathbb{E}[h(\sigma^2 t, \sigma W_t)|\mathcal{F}_s] \\
&= \mathbb{E}[h(\sigma^2 t, \sigma(W_t - W_s) + \sigma W_s)|\mathcal{F}_s] \\
&= \mathbb{E}[h(\sigma^2 t, \sqrt{\sigma^2(t-s)}X + b)|\mathcal{F}_s]|_{b=\sigma W_s} \\
&= h(\sigma^2 s, b)|_{b=\sigma W_s} \\
&= h(\sigma^2 s, \sigma W_s).
\end{aligned}$$

Therefore $(h(\sigma^2 s, \sigma W_s))_{s\geq 0}$ is $(\mathcal{F}_s)$- martingale. □

The following definitions are adopted from [24].

Definition 5.4. *Let X and Y be two real-valued random variables. X is said to be dominated by Y for the convex order if, for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $E[|\psi(X)|] < \infty$ and $E[|\psi(Y)|] < \infty$, one has:*

$$E[\psi(X)] \leq E[\psi(Y)],$$

and this order is denoted by:

$$X \overset{(c)}{\leq} Y.$$

Definition 5.5. *A process $(X_t)_{t\geq 0}$ is said to be increasing for the convex order if, for every*
$s \leq t, X_s \overset{(c)}{\leq} X_t$.

Definition 5.6. *A process $(X_t)_{t\geq 0}$ is said to be integrable if, for every $t \geq 0$, $E[|X_t|] < \infty$. An integrable process which is increasing in the convex order will be called a PCOC (from the French: Processus Croissant pour l'Ordre Convexe, and maybe pronounced as peacock.)*

5.2 Main Results

Lemma 5.7. *A Gaussian field $(W^H_{t,s})$ with*

$$E[W^H_{t,s} W^H_{t',s'}] = \min(t,t')\left(\frac{|s|^{2H} + |s'|^{2H} - |s-s'|^{2H}}{2}\right), \quad t,t',s,s' \geq 0 \tag{5.1}$$

exists, and satisfies the following properties, for $t > t'$:

(i) $W^H_{t,s} - W^H_{t',s}$ *is independent of* $\sigma\{W^H_{u,s'}; u \leq t', s' \in \mathbb{R}\}$,

(ii) $E[(W^H_{t,s} - W^H_{t',s})^2] = s^{2H}(t - t')$.

Proof. The existence of $(W^H_{t,s})$ is clear since the right-hand side of (5.1) is a positive definite function (see e.g. [40, p.34]). In fact, for $(\xi_1, ..., \xi_n) \in \mathbb{R}^n$, $0 \leq t_1 < ... < t_n$ and $0 \leq s_1 < ... < s_n$,

$$\begin{aligned} F((t_j, s_j, \xi_j)_{j=1}^n) &:= \sum_{i,j} \min(t_i, t_j) \left(\frac{|s_i|^{2H} + |s_j|^{2H} - |s_i - s_j|^{2H}}{2} \right) \xi_i \xi_j \\ &= \sum_{i,j} E[W_{t_i} W_{t_j}] E[B^H_{s_i} B^H_{s_j}] \xi_i \xi_j, \end{aligned}$$

where (W_t) and (B^H_t) are mutually independent standard and fractional Brownian motions, respectively. Then, we see that

$$\begin{aligned} F((t_j, s_j, \xi_j)_{j=1}^n &= \sum_{i,j} E\left[W_{t_i} B^H_{s_i} \xi_i\right] E\left[W_{t_j} B^H_{s_j} \xi_j\right] \\ &= \left(\sum_i E\left[W_{t_i} B^H_{s_i} \xi_i\right] \right)^2 \geq 0. \end{aligned}$$

For $t > t'$,

$$\begin{aligned} E[(W^H_{t,s} - W^H_{t',s}) W^H_{u,s'}] &= E[W^H_{t,s} W^H_{u,s'} - W^H_{t',s} W^H_{u,s'}] = 0 \\ &= E[(W^H_{t,s} - W^H_{t',s})] E[W^H_{u,s'}], \end{aligned}$$

and

$$\begin{aligned} E[(W^H_{t,s} - W^H_{t',s})^2] &= E[(W^H_{t,s})^2 - 2W^H_{t,s} W^H_{t',s} + (W^H_{t',s})^2] \\ &= ts^{2H} - 2t's^{2H} + t's^{2H} \\ &= s^{2H}(t - t'), \end{aligned}$$

as desired. □

Theorem 5.8. *Let h be a space-time harmonic function. Then the process*

$$t^{-\frac{1}{2H}} \int_0^{t^{1/2H}} h(u^{2H}, B^H_u) du, \quad t \geq 0$$

is a PCOC.

Proof. Since

$$t^{-\frac{1}{2H}} \int_0^{t^{1/2H}} h(u^{2H}, B^H_u) du = \int_0^1 h\Big((t^{1/2H} u)^{2H}, B^H_{t^{1/2H} u}\Big) du,$$

then by the scaling property of the fractional Brownian motion, it is equivalent in law as a stochastic process with

$$\int_0^1 h(tu^{2H}, t^{\frac{1}{2}} B_u^H)du.$$

On the other hand, we know that

$$\int_0^1 h(tu^{2H}, t^{\frac{1}{2}} B_u^H)du \stackrel{d}{=} \int_0^1 h(tu^{2H}, W_{t,u}^H)du$$

for fixed t, where $W_{t,u}^H$ is a fractional Brownian sheet introduced in Lemma 5.7. Let $(\mathcal{F}_t) = \sigma\{W_{t',u}; t' \leq t, u \in \mathbb{R}_{>0}\}$. We claim that $\big(h(tu^{2H}, W_{t,u}^H)\big)_{t\geq 0}$ is a $(\mathcal{F}_t)$-martingale. If this the case, clearly the assertion is valid. Since we know that $h(u^{2H}t, u^H W_t)$ is a martingale, it now suffices to show the following equivalence in law:

$$(W_{t,u}^H)_{t\geq 0} \stackrel{d}{=} (u^H W_t)_{t\geq 0} \tag{5.2}$$

where W_t is a standard Brownian motion starting at 0. This can be seen from the covariance function

$$E[(W_{t,u}^H)(W_{t',u}^H)] = \min(t,t')u^{2H},$$

which implies $\frac{1}{u^H}(W_{t,u}^H)_{t\geq 0}$ is a $(\mathcal{F}_t)$-Brownian motion, proving (5.2). □

Corollary 5.9.

$$t^{-1}\int_0^t h(u^{2H}, B_u^H)du, \quad t \geq 0$$

is a PCOC

Proof. For any convex function $\psi :\to$ such that

$$I(t) := E[|\psi(t^{-1}\int_0^t h(u^{2H}, B_u^H)du)|] < \infty$$

for any $t > 0$, I is increasing in t. □

As an almost straightforward extension, we have

Theorem 5.10. *Let $(W_t^{H'})$ be another fractional Brownian motion that is independent of (B_t^H) with Hurst parameter H'. Then, the process*

$$t^{-\frac{H'}{2H}}\int_0^{t^{1/2H}} h(u^{2H}, B_u^H)dW_u^{H'}, \quad t \geq 0$$

is a PCOC. Here we understand the stochastic integral as a path-wise Young integral when $H' \in (1/2, 1)$, while it is the standard one when $H' = 1/2$.

Proof. The proof is almost the same as the one for the previous theorem except that we use scaling property of $(W_t^{H'})$:

$$\begin{aligned}
& t^{-\frac{H'}{2H}} \int_0^{t^{1/2H}} h(u^{2H}, B_u^H) dW_u^{H'} \\
& \sim t^{-\frac{H'}{2H}} \sum_i h\Big((t^{1/2H} u_i)^{2H}, B^H_{t^{1/2H} u_i}\Big)(W^{H'}_{t^{1/2H} u_{i+1}} - W^{H'}_{t^{1/2H} u_i})
\end{aligned}$$

then by the scaling property of fractional Brownian motion:

$$\begin{aligned}
& \stackrel{d}{=} \sum_i h\Big((t^{1/2H} u_i)^{2H}, t^{\frac{1}{2}} B^H_{u_i}\Big)(W^{H'}_{u_{i+1}} - W^{H'}_{u_i}) \\
& \sim \int_0^1 h(tu^{2H}, t^{\frac{1}{2}} B_u^H)\, dW_u^{H'} \\
& \stackrel{d}{=} \int_0^1 h(tu^{2H}, W_{t,u}^H)\, dW_u^{H'}.
\end{aligned}$$

Here we understand $\sim$ to be a relation that an almost-sure limit of the right-hand-side is the left when $H' \in (1/2, 1)$ while it is limit in probability when $H' = 1/2$. Since we know that $h(tu^{2H}, W_{t,u}^H)$ is a $(\mathcal{F}_t')$-martingale, where

$$\mathcal{F}_t' = \sigma(W_u : u \le t) \vee \sigma(W_s^{H'} : s \in [0, \infty)),$$

we have the assertion. □

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