Doctoral Thesis

Numerical analysis for stochastic differential equations with irregular coefficients

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(滑らかでない係数を持つ確率微分方程式の数値解析)

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Abstract

Numerical analysis for stochastic differential equations has been studied by many authors from both sides of the theory and application. Recently, many numerical schemes have been proposed and a lot of interesting results have been discovered. The aim of this thesis is to study a numerical analysis for stochastic differential equations with irregular coefficients.

In Chapter 1, we present the background and outline of this thesis. This chapter contains History of the existence and uniqueness for solution of stochastic differential equations under general settings for the coefficients. We also discuss the relation between stochastic differential equations and Kolmogorov equations. Moreover, we consider several numerical scheme which contains the Euler-Maruyama approximation with historical frame work.

In Chapter 2, we consider the Euler-Maruyama approximation for multi-dimensional stochastic differential equations with irregular coefficients. We provide the rate of strong convergence where the possibly discontinuous drift coefficient satisfies a one-sided Lipschitz condition and the diffusion coefficient is Hölder continuous and uniformly elliptic. Yamada and Watanabe approximation technique in the celebrated paper [120] plays a crucial role in our argument.

In Chapter 3, we study the Euler-Maruyama approximation for one-dimensional stochastic differential equations with discontinuous coefficients. Using the method of removal of drift, we present the rate of strong convergence when the drift coefficient is the sum of Hölder continuous and bounded variation functions, and the diffusion coefficient is Hölder continuous or more generally, bounded 2-variation function.

In Chapter 4, we study the weak approximation for stochastic differential equations with unbounded, irregular drift and constant diffusion coefficients. We apply Girsanov theorem to obtain the weak rate of the Euler-Maruyama approximation for the expectation of various non-smooth functionals of stochastic differential equations, its maximum and killed diffusion. We also apply our method to the study the weak approximation of reflected stochastic differential equations.

In Chapter 5, we consider stability problem for one-dimensional stochastic differential equations with irregular coefficients. The goal of this chapter is to estimate the L^p-difference between two SDEs using a norm associated to the difference of coefficients. In our setting, the discontinuous drift coefficient satisfies a one-sided Lipschitz condition and the diffusion coefficient is bounded, uniformly elliptic and Hölder continuous.

In Chapter 6, we study an unbiased simulation scheme for skew diffusion processes. We apply the parametrix method in order to obtain the existence and the regularity properties of the density of a skew diffusion and provide a Gaussian upper bound. The parametrix method leads to a probabilistic representation in order to use Monte Carlo simulation.

Contents

Chapter 1

Introduction

1.1 Overview

1.1.1 Stochastic differential equations and Kolmogorov equations

Stochastic differential equations

The theory of *stochastic calculus* and *stochastic differential equations* (SDEs) introduced by Kiyosi Itô [52, 53, 54] is used to model a random dynamical phenomena in many fields of applications, for example, mathematical finance, physics, biology, optimal control problem and filtering problem. In particular, it is indispensable to the theory of option pricing in mathematical finance developed by Black, Scholes [15] and Merton [91].

We shall first consider history of the *existence* and *uniqueness* for solution of SDEs. Let $X = (X_t)_{0\leq t\leq T}$ be a solution of d-dimensional SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{1.1}$$

where $W = (W_t)_{0\leq t\leq T}$ is a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $b = (b^{(1)}, \ldots, b^{(d)})^* : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = (\sigma_{i,j})_{1\leq i,j\leq d} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. The method to prove the existence and uniqueness which Itô uses is *Picard's successive approximation* (see [52, 53]). Under the Lipschitz condition, we can use the framework of the contraction mapping of L^2-space in order to show the convergence of the successive approximation and *the pathwise uniqueness* for the solution of SDE. It is worth noting that under Osgood's and Nagumo's type condition for the coefficients, Picard's successive approximation converges to the unique solution to the corresponding SDE in L^2-sense (see [62, 119]). In contrast to the existence and uniqueness of ordinary differential equations, we can discuss them under *"weak regularity condition for the coefficients"* of the SDE (1.1). It is well-known that the *martingale problem* introduced by Stroock and Varadhan [110] plays a crucial role in the proof of the weak existence and uniqueness in the sense of probability law for the SDE (1.1) under the condition that the drift coefficient b is bounded measurable, and the diffusion coefficient σ is bounded, uniformly elliptic and continuous. Krylov [71] extended the results in [110] to measurable diffusion coefficient. Moreover, for the one-dimensional SDE $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t$, by using the *method of time-change*, Engelbert and Schmidt gave the equivalent conditions for the weak existence and uniqueness in the sense of probability law (see e.g., Theorem 5.5.4 and 5.5.7 in [60]). The relation between *pathwise uniqueness* and *uniqueness in the sense of probability law* is considered by Yamada and Watanabe [120]. Yamada and Watanabe prove that *pathwise uniqueness implies uniqueness in the sense of probability law*, and *weak existence and pathwise uniqueness implies the solution is a strong solution*. Moreover, they also provide that under one-dimensional setting, if the drift coefficient is Lipschitz continuous and diffusion coefficient is 1/2-Hölder continuous, the pathwise uniqueness holds, so the solution is a strong solution. The problem of the pathwise uniqueness is considered by many authors, and the result in [120] is extended to some SDE with discontinuous coefficients, [77, 94, 116, 122] (see below in detail). We remark here that Girsanov gives an example of SDE with α-Hölder continuous diffusion coefficient with $\alpha \in (0, 1/2)$ which does not satisfy the pathwise uniqueness (see Example 1.22 in [18]). Tanaka also provides that if the diffusion coefficient is a signed function (discontinuous), the weak solution of the corresponding SDE (Tanaka's equation) is not a strong solution but satisfies uniqueness in the sense of probability law, (see e.g., Example 5.3.5 in [60]). Finally we notice here that the pathwise uniqueness arguments was developed to some "*comparison theorem*" which is the monotonicity property for solutions of one-dimensional SDEs (see e.g. [50, 117]), and "*stability problem*" which is the error analysis between different two solutions of SDEs (see [59, 61]).

Kolmogorov equations

Now we consider the relation between SDE (1.1) and the Kolmogorov equation:

$$\begin{aligned} &\frac{\partial u}{\partial s} + Lu = 0; && \text{in } [0, T) \times \mathbb{R}^d, \\ &u(T, x) = f(x); && x \in \mathbb{R}^d. \end{aligned} \tag{1.2}$$

Here the differential operator L corresponding to the process X is defined by

$$Lf(x) := \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^{(i)}(x) \frac{\partial f(x)}{\partial x_i},$$

where $a \equiv \sigma\sigma^* = (a_{i,j})_{1 \leq i,j \leq d}$. It is well-known that the Feynman-Kac formula implies that under the assumption that the coefficients b and σ are bounded, Hölder-continuous and σ is uniformly elliptic then the solution u of (1.2) admits the stochastic representation

$$u(s, x) = \mathbb{E}_x[f(X_{T-s})], \tag{1.3}$$

on $[0, T] \times \mathbb{R}^d$, for any continuous and polynomial growth function $f : \mathbb{R}^d \to \mathbb{R}$, (e.g., Theorem 5.7.6 in [60] and see also [28]). Moreover, in the case of bounded, Hölder continuous coefficients, it is known that there exists the transition density function of X_t which is also called *fundamental solution* of the equation (1.2). The idea of proof is based on Levi's *parametrix method* which is a classical method in order to construct fundamental solution for parabolic type partial differential equations using a "Taylor-like" expansion argument (see [28]). This method allows for coefficients to be less regular than in the Malliavin Calculus [101] approach for the study of the density. On the other hand, we know that the stochastic differential equation (1.1) has a unique solution under weak regularity condition for the coefficients, and the operator L is well-defined for any measurable functions b and a. Stroock and Varadhan (Corollary 11.1 in [110]) show that if the drift coefficient b is bounded, measurable and the diffusion coefficient σ is bounded, uniformly elliptic and continuous, then the solution to the equation (1.2) in the class $W_p^{1,2}$ with $p > (d+2)/2$ also admits the stochastic representation (1.3) (see also Theorem 1 in [122]). We refer to [76] for the definition and properties of solutions in the class $W_p^{1,2}$. We also refer to [27, 75, 85] for the existence of the density of general SDEs, and [46, 47, 75] for the Hölder continuity property of the density. Finally, we remark that Fabes and Kenig [24] give an example that there exists, bounded, uniformly elliptic and uniformly continuous coefficient σ such that the law of the solution to $\mathrm{d}X_t = \sigma(t, X_t)\mathrm{d}W_t$ is *purely singular* with respect to Lebesgue measure on $\mathbb{R}^d$, it means that X_t does not have the probability density function.

1.1.2 Problem of numerical analysis for stochastic differential equations

Euler-Maruyama approximation

In many fields of applications, the theory of a numerical analysis for SDEs is quite important topic. In particular, it is essential for quantitative analysts in banking facilities to price complex financial derivative products.

Historically, the *Euler-Maruyama scheme* introduced by Maruyama [88] (which is a polygonal line approximation) is the most standard discrete approximation for solution of SDEs. Maruyama introduced it in order to prove Girsanov's theorem for the solution of one-dimensional SDE $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t$ (see [87, 88]). The Euler-Maruyama scheme for SDE is a stochastic analogue of the *Euler scheme* for ordinary differential equation, and is recognized as a way to have vast range of applications.

Let us define the Euler-Maruyama approximation introduced by Maruyama [88]. The Euler-Maruyama approximation for the SDE (1.1) denoted by $X^{(n)} = (X_t^{(n)})_{0 \le t \le T}$ is defined by

$$\mathrm{d}X_t^{(n)} = b(X_{\eta_n(t)}^{(n)})\mathrm{d}t + \sigma(X_{\eta_n(t)}^{(n)})\mathrm{d}W_t,\ X_0^{(n)} = X_0, \tag{1.4}$$

where $\eta_n(t) := kT/n$ if $t \in [kT/n, (k+1)T/n)$. For the Euler-Maruyama approximation, there are the following two types of error:

$$\mathbb{E}[\sup_{0 \le t \le T} |X_t - X_t^{(n)}|^p]^{1/p}, \quad \text{(strong error)}, \tag{1.5}$$

and

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{(n)})]|, \quad \text{(weak error)}. \tag{1.6}$$

Maruyama prove that under the Lipschitz conditions for b and σ, the process $X^{(n)}$ converges to the unique solution to the corresponding SDE in L^2-sense. Moreover, it is well-known that the Euler-Maruyama approximation has strong rate of convergence 1/2, (see [25, 58, 64]), that is, for any $p \ge 1$, if the coefficients b and σ are Lipschitz continuous,

$$\mathbb{E}[\sup_{0 \le t \le T} |X_t - X_t^{(n)}|^p]^{1/p} \le \frac{C}{n^{1/2}}, \tag{1.7}$$

and has weak rate of convergence 1, (see [10, 35, 37, 69, 113]), that is, for any bounded and smooth function $f : \mathbb{R}^d \to \mathbb{R}$, if b and σ are smooth functions,

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{(n)})]| \le \frac{C}{n}. \tag{1.8}$$

On the other hand, as mentioned above, the existence and uniqueness are true for SDE with "irregular coefficient". Therefore we now consider the following two questions:

Q-(i) *Under the assumption of existence and uniqueness for SDE, does the Euler-Maruyama approximation converge to the unique solution?*

Q-(ii) *If the Euler-Maruyama approximation converges to the unique solution, how is the convergence rate?*

The question Q-(i) is considered by Yamada in [118], and he show that under the same assumptions for the coefficients considered in [120], the Euler-Maruyama approximation converges to the unique solution to the corresponding SDE in L^1-sense and then extend to L^2-sense. Moreover, the question Q-(i) is solved by Kaneko and Nakao [59] in continuous coefficients. They show that the pathwise uniqueness, and continuity and linear growth property for the coefficients imply the Euler-Maruyama approximation converges to the unique solution in L^2-sense. Recently, Yan [121] show that for linear growth coefficients b and σ, under some assumptions on the discontinuity points of b and σ, the Euler-Maruyama approximation weakly converges to the unique weak solution. In the proof of [59] and [121], by using the *tightness* of the Euler-Maruyama approximation, we can deal with a convergence without having to use the smoothness of the coefficients. Therefore, the question Q-(i) has been solved in some sense. Finally we remark the result of Hutzenthaler, Jentzen and Kloeden [48]. They show that under a super-linear growth condition for the coefficients, the Euler-Maruyama scheme does not converge to the unique solution in the both weak and strong sense.

For the question Q-(ii), the strong rate in the case of non-Lipschitz coefficients have been studied recently. Yan [121] prove that for one-dimensional SDEs by using the Itô-Tanaka formula and estimation of local time, if the diffusion coefficient σ is $(\alpha+1/2)$-Hölder continuous with $\alpha \in (0,1/2]$ and the drift coefficient is Lipschitz continuous, then

$$\mathbb{E}[|X_T - X_T^{(n)}|] \leq \frac{C}{n^{\alpha}}.$$

This result extended by Gyöngy and Rásonyi [40]. They show that by using the approximation method of Yamada and Watanabe considered in [120] (see below in detail), for one-dimensional SDEs, if the diffusion coefficient is $(\alpha+1/2)$-Hölder continuous with $\alpha \in [0,1/2]$ and the drift is the sum of a Lipschitz and a non-increasing γ-Hölder continuous function with $\gamma \in (0,1)$ then

$$\mathbb{E}[|X_T - X_T^{(n)}|] \leq \begin{cases} \dfrac{C}{\log n} & \text{if } \alpha = 0, \\ \dfrac{C}{n^{\alpha}} + \dfrac{C}{n^{\gamma/2}} & \text{if } \alpha \in (0,1/2]. \end{cases}$$

Therefore, we know the strong rate of convergence for the Euler-Maruyama scheme, under the assumption for the pathwise uniqueness considered by Yamada and Watanabe [120].

The weak rate of convergence in the case of Hölder continuous coefficients are considered by Mikulevicius and Platen [92]. They prove that by using the property of the Kolmogorov equation (1.2) and stochastic representation (1.3), for any bounded, smooth function $f : \mathbb{R}^d \to \mathbb{R}$,

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{(n)})]| \leq \frac{C}{n^{\beta/2}},$$

where $\beta \in (0,1]$ is the Hölder exponent of the coefficients b and σ. Recently, Kohatsu-Higa, Lejay and Yasuda [65] provide the weak rate of convergence for the Euler-Maruyama scheme under non-regular drift coefficient. Martinez and Talay [86] prove that the weak rate for the Euler-Maruyama scheme with some discontinuous diffusion coefficients is $(1-\varepsilon)/2$ for any $\varepsilon \in (0,1/2)$. We note that the weak convergence rate presented in [65, 86, 92] is less that 1/2 for the Euler-Maruyama scheme for SDEs with irregular coefficients. Finally, we remark the result of Hairer, Hutzenthaler and Jentzen [41] (see also [56]). In [41], it has been shown that there is an example of SDE with infinitely often differentiable and globally bounded coefficients (the derivatives of drift are not bounded and diffusion coefficient is not uniformly elliptic) such that for any $\alpha \geq 0$,

$$\lim_{n\to\infty} n^{\alpha}\mathbb{E}[|X_T - X_T^{(n)}|] = \lim_{n\to\infty} n^{\alpha}|\mathbb{E}[X_T] - \mathbb{E}[X_T^{(n)}]| = \begin{cases} 0, & \text{if } \alpha = 0, \\ \infty, & \text{if } \alpha > 0. \end{cases}$$

Therefore the "polynomial type" convergence rates (1.7) and (1.8) are not satisfied in general, even if the coefficients are bounded and smooth enough.

Unbiased simulation and the parametrix method

Recently, many numerical schemes alternative to the Euler-Maruyama method have been studied by using a variety of techniques, and it has been discovered a lot of interesting results (see [4, 5, 9, 23, 26, 29, 49, 67, 74, 79, 81, 95, 105, 106]). As one of them, Bally and Kohatsu-Higa introduce in [9] an unbiased simulation method for the expectation $\mathbb{E}[f(X_T)]$ based on the parametrix method (see also [4, 67]).

Let us first recall Levi's parametrix method. For more detail, we refer to [28]. As we mentioned above, the parametrix method is used to construct a fundamental solution of the Kolmogorov equation (1.2). A fundamental solution of (1.2) is a function $p(s,x;t,y)$, $0 \leq s < t \leq T$ and $x, y \in \mathbb{R}^d$ which satisfies the following conditions:

(i) For fixed $(t,y) \in (0,T] \times \mathbb{R}^d$, it satisfies the equation $(\partial_s + L)p(s,x;t,y) = 0$.

(ii) For any continuous function $f : \mathbb{R}^d \to \mathbb{R}$, $\lim_{s \nearrow t} \int_{\mathbb{R}^d} p(s,x;t,y)f(y)\mathrm{d}y = f(x)$.

In order to construct the fundamental solution $p(s,x;t,y)$, we introduce an "approximation function" $p^z(s,x;t,y)$ defined by

$$p^z(s,x;t,y) = \frac{\exp\left(-\frac{1}{2(t-s)}\langle a(z)^{-1}(y-x), y-x\rangle_{\mathbb{R}^d}\right)}{(2\pi(t-s))^{d/2}\sqrt{\det a(z)}},$$

and we denote $\overline{p}(s,x;t,y) := p^y(s,x;t,y)$ which satisfies the equation $(\partial_s + L^y)\overline{p}(s,x;t,y) = 0$ where

$$L^y f(x) := \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{i,j}(y)\frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

Then the fundamental solution $p(s,x;t,y)$ can be constructed as the solution to the integral equation

$$p(s,x;t,y) = \overline{p}(s,x;t,y) + \int_s^t \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}z p(s,x;u,z)\Phi(u,z;t,y), \tag{1.9}$$

where $\Phi(s,x;t,y) = (L - L^y)\overline{p}(s,x;t,y)$. The equation (1.9) can be solved "formally" as follows: We define the convolution $\circledast$ for functions f, g as

$$f \circledast g(s,x;t,y) := \int_s^t \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}z f(s,x;u,z) g(u,z;t,y).$$

We denote $f^{\circledast 1} = f$, $f^{\circledast k} = f \circledast f^{\circledast (k-1)}$ and $f \circledast g^{\circledast 0} = f$. Then the equation (1.9) is linear equation, thus it satisfies the formal expansion

$$p(s,x;t,y) = \sum_{n=0}^{\infty} \overline{p} \circledast \Phi^{\circledast n}(s,x;t,y). \tag{1.10}$$

This procedure is called the *parametrix method* and $\overline{p}(s,x;t,y)$ is called the *parametrix*. The expansion (1.10) converges absolutely and uniformly for $x, y \in \mathbb{R}^d$ if the coefficients b and σ are bounded, Hölder continuous and σ is uniformly elliptic.

Now we provide a probabilistic representation for the density function $p(0,x;t,\cdot)$ of the solution to the SDE (1.1) which is introduced by Bally and Kohatsu-Higa [9]. Let $N = (N_t)_{t \geq 0}$ be a Poisson process with parameter $\lambda > 0$, that is $N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_n \leq t\}}$ where $(\tau_n - \tau_{n-1})_{n \in \mathbb{N}}$ with τ_0 are independent and identically exponential distribution with parameter λ. We also define

$$\hat{\theta}_t(x,y) := \frac{1}{2} \sum_{1 \leq i,j \leq d} (a_{i,j}(x) - a_{i,j}(y)) H^{i,j}_{ta(y)}(y-x) - \sum_{i=1}^{d} b_i(x) H^i_{ta(y)}(y-x),$$

where H^i and $H^{i,j}$ are Hermite polynomials of order 1 and 2, that is,

$$H^i_a(y) := -(a^{-1}y)_i \text{ and } H^{i,j}_a(y) := (a^{-1}y)_i(a^{-1}y)_j - (a^{-1})_{i,j}.$$

Then it is easy to prove that

$$\Phi(0,x;t,y) = \hat{\theta}_t(x,y)\overline{p}(0,x;t,y). \tag{1.11}$$

Using the expansion (1.10), the equation (1.11) and the Markov property of the stochastic process $y + \sigma(y)W_t$ whose density is $\overline{p}(0,\cdot;t,y)$, we have the following probabilistic representation for the density of X_t:

$$p(0,x;t,y) = e^{\lambda t}\mathbb{E}\left[\lambda^{-N_t}\overline{p}(\tau_t, x, ; t, X^{*,\pi}_{\tau_t}(y)) \prod_{j=0}^{N_t-1} \hat{\theta}_{\tau_{j+1}-\tau_j}(X^{*,\pi}_{\tau_{j+1}}(y), X^{*,\pi}_{\tau_j}(y))\right], \tag{1.12}$$

where $\tau_t := \tau_{N_t}$ and $X^{*,\pi}(y)$ is the Euler-Maruyama scheme with $X^{*,\pi}_0(y) = y$ and a random partition $\pi = (\tau_j \wedge T)_{j \in \mathbb{N}}$ whose diffusion coefficient is σ, that is $X^{*,\pi}_0(y) := y$ and for $j \geq 1$,

$$X^{*,\pi}_{\tau_j}(y) := X^{*,\pi}_{\tau_{j-1}}(y) + \sigma(X^{*,\pi}_{\tau_{j-1}}(y))(W_{\tau_j} - W_{\tau_{j-1}}).$$

The stochastic representation (1.12) can be used for Monte Carlo simulation in order to compute the expectation $\mathbb{E}[f(X_T)]$ for some Borel measurable function f.

1.2 Outline of the thesis

In this thesis, we consider numerical analysis for stochastic differential equations with irregular coefficients. This thesis consists of six chapters and references. In Chapter 2, 3, 4, we consider the above two questions Q-(i) and Q-(ii) for the Euler-Maruyama approximation with irregular coefficients. We provide the strong and weak rate of convergence. In Chapter 5, we consider the L^p-difference between two SDEs using a norm associated to the difference of irregular coefficients. In Chapter 6, we introduce an unbiased simulation scheme for SDEs with symmetric local time by using the parametrix method.

In Chapter 2, we consider the Euler-Maruyama approximation for multi-dimensional SDEs. We introduce some basic properties and the Gaussian estimate for the density of the Euler-Maruyama approximation. We will see the proof of a well-known result that if the coefficients are Lipschitz continuous then the Euler-Maruyama approximation has strong rate of convergence $1/2$, which was proved by Kanagawa [58] and Faure [25]. We introduce Yamada and Watanabe approximation technique in order to deal with Hölder continuous diffusion coefficients and one-sided Lipschitz drift coefficient. We introduce a class of irregular functions $\mathcal{A}$ in subsection 2.4.1 for the drift coefficient. The properties of the class $\mathcal{A}$ play a crucial role in our argument to obtain the strong rate of convergence for the Euler-Maruyama scheme. The results presented in section 2.4 are based on the paper [97] by Ngo and Taguchi published in *Mathematics of Computation*.

In Chapter 3, we consider the Euler-Maruyama approximation for one-dimensional SDEs with Hölder continuous and discontinuous coefficients. We apply the method of removal of drift to the solution of SDE and the Euler-Maruyama scheme, for providing the strong rate of convergence. In the case of discontinuous diffusion coefficient, we use an argument with local time and tightness of the Euler-Maruyama scheme. The results presented in section 3.2 are based on the paper [99] accepted for publication at *IMA Journal of Numerical Analysis*, and those in section 3.3 are based on the paper [100] accepted for publication at *Statistics and Probability Letters* by Ngo and Taguchi.

In Chapter 4, we aim at developing a systematic study for the weak rate of convergence of the Euler-Maruyama scheme for SDEs with unbounded, irregular drift and constant diffusion coefficients. We apply Girsanov's theorem to obtain the rates of approximation for the expectation of various non-smooth functionals. We also discuss approximation for the density of SDE and the weak rate of convergence for reflected stochastic differential equations. The results presented in this chapter are based on the preprint [98] by Ngo and Taguchi.

In Chapter 5, we consider stability problem for one-dimensional SDEs with irregular coefficients. The goal of this chapter is to estimate the L^p-difference between two SDEs using a norm associated to the difference of coefficients. In our setting, the (possibly) discontinuous drift coefficient satisfies a one-sided Lipschitz condition and the diffusion coefficient is bounded, uniformly elliptic and Hölder continuous. As an application of this result, we consider the stability problem for this class of SDEs. The results presented in this chapter are based on the paper [114] by Taguchi, published in *Séminaire de Probabilités*.

In Chapter 6, we apply the parametrix method in order to obtain the existence and the regularity properties of the density of a skew diffusion which is the solution of SDE with symmetric local time, and provide a Gaussian upper bound. We will provide a probabilistic representation similar to (1.12) which can be used to Monte Carlo simulation. The results presented in this chapter are based on the paper [67] by Kohatsu-Higa, Taguchi and Zhong, accepted for publication at *Potential Analysis*.

Table 1

The following is a table of the results on the strong rate of convergence for $\sup_{0\le t\le T}\mathbb{E}[|X_t - X_t^{(n)}|]$ presented in Chapter 2, 3 and several known results.

Multidimensional case:

b \ σ	Lipschitz conti.	Lipschitz conti. bdd, UE
Lipschitz conti.	$\frac{1}{n^{1/2}}$, [25], [58]	
$b \in \mathcal{A} \cap \mathcal{L}$		$\frac{1}{n^{1/2}}$, Sec. 2.4, [97]

One-dimensional case:

b \ σ	$(\alpha+\frac{1}{2})$-Hölder conti. $\alpha \in (0, \frac{1}{2}]$	$\frac{1}{2}$-Hölder conti.	$\sigma \in H^{\beta,\kappa}$
Lipschitz conti.	$\frac{1}{n^{\alpha}}$, [40, 121]	$\frac{1}{\log n}$, [40]	
$b \in \mathcal{A} \cap \mathcal{L}$	$\frac{1}{n^{\alpha}}$, Sec. 2.4, [97]	$\frac{1}{\log n}$, Sec. 2.4, [97]	
$b \in \mathcal{A} + H^{\beta}$	$\frac{e^{C\sqrt{\log n}}}{n^{\frac{\beta}{2}\wedge\alpha}}$, Sec. 3.2, [99]	$\frac{e^{C\sqrt{\log(\log n)}}}{\log n}$, Sec. 3.2, [99]	
$b \in (\mathcal{A} + H^{\beta}) \cap L^1(\mathbb{R})$	$\frac{1}{n^{\frac{\beta}{2}\wedge\alpha}}$, Sec. 3.2, [99]	$\frac{1}{\log n}$, Sec. 3.2, [99]	
$b \in H^{\beta,\kappa}$			$\frac{e^{C\sqrt{\log(\log n)}}}{\log n}$, Sec. 3.3, [100],
$b \in H^{\beta,\kappa} \cap L^1(\mathbb{R})$			$\frac{1}{\log n}$, Sec. 3.3, [100],

b \ σ	monotone β-Hölder conti, $\beta \in (0,1)$
$b = 0$	$\frac{1}{n^{\beta/2}}$, Sec. 3.4

Table 2

The following is a table of the results on the strong rate of convergence for $\mathbb{E}[\sup_{0\le t\le T}|X_t - X_t^{(n)}|]$ presented in Chapter 2, 3 and several known results.

Multidimensional case:

b \ σ	Lipschitz conti.	Lipschitz conti. bdd, UE
Lipschitz conti.	$\frac{1}{n^{1/2}}$, [25], [58]	
$b \in \mathcal{A}\cap\mathcal{L}$		$\frac{1}{n^{1/2}}$, Sec. 2.4, [97]

One-dimensional case:

b \ σ	$(\alpha+\frac{1}{2})$-Hölder conti. $\alpha\in(0,\frac{1}{2}]$	$\frac{1}{2}$-Hölder conti.
Lipschitz conti.	$\frac{1}{n^{2\alpha^2}}$, [40]	$\frac{1}{\sqrt{\log n}}$, [40]
$b \in \mathcal{A}\cap\mathcal{L}$	$\frac{1}{n^{2\alpha^2}}$, Sec. 2.4, [97]	$\frac{1}{\sqrt{\log n}}$, Sec. 2.4, [97]
$b \in \mathcal{A}+H^{\beta}$	$\frac{e^{C\sqrt{\log n}}}{n^{\alpha(\beta\wedge 2\alpha)}}$, Sec. 3.2, [99]	$\frac{e^{C\sqrt{\log(\log n)}}}{\sqrt{\log n}}$, Sec. 3.2, [99]
$b \in (\mathcal{A}+H^{\beta})\cap L^1(\mathbb{R})$	$\frac{1}{n^{\alpha(\beta\wedge 2\alpha)}}$, Sec. 3.2, [99]	$\frac{1}{\sqrt{\log n}}$, Sec. 3.2, [99]

Table 3

The following is a table of the results on the strong rate of convergence for $\mathbb{E}[\sup_{0\le t\le T}|X_t - X_t^{(n)}|^p]$, $p \geq 2$ presented in Chapter 2, 3 and several known results.

Multidimensional case:

b \ σ	Lipschitz conti.	Lipschitz conti. bdd, UE
Lipschitz conti.	$\frac{1}{n^{p/2}}$, [25], [58]	
$b \in \mathcal{A} \cap \mathcal{L}$		$\frac{1}{n^{1/2}}$, Sec. 2.4, [97]

One-dimensional case:

b \ σ	Lipschitz conti.	$(\alpha+\frac{1}{2})$-Hölder conti. $\alpha \in (0, \frac{1}{2})$	$\frac{1}{2}$-Hölder conti.
Lipschitz conti.	$\frac{1}{n^{p/2}}$, [25], [58]	$\frac{1}{n^{\alpha}}$, [40]	$\frac{1}{\log n}$, [40]
$b \in \mathcal{A} \cap \mathcal{L}$	$\frac{1}{n^{1/2}}$, Sec. 2.4, [97]	$\frac{1}{n^{\alpha}}$, Sec. 2.4, [97]	$\frac{1}{\log n}$, Sec. 2.4, [97]
$b \in (\mathcal{A}+H^{\beta}) \cap L^1(\mathbb{R})$	$\frac{1}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}}}$ Sec. 3.2, [99]	$\frac{1}{n^{\frac{\beta}{2}\wedge\alpha}}$, Sec. 3.2, [99]	$\frac{1}{\log n}$, Sec. 3.2, [99]
$b \in H^{\beta} \cap L^1(\mathbb{R})$	$\frac{1}{n^{p\beta/2}}$ Sec. 3.2, [99]		

Table 4

The following is a table of the results on the weal rate of convergence presented in Chapter 4 and several known results.

(i) $|\mathbb{E}[f(X_\cdot)] - \mathbb{E}[f(X_\cdot^{(n)})]|$

b, σ / $f : C[0,T] \to \mathbb{R}$	m'ble, linear growth and some assumptions on the discontinuity points	$b \in \mathcal{B}(\alpha)$ and sub-linear growth, $\sigma(x) = \sigma$
f is m'ble and bdd	converges to zero as $n \to \infty$, [121]	
$\mathbb{E}[\lvert f(x_0 + \sigma W_\cdot)\rvert^r] < \infty$, for some $r > 2$		$\frac{1}{n^{\frac{\alpha}{2} \wedge \frac{1}{4}}}$, Sec. 4.2, [98]

(ii) $|\mathbb{E}[f(X_\cdot)] - \mathbb{E}[f(X_{\eta_n(\cdot)}^{(n)})]|$

b, σ / $f : C[0,T] \to \mathbb{R}$	$b, \sigma \in C_b^3$ or C_b^∞, σ is UE	$b \in \mathcal{B}(\alpha)$, and sub-linear growth, $\sigma(x) = \sigma$
$f(x_\cdot) = g(\max_{0 \le s \le T} \lvert x_s\rvert)$, g is β-Hölder conti.		$\frac{1}{n^{\frac{\alpha}{2} \wedge \frac{1}{4}}} + \left\{ \frac{\log n}{n} \right\}^{\beta/2}$, Sec. 4.2, [98]
$f(x_\cdot) = \int_0^T g(x_s) \mathrm{d}s$, $\lvert g(x)\rvert \le Ce^{C\lvert x\rvert}$	$\frac{1}{n}$, [35]	
$f(x_\cdot) = f\Big(\int_0^T g(x_s) \mathrm{d}s \Big)$, f is Lipschitz conti., $g \in \mathcal{B}(\beta)$		$\frac{1}{n^{\frac{\alpha}{2} \wedge \frac{\beta}{2} \wedge \frac{1}{4}}}$, if b is bdd, Sec. 4.2, [98]
$f(x_\cdot) = g(x_T) \mathbf{1}_{(\tau_D > T)}$, $\tau_D = \inf\{t > 0 : x_t \notin D\}$, g is bdd, m'ble with $\mathrm{d}(\mathrm{Supp}(g), \partial D) \ge 2\varepsilon$	$\frac{1}{(1 \wedge \varepsilon^4) n^{1/2}}$, [34]	$\frac{1}{n^{\frac{\alpha}{2} \wedge \frac{1}{4}}} + \frac{1}{(1 \wedge \varepsilon^{4/p}) n^{1/(2p)}}$, for any $p > 1$, Sec. 4.2, [98]
$f(x_\cdot) = \delta_y(x_T)$, δ_y is the Dirac delta function at $y \in \mathbb{R}^d$	$\frac{g_{cT}(x_0, y)}{T^{1/2}} \frac{1}{n}$, [9],[35],[37],[69]	$g_{cT}(x_0, y) \left\{ \frac{\sqrt{T}}{n} + \frac{1}{n^{\alpha/2}} + \frac{1}{n^{1/(2pr)}} \right\}$, for any $p > d$ and $r > 1$, if b is bdd, Sec. 4.2, [98]

Chapter 2

Strong rate of convergence for the Euler-Maruyama approximation

2.1 Introduction

Let us consider the d-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s,\ x_0 \in \mathbb{R}^d,\ t \in [0, T], \tag{2.1}$$

where $W := (W_t)_{0\le t\le T}$ is a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions. The drift coefficient b is a Borel-measurable function from $[0, T] \times \mathbb{R}^d$ into $\mathbb{R}^d$ and the diffusion coefficient σ is a Borel-measurable function from $[0, T] \times \mathbb{R}^d$ into $\mathbb{R}^{d\times d}$. In this thesis, we consider that elements of $\mathbb{R}^d$ are column vectors. The diffusion process $X := (X_t)_{0\le t\le T}$ is used to model many random dynamical phenomena in many fields of application, for example, mathematical finance, optimal control problem and filtering.

Since the solution of (2.1) is rarely analytically tractable, one often approximates X by using the Euler-Maruyama scheme given by

$$X_t^{(n)} = x_0 + \int_0^t b\left(\eta_n(s), X^{(n)}_{\eta_n(s)}\right)\mathrm{d}s + \int_0^t \sigma\left(\eta_n(s), X^{(n)}_{\eta_n(s)}\right)\mathrm{d}W_s,\ t \in [0, T], \tag{2.2}$$

where $\eta_n(s) = kT/n =: t_k^{(n)}$ if $s \in [kT/n, (k+1)T/n)$. It is well-known that if the coefficients b and σ are Lipschitz continuous in space and 1/2-Hölder continuous in time then the Euler-Maruyama scheme has strong rate of convergence 1/2, (see [64]), i.e. for any $p > 0$, there exists $C_p > 0$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t - X_t^{(n)}|^p\right] \le \frac{C_p}{n^{p/2}}.$$

The strong rate in the case of non-Lipschitz coefficients have been studied recently by using the approximation method of Yamada and Watanabe ([120], Theorem 1) in Gyöngy and Rásonyi [40]. They have proven that for a one-dimensional SDE, if the diffusion coefficient is $(\alpha+1/2)$-Hölder continuous in space and the drift is the sum of a Lipschitz and a non-increasing γ-Hölder continuous function then

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t-\tilde{X}_t^{(n)}|\right]\le\begin{cases}C(\log n)^{-1/2} & \text{if } \alpha=0,\\ C(n^{-2\alpha^2}+n^{-\alpha\gamma}) & \text{if } \alpha\in(0,1/2],\end{cases}\tag{2.3}$$

where $\tilde{X}$ is the Euler's "polygonal" approximation of X given by

$$\tilde{X}_t^{(n)}=x_0+\int_0^t b\left(s,\tilde{X}_{\eta_n(s)}^{(n)}\right)\mathrm{d}s+\int_0^t\sigma\left(s,\tilde{X}_{\eta_n(s)}^{(n)}\right)\mathrm{d}W_s,\ \ t\in[0,T].\tag{2.4}$$

Yan [121] has obtained a result similar to (2.3) for the Euler-Maruyama scheme applied to a one-dimensional SDE with a drift which is Lipschitz continuous in space and Hölder continuous in time by using Tanaka's formula and some estimates for the local time. When the drift b is not supposed to be continuous, Halidias et al. (Theorem 3.1 in [42]) have shown the convergence of Euler-Maruyama approximation in L^2-norm (see also Theorem 2.8 in [39]). Regarding the rates of convergence, Gyöngy has shown that if b satisfies the one-sided Lipschitz condition (see Definition 2.4.7) and σ is locally Lipschitz then the rate of almost sure convergence for the Euler-Maruyama's polygonal approximation is of order 1/4 (see [38] Theorem 2.6). Moreover, Bastani et al. have recently proven strong L^p-rate 1/4 for $p\ge 2$ for split-step backward Euler approximations of SDEs with discontinuous drift and Lipschitz continuous diffusion coefficients (see Theorem 5.2 in [12]).

Besides the strong approximation problem, the weak approximation for non-Lipschitz coefficients SDE has also received a lot of attention. The weak rate of the Euler-Maruyama approximation when both drift and diffusion coefficients as well as payoff functions are Hölder continuous has been studied in [40, 64, 92]. Kohatsu-Higa et al. studied weak approximation errors for SDEs with discontinuous drift by using a perturbation method in [65]. The case of locally Lipschitz coefficients has been studied extensively, too, see [48] and the references therein. It should be noted that the strong rate of approximation is very useful to implement an effective Multi-level Monte Carlo simulation scheme for approximating expectations of some functionals of X (see [32]).

The goal of this chapter is to show that the strong rates obtained in [40] and [121] still hold even when b is discontinuous. More precisely, we will investigate the strong rate of the Euler-Maruyama approximation under the assumption that the diffusion coefficient σ is $(\alpha+1/2)$-Hölder continuous and the drift b is one-sided Lipschitz and belongs to the class $\mathcal{A}$ of functions which is, roughly speaking, of bounded variation with respect to a Gaussian measure on $\mathbb{R}^d$. In particular, our result implies that the Euler-Maruyama approximation has the optimal strong rate 1/2 in the case of Lipschitz continuous diffusion coefficient and discontinuous drift. Hence our result partly improves upon the ones in [38, 40, 121]. In this article, Lemma 2.4.12 is the key estimation. If the drift coefficient b is a Lipschitz continuous function, it is easy to prove this lemma. To obtain the same estimate with discontinuous drift, we use the result of Lemaire and Menozzi which is the Gaussian bound for the density of the Euler-Maruyama approximation (see [80], Theorem 2.1).

Finally we note that SDEs with discontinuous drift appear in many applications such as optimal control and interacting infinite particle systems, see e.g. [14, 16, 72].

In this chapter, we use the following notations. We define

$$Y_t^{(n)}:=X_t-X_t^{(n)}\ \text{and}\ U_t^{(n)}:=X_t^{(n)}-X_{\eta_n(t)}^{(n)}.\tag{2.5}$$

We denote $b=(b^{(1)},\dots,b^{(d)})^*$ and $\sigma=(\sigma_{i,j})_{1\le i,j\le d}$. Here * means transpose for the matrix.

2.2 Basic properties for the Euler-Maruyama scheme

2.2.1 Some auxiliary estimates

We first introduce the following two useful inequalities for proving the rate of convergence of the Euler-Maruyama approximation.

Lemma 2.2.1. *Assume that the coefficients b and σ are measurable and linear growth in space, i.e. there exists a positive constant K such that for any $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$|b(t,x)| + |\sigma(t,x)| \leq K(1+|x|).$$

Then for any $q>0$,

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^{(n)}|^q\right] < +\infty.$$

Proof. The case $0<q<2$ can be obtained from the case $q \geq 2$, so we assume that $q \geq 2$. Define a stopping time $\tau_N := \inf\{t>0 : |X_t^{(n)}| = N\}$ for $N \in \mathbb{N}$. By using the inequality $|a+b+c|^q \leq 3^{q-1}(|a|^q+|b|^q+|c|^q)$ for $a,b,c \in \mathbb{R}$, it holds that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_{t \wedge \tau_N}^{(n)}|^q\right]$$
$$\leq 3^{q-1}|x_0|^q + (3T)^{q-1}\mathbb{E}\left[\int_0^{T \wedge \tau_N} \left|b(\eta_n(s), X_{\eta_n(s)}^{(n)})\right|^q \mathrm{d}s\right] + 3^{q-1}\mathbb{E}\left[\sup_{0 \leq t \leq T}\left|\int_0^{t \wedge \tau_N} \sigma(\eta_n(s), X_{\eta_n(s)}^{(n)})\mathrm{d}W_s\right|^q\right].$$

By using Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0 \leq t \leq T}\left|\int_0^{t \wedge \tau_N} \sigma(\eta_n(s), X_{\eta_n(s)}^{(n)})\mathrm{d}W_s\right|^q\right] &\leq c(q)\mathbb{E}\left[\left\{\sum_{k=1}^d \int_0^{T \wedge \tau_N} \left|\sum_{i=1}^d \sigma_{i,k}(\eta_n(s), X_{\eta_n(s)}^{(n)})\right|^2 \mathrm{d}s\right\}^{q/2}\right] \\ &\leq c(q)(dT)^{q/2-1}\mathbb{E}\left[\int_0^{T \wedge \tau_N} \left|\sigma(\eta_n(s), X_{\eta_n(s)}^{(n)})\right|^q \mathrm{d}s\right].\end{aligned}$$

Since b and σ are linear growth in space, we have

$$|b(\eta_n(s \wedge \tau_N), X_{\eta_n(s \wedge \tau_N)}^{(n)})|^q + |\sigma(\eta_n(s \wedge \tau_N), X_{\eta_n(s \wedge \tau_N)}^{(n)})|^q \leq (2K)^q \left\{1 + \sup_{0 \leq u \leq s} |X_{u \wedge \tau_N}^{(n)}|^q\right\}.$$

Therefore there exists a positive constant C such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_{t \wedge \tau_N}^{(n)}|^q\right] \leq C + C\int_0^T \mathbb{E}\left[\sup_{0 \leq u \leq s} |X_{u \wedge \tau_N}^{(n)}|^q\right] \mathrm{d}s.$$

Applying the Gronwall's inequality we obtain

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_{t \wedge \tau_N}^{(n)}|^q\right] \leq C\exp(CT).$$

By taking $N \to \infty$, we conclude the statement. $\square$

Using the above Lemma, we have the following estimation.

Lemma 2.2.2. *Let the coefficients b and σ are measurable and satisfy linear growth in space. Then for any $q > 0$, there exists a positive constant C_q such that*

$$\sup_{0 \le t \le T} \mathbb{E}\left[|U_t^{(n)}|^q \right] \le \frac{C_q}{n^{q/2}}.$$

Proof. It is sufficient to prove the case $q \ge 2$. From the definition of $U_t^{(n)}$ and linear growth condition for the coefficient, we have

$$\begin{aligned}
|U_t^{(n)}|^q &= \left\{ \sum_{i=1}^d \left| X_t^{(n,i)} - X_{\eta_n(t)}^{(n,i)} \right|^2 \right\}^{q/2} \\
&= \left\{ \sum_{i=1}^d \left| (t-\eta_n(t)) b^{(i)}(\eta_n(t), X_{\eta_n(t)}^{(n)}) + \sum_{j=1}^d \sigma_{i,j}(\eta_n(t), X_{\eta_n(t)}^{(n)})(W_t^j - W_{\eta_n(t)}^j) \right|^2 \right\}^{q/2} \\
&\le C \left\{ |t-\eta_n(t)|^q \, (1 + |X_{\eta_n(t)}^{(n)}|^q) + \sum_{j=1}^d (1 + |X_{\eta_n(t)}^{(n)}|^q) |W_t^j - W_{\eta_n(t)}^j|^q \right\}.
\end{aligned}$$

Since $X_{\eta_n(t)}^{(n)}$ and $W_t^j - W_{\eta_n(t)}^j$ are independent, by using Lemma 2.2.1, we have

$$\begin{aligned}
\mathbb{E}\left[|U_t^{(n)}|^q \right] &\le C \left\{ |t-\eta_n(t)|^q + \sum_{j=1}^d \mathbb{E}\left[|W_t^j - W_{\eta_n(t)}^j|^q \right] \right\} \\
&\le C \left\{ |t-\eta_n(t)|^q + |t-\eta_n(t)|^{q/2} \right\} \le \frac{C}{n^{q/2}}.
\end{aligned}$$

This concludes Lemma 2.2.2. □

2.2.2 SDEs with Lipschitz continuous coefficients

In this section, we consider the strong rate of convergence for the Euler-Maruyama approximation for SDE (2.1) with Lipschitz continuous coefficients. Maruyama [88, Theorem 1] prove the Euler-Maruyama approximation converges to the unique solution to the corresponding SDE in L^2-sense under the assumption that the coefficients are Lipschitz continuous. Moreover, Kanagawa [58] and Faure [25] prove that the Euler-Maruyama approximation has strong rate of convergence 1/2. For the convenience of the reader, we will give a proof below.

Theorem 2.2.3 ([25, 58, 64])**.** *We assume that the coefficients b and σ are Lipschitz continuous in space and 1/2-Hölder continuous in time, that is*

$$\sup_{t \in [0,T], x,y \in \mathbb{R}^d, x \neq y} \frac{|b(t,x)-b(t,y)|}{|x-y|} + \sup_{t,s \in [0,T], t \neq s, x \in \mathbb{R}^d} \frac{|b(t,x)-b(s,y)|}{|t-s|^{1/2}} < +\infty$$

and

$$\sup_{t\in[0,T],x,y\in\mathbb{R}^d,x\neq y}\frac{|\sigma(t,x)-\sigma(t,y)|}{|x-y|}+\sup_{t,s\in[0,T],t\neq s,x\in\mathbb{R}^d}\frac{|\sigma(t,x)-\sigma(s,y)|}{|t-s|^{1/2}}<+\infty.$$

Then for any $p>0$, there exists a positive constant C such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t-X_t^{(n)}|^p\right]\le\frac{C}{n^{p/2}}.$$

Proof. By using Jensen's inequality, the case $0<p<2$ is obtained from the case $p\geq 2$, so we assume that $p\geq 2$. Applying the simple inequality $|a+b|^p\le 2^{p-1}(|a|^p+|b|^p)$ for $a,b\in\mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le T}|X_s-X_s^{(n)}|^p\right]\\
&\le C\int_0^T\mathbb{E}\left[|b(s,X_s)-b(\eta_n(s),X^{(n)}_{\eta_n(s)})|^p\right]\mathrm{d}s+C\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\sigma(s,X_s)-\sigma(\eta_n(s),X^{(n)}_{\eta_n(s)})\mathrm{d}W_s\right|^p\right]\\
&\le C\int_0^T|s-\eta_n(s)|^{p/2}\mathrm{d}s+C\int_0^T\mathbb{E}\left[|b(\eta_n(s),X_s)-b(\eta_n(s),X^{(n)}_{\eta_n(s)})|^p\right]\mathrm{d}s\\
&+C\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\sigma(s,X_s)-\sigma(\eta_n(s),X^{(n)}_{\eta_n(s)})\mathrm{d}W_s\right|^p\right].
\end{aligned}$$

It follows from Burkholder-Davis-Gundy's inequality that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\sigma(s,X_s)-\sigma(\eta_n(s),X^{(n)}_{\eta_n(s)})\mathrm{d}W_s\right|^p\right]\\
&\le C\mathbb{E}\left[\left\{\sum_{k=1}^d\int_0^T\left|\sum_{i=1}^d\left\{\sigma_{i,k}(s,X_s)-\sigma_{i,k}(\eta_n(s),X^{(n)}_{\eta_n(s)})\right\}\right|^2\mathrm{d}s\right\}^{p/2}\right]\\
&\le C\mathbb{E}\left[\int_0^T\left|\sigma(s,X_s)-\sigma(\eta_n(s),X^{(n)}_{\eta_n(s)})\right|^p\mathrm{d}s\right]\\
&\le C\int_0^T\mathbb{E}\left[|\sigma(s,X_s)-\sigma(\eta_n(s),X_s)|^p\right]\mathrm{d}s+C\int_0^T\mathbb{E}\left[\left|\sigma(\eta_n(s),X_s)-\sigma(\eta_n(s),X^{(n)}_{\eta_n(s)})\right|^p\right]\mathrm{d}s.
\end{aligned}$$

Therefore from the Lipschitz continuity of the coefficients and Lemma 2.2.2 with $q=p$, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le T}|X_s-X_s^{(n)}|^p\right]&\le C\int_0^T|s-\eta_n(s)|^{p/2}\mathrm{d}s+C\int_0^T\mathbb{E}\left[|X_s-X^{(n)}_{\eta_n(s)}|^p\right]\mathrm{d}s\\
&\le\frac{C}{n^{p/2}}+C\int_0^T\mathbb{E}\left[|X_s-X_s^{(n)}|^p\right]\mathrm{d}s+C\int_0^T\mathbb{E}\left[|X_{\eta_n(s)}-X^{(n)}_{\eta_n(s)}|^p\right]\mathrm{d}s\\
&\le C\int_0^T\mathbb{E}\left[\sup_{0\le u\le s}|X_u-X_u^{(n)}|^p\right]\mathrm{d}s+\frac{C}{n^{p/2}}.
\end{aligned}$$

Hence by the Gronwall's inequality, it holds that

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|X_s - X_s^{(n)}|^p\right] \leq \frac{C}{n^{p/2}},$$

which concludes the proof of Theorem 2.2.3. $\square$

2.2.3 Gaussian bound for the density of the Euler-Maruyama approximation

In this section, we consider the Gaussian estimate for the transition density of the Euler-Maruyama approximation. If the diffusion coefficient satisfies the uniformly elliptic condition, then the transition density $p^{(n)}(s,t,x,x')$ of $X^{(n)}$ between times s and t exists. Lemaire and Menozzi [80] prove that if the drift coefficient is bounded measurable and the diffusion coefficient is a Hölder continuous function, then the transition density of the Euler-Maruyama approximations has the upper and lower bound by the density of Gaussian distribution. The idea of their proof is to use Levi's parametrix method (see [28]).

Theorem 2.2.4 ([80], Theorem 2.1)**.** *Suppose that the drift coefficient b and diffusion coefficient σ satisfies the following conditions:*

(UE) *$a = \sigma\sigma^*$ is bounded and uniformly elliptic, i.e., there exists $\lambda_0 \geq 1$ such that for any $(t,x,\xi) \in [0,T]\times\mathbb{R}^{2d}$,*

$$\lambda_0^{-1}|\xi|^2 \leq \langle a(t,x)\xi,\xi\rangle_{\mathbb{R}^d} \leq \lambda_0|\xi|^2.$$

(SB) *The drift b is bounded measurable and the diffusion coefficient σ is uniformly η-Hölder continuous with $\eta \in (0,1]$ in space and uniformly in time. More precisely, there exists $K > 0$ such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|b(t,x)| + \sup_{(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d, x\neq y}\frac{|\sigma(t,x)-\sigma(t,y)|}{|x-y|^\eta} \leq K.$$

Then there exist constant $c > 0$ and $C \geq 1$ such that for any $0 \leq j < j' \leq n$ and $x,x' \in \mathbb{R}$,

$$C^{-1}p_{c^{-1}}\left(t_{j'}^{(n)} - t_j^{(n)}, x, x'\right) \leq p^{(n)}\left(t_j^{(n)}, t_{j'}^{(n)}, x, x'\right) \leq Cp_c\left(t_{j'}^{(n)} - t_j^{(n)}, x, x'\right), \tag{2.6}$$

where $p_c(t-s,x,x') := \left(\frac{c}{2\pi(t-s)}\right)^{d/2}\exp\left(-c\frac{|x'-x|^2}{2(t-s)}\right)$.

Note that the constant C depends on K, λ_0, η, d, T and the constant c depends on K, λ_0, η, d but not on T.

The following lemma plays a crucial role in our argument.

Lemma 2.2.5. *Suppose that (UE) and (SB) hold. Then there exist $C \geq 1$ and $c > 0$ such that for any $x, x' \in \mathbb{R}^d$, $0 \leq j < j' \leq n$ and $t \in (t_{j'-1}^{(n)}, t_{j'}^{(n)}]$, we have*

$$p^{(n)}\left(t_j^{(n)}, t, x, x'\right) \leq Cp_c\left(t - t_j^{(n)}, x, x'\right).$$

Proof. Note that for any $u \in \mathbb{R}^d$,

$$p^{(n)}\left(t^{(n)}_{j'-1}, t, u, x'\right) = \left(\frac{1}{2\pi}\right)^{d/2} \frac{1}{\sqrt{(t - t^{(n)}_{j'-1})^d \det a(t^{(n)}_{j'-1}, u)}}$$
$$\times \exp\left(-\frac{\langle a^{-1}(t^{(n)}_{j'-1}, u)(x' - u - (t - t^{(n)}_{j'-1})b(t^{(n)}_{j'-1}, u)), (x' - u - (t - t^{(n)}_{j'-1})b(t^{(n)}_{j'-1}, u))\rangle_{\mathbb{R}^d}}{2(t - t^{(n)}_{j'-1})}\right).$$

Since a^{-1} is uniformly elliptic, using the inequality $|x - y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$ for any $x, y \in \mathbb{R}^d$, we obtain

$$-\frac{\langle a^{-1}(t^{(n)}_{j'-1}, u)(x' - u - (t - t^{(n)}_{j'-1})b(t^{(n)}_{j'-1}, u)), (x' - u - (t - t^{(n)}_{j'-1})b(t^{(n)}_{j'-1}, u))\rangle_{\mathbb{R}^d}}{2(t - t^{(n)}_{j'-1})}$$
$$\leq -\frac{\lambda_0^{-1}|x' - u - (t - t^{(n)}_{j'-1})b(t^{(n)}_{j'-1}, u)|^2}{2(t - t^{(n)}_{j'-1})}$$
$$\leq -\frac{(2\lambda_0)^{-1}|x' - u|^2}{2(t - t^{(n)}_{j'-1})} + \frac{\lambda_0^{-1}(t - t^{(n)}_{j'-1})^2|b(t^{(n)}_{j'-1}, u)|^2}{2(t - t^{(n)}_{j'-1})} \leq -c_1\frac{|x' - u|^2}{2(t - t^{(n)}_{j'-1})} + C_1.$$

Hence we have

$$p^{(n)}\left(t^{(n)}_{j'-1}, t, u, x'\right) \leq C_2 \frac{\exp\left(-c_1 \frac{|x'-u|^2}{2(t-t^{(n)}_{j'-1})}\right)}{\left(2\pi(t - t^{(n)}_{j'-1})\right)^{d/2}}.$$

This estimate together with the Chapman-Kolmogorov equation and (2.6) yield

$$p^{(n)}\left(t^{(n)}_j, t, x, x'\right) = \int_{\mathbb{R}^d} p^{(n)}\left(t^{(n)}_j, t^{(n)}_{j'-1}, x, u\right) p^{(n)}\left(t^{(n)}_{j'-1}, t, u, x'\right) \mathrm{d}u$$
$$\leq C_3 \int_{\mathbb{R}^d} \frac{\exp\left(-c_2 \frac{|u-x|^2}{2(t^{(n)}_{j'-1}-t^{(n)}_j)}\right)}{\left(2\pi(t^{(n)}_{j'-1} - t^{(n)}_j)\right)^{d/2}} \frac{\exp\left(-c_2 \frac{|x'-u|^2}{2(t-t^{(n)}_{j'-1})}\right)}{\left(2\pi(t - t^{(n)}_{j'-1})\right)^{d/2}} \mathrm{d}u = C_4 p_{c_2}\left(t - t^{(n)}_j, x, x'\right).$$

We therefore obtain the desired estimate. □

Corollary 2.2.6. *Let $p^{(n)}_t$ be a density for $X^{(n)}_t$. From Lemma 2.2.5 with $j = 0$, there exist $C \geq 1$ and $c > 0$ such that for any $t \in (0, T]$ and $x \in \mathbb{R}^d$ we have*

$$p^{(n)}_t(x) = p^{(n)}(0, t, x_0, x) \leq C p_c(t, x_0, x). \tag{2.7}$$

2.3 Yamada and Watanabe approximation technique

Inspired by the paper [40], we will use the approximation technique of Yamada and Watanabe (see [120], Theorem 1). For each $\delta \in (1, \infty)$ and $\varepsilon \in (0, 1)$, we can define a continuous function $\psi_{\delta,\varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with

$\operatorname{supp}\psi_{\delta,\varepsilon} \subset [\varepsilon/\delta, \varepsilon]$ such that

$$\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta,\varepsilon}(z)\mathrm{d}z = 1 \text{ and } 0 \le \psi_{\delta,\varepsilon}(z) \le \frac{2}{z\log\delta},\ z > 0.$$

Since $\int_{\varepsilon/\delta}^{\varepsilon} \frac{2}{z\log\delta}\mathrm{d}z = 2$, there exists such a function $\psi_{\delta,\varepsilon}$. We define a function $\phi_{\delta,\varepsilon} \in C^2(\mathbb{R};\mathbb{R})$ by

$$\phi_{\delta,\varepsilon}(x) := \int_0^{|x|}\int_0^{y} \psi_{\delta,\varepsilon}(z)\mathrm{d}z\mathrm{d}y.$$

It is easy to verify that $\phi_{\delta,\varepsilon}$ has the following useful properties:

$$\phi'_{\delta,\varepsilon}(x) = \frac{x}{|x|}\phi'_{\delta,\varepsilon}(|x|),\ \text{for any } x \in \mathbb{R}\setminus\{0\}. \tag{2.8}$$

$$0 \le |\phi'_{\delta,\varepsilon}(x)| \le 1,\ \text{for any } x \in \mathbb{R}. \tag{2.9}$$

Moreover, we define the function $\Phi_{\delta,\varepsilon} : \mathbb{R}^d \to \mathbb{R}$ by

$$\Phi_{\delta,\varepsilon}(x) := \phi_{\delta,\varepsilon}(|x|).$$

Then we also have the following useful properties:

$$|x| \le \varepsilon + \Phi_{\delta,\varepsilon}(x),\ \text{for any } x \in \mathbb{R}^d. \tag{2.10}$$

$$\frac{\phi'_{\delta,\varepsilon}(|x|)}{|x|} \le \frac{\delta}{\varepsilon},\ \text{for any } x \in \mathbb{R}^d\setminus\{0\}. \tag{2.11}$$

$$\phi''_{\delta,\varepsilon}(\pm|x|) = \psi_{\delta,\varepsilon}(|x|) \le \frac{2}{|x|\log\delta}\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|x|),\ \text{for any } x \in \mathbb{R}^d\setminus\{0\}. \tag{2.12}$$

Note that partial differentiations of $\Phi_{\delta,\varepsilon}$ give the following: for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\partial_i\Phi_{\delta,\varepsilon}(x) = \phi'_{\delta,\varepsilon}(|x|)\frac{x_i}{|x|}, \tag{2.13}$$

$$\partial_i^2\Phi_{\delta,\varepsilon}(x) = \phi''_{\delta,\varepsilon}(|x|)\frac{x_i^2}{|x|^2} + \phi'_{\delta,\varepsilon}(|x|)\left(\frac{|x|^2 - x_i^2}{|x|^3}\right), \tag{2.14}$$

$$\partial_i\partial_j\Phi_{\delta,\varepsilon}(x) = \phi''_{\delta,\varepsilon}(|x|)\frac{x_ix_j}{|x|^2} - \phi'_{\delta,\varepsilon}(|x|)\left(\frac{x_ix_j}{|x|^3}\right). \tag{2.15}$$

Notice also that all derivatives of $\phi_{\delta,\varepsilon}$ and $\Phi_{\delta,\varepsilon}$ at the origin equal to 0. In particular, note that for any $x \in \mathbb{R}^d$ and $i = 1, \dots, d$, using (2.9) and (2.13),

$$|\partial_i\Phi_{\delta,\varepsilon}(x)| \le 1.$$

Remark 2.3.1. Note that Yamada and Watanabe [120] prove that under the one-dimensional setting, if the diffusion coefficient σ is $1/2+\alpha$-Hölder continuous with $\alpha \in [0, 1/2]$ and drift coefficient b is Lipschitz continuous, then the pathwise uniqueness holds. Let us prove this. To simplify the discussion, we consider the one-dimensional SDE $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t$ with $X_0 = x_0$. It suffices to prove that $\mathbb{E}[|X_t - \widetilde{X}_t|] = 0$ for all $t \in [0,\infty)$ where X and $\widetilde{X}$ are two solutions. For any $\varepsilon \in (0,1)$ and $\delta \in (1,\infty)$, by using the inequality (2.10) and Itô's formula, we have,

$$\begin{aligned} |X_t - \widetilde{X}_t| &\leq \varepsilon + \phi_{\delta,\varepsilon}(X_t - \widetilde{X}_t) \\ &= \varepsilon + \int_0^t \phi'_{\delta,\varepsilon}(X_s - \widetilde{X}_s)\{\sigma(X_s) - \sigma(\widetilde{X}_s)\}\mathrm{d}W_s + \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(X_s - \widetilde{X}_s)|\sigma(X_s) - \sigma(\widetilde{X}_s)|^2\mathrm{d}s. \end{aligned} \tag{2.16}$$

Since $|\phi'(x)| \leq 1$ and σ is Hölder continuous, the stochastic integral of right hand side of (2.16) is a martingale, thus its expectation equals to zero. Therefore, we only consider the last part of (2.16). Since σ is $1/2+\alpha$-Hölder continuous, it follows from (2.12) that

$$\begin{aligned} \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(X_s - \widetilde{X}_s)|\sigma(X_s) - \sigma(\widetilde{X}_s)|^2\mathrm{d}s &\leq K^2\int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|X_s - \widetilde{X}_s|)}{|X_s - \widetilde{X}_s|\log\delta}|X_s - \widetilde{X}_s|^{1+2\alpha}\mathrm{d}s \\ &\leq \frac{K^2T\varepsilon^{2\alpha}}{\log\delta}, \end{aligned} \tag{2.17}$$

where $K := \sup_{x\neq y}\frac{|\sigma(x)-\sigma(y)|}{|x-y|^{1/2+\alpha}}$. Therefore, by taking the expectation, we have

$$\mathbb{E}[|X_t - \widetilde{X}_t|] \leq \varepsilon + \frac{K^2T\varepsilon^{2\alpha}}{\log\delta}. \tag{2.18}$$

Since $\alpha \in [0,1/2]$, by letting $\varepsilon \to 0$ and then $\delta \to \infty$, we conclude $\mathbb{E}[|X_t - \widetilde{X}_t|] = 0$.

From the next section, we will see the Yamada and Watanabe approximation technique works well for the error analysis of the Euler-Maruyama scheme.

2.4 SDEs with one-sided Lipschitz drift and Hölder continuous diffusion coefficient

In this section, we provide the rate of strong convergence where the possibly discontinuous drift coefficient and the diffusion coefficient is Hölder continuous and uniformly elliptic.

Veretennikov [116] has shown the following result. Assume that b and σ are bounded measurable functions such that $\sigma\sigma^*$ is bounded and uniformly elliptic. If σ is $1/2$-Hölder continuous in $x \in \mathbb{R}$ when $d=1$ and it is Lipschitz in $x \in \mathbb{R}^d$ when $d \geq 2$, then there exists a unique strong solution to the stochastic differential equation (2.1) (see also [18, 39, 71, 72, 122] for other criteria for the existence and uniqueness of solution of SDE with non-Lipschitz coefficients).

2.4.1 Class of irregular functions

In this section, we introduce two classes of functions, $\mathcal{A}$ and $\mathcal{L}$, for the drift coefficient.

Class of irregular functions $\mathcal{A}$

We first define the class of functions $\mathcal{A}$ which is first introduced in [66] (see also [97]). We also provide some properties of class $\mathcal{A}$.

Definition 2.4.1. *Let $\mathcal{A}$ be the class of all bounded measurable functions $\zeta : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that there exists a sequence $(\zeta_N)_{N\in\mathbb{N}}$ of functions satisfying $\zeta_N(t,\cdot) \in C^1(\mathbb{R}^d;\mathbb{R})$ for any $t \in [0,T]$ and the following conditions:*

$\mathcal{A}$(i) For any $L > 0$, $\sup_{t\in[0,T]} \int_{|x|\le L} |\zeta_N(t,x) - \zeta(t,x)|\mathrm{d}x \to 0$ as $N \to \infty$.

$\mathcal{A}$(ii) There exists a positive constant K such that for any $x \in \mathbb{R}^d$,

$$\sup_{t\in[0,T]} \sup_{N\in\mathbb{N}} |\zeta_N(t,x)| \le K.$$

$\mathcal{A}$(iii) There exists a positive constant K such that for any $a \in \mathbb{R}^d$ and $u > 0$,

$$\sup_{t\in[0,T]} \sup_{N\in\mathbb{N}} \sum_{i=1}^{d} \int_{\mathbb{R}^d} |\partial_i \zeta_N(t,x+a)| \frac{e^{-\frac{|x|^2}{u}}}{u^{(d-1)/2}} \mathrm{d}x \le K(1+\sqrt{u}),$$

where ∂_i is partial derivative in space with respect to the i-th coordinate.

We call $(\zeta_N)_{N\in\mathbb{N}}$ an $\mathcal{A}$-approximation sequence of ζ. This class of function $\mathcal{A}$ is similar to the one introduced in [66]. The following proposition shows that this class is quite large.

Proposition 2.4.2. *(i) If $\xi, \zeta \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$, then $\xi\zeta \in \mathcal{A}$ and $\alpha\xi + \beta\zeta \in \mathcal{A}$.*
(ii) If $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a bounded measurable function and $g(t,\cdot) : \mathbb{R}^d \to \mathbb{R}$ is monotone in each variable separately, then $g \in \mathcal{A}$.
(iii) If g is bounded and Lipschitz continuous in space, then $g \in \mathcal{A}$.

Proof. (i) It is easy to prove that $\mathcal{A}$ is a vector space over $\mathbb{R}$.
(ii) Let $g : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function and let $g(t,\cdot) : \mathbb{R}^d \to \mathbb{R}$ be monotone in each variable separately. Let $\rho(x)$ be the density function of the d-dimensional standard normal distribution, i.e. $\rho(x) := e^{-|x|^2/2}/(2\pi)^{d/2}$ and a sequence $(\rho_N)_{N\in\mathbb{N}}$ be defined by $\rho_N(x) := N^d\rho(Nx)$. Finally, we set $g_N(t,x) := \int_{\mathbb{R}^d} g(t,y)\rho_N(x-y)\mathrm{d}y$ and $\|g\|_\infty := \sup_{t\in[0,T],x\in\mathbb{R}^d} |g(t,x)|$. We will show that (g_N) is a $\mathcal{A}$-approximation sequence of g. Indeed, since $\int_{\mathbb{R}^d} \rho_N(y)\mathrm{d}y = 1$, we have $|g_N(t,x)| \le \|g\|_\infty$. Thus (g_N) satisfies $\mathcal{A}$(ii). Moreover, for any $L > 0$, we have

$$\begin{aligned}
\int_{|x|\le L} |g_N(t,x) - g(t,x)|\mathrm{d}x &\le \int_{|x|\le L} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y |g(t,y) - g(t,x)|\rho_N(x-y) \\
&= \int_{|x|\le L} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}z |g(t,x-z) - g(t,x)|\rho_N(z) \\
&= \int_{\mathbb{R}^d} \mathrm{d}z \int_{|x|\le L} \mathrm{d}x |g(t,x-\frac{z}{N}) - g(t,x)|\rho(z).
\end{aligned}$$

For each $z \in \mathbb{R}^d$, we write $z = (z_1, \ldots, z_d)^*$, $z^{(0)} = 0$ and $z^{(k)} = (z_1, \ldots, z_k, 0, \ldots, 0)^*$ for $k = 1, \ldots, d$. We have

$$\int_{|x|\le L} |g(t, x - \frac{z}{N}) - g(t,x)|\mathrm{d}x \le \sum_{k=1}^{d} \int_{|x_1|\le L} \mathrm{d}x_1 \ldots \int_{|x_{k-1}|\le L} \mathrm{d}x_{k-1} \int_{|x_{k+1}|\le L} \mathrm{d}x_{k+1} \ldots \int_{|x_d|\le L} \mathrm{d}x_d$$
$$\times \int_{|x_k|\le L} \left| g(t, x - \frac{z^{(k)}}{N}) - g(t, x - \frac{z^{(k-1)}}{N}) \right| \mathrm{d}x_k.$$

Since $g(t, \cdot)$ is monotone in each variable,

$$\int_{|x_k|\le L} \left| g(t, x - \frac{z^{(k)}}{N}) - g(t, x - \frac{z^{(k-1)}}{N}) \right| \mathrm{d}x_k = \left| \int_{|x_k|\le L} \Big(g(t, x - \frac{z^{(k)}}{N}) - g(t, x - \frac{z^{(k-1)}}{N}) \Big) \mathrm{d}x_k \right|.$$

By the change of variable, we have

$$\left| \int_{|x_k|\le L} \Big(g(t, x - \frac{z^{(k)}}{N}) - g(t, x - \frac{z^{(k-1)}}{N}) \Big) \mathrm{d}x_k \right| = \left| \left(\int_{-L-z_k/N}^{-L} + \int_{L-z_k/N}^{L} \right) g(t, x - \frac{z^{(k-1)}}{N}) \mathrm{d}x_k \right|$$
$$\le \frac{2|z_k| \|g\|_\infty}{N}.$$

Therefore

$$\sup_{t\in[0,T]} \int_{|x|\le L} |g(t, x - \frac{z}{N}) - g(t,x)|\mathrm{d}x \le \sum_{k=1}^{d} \frac{2|z_k| \|g\|_\infty L^{d-1}}{N}.$$

This implies that

$$\sup_{t\in[0,T]} \int_{|x|\le L} |g_N(t,x) - g(t,x)|\mathrm{d}x \le \int_{\mathbb{R}^d} \sum_{k=1}^{d} \frac{2|z_k| \|g\|_\infty L^{d-1}}{N} \rho(z)\mathrm{d}z \to 0.$$

as $N \to \infty$. Thus $(g_N)_{N\in\mathbb{N}}$ satisfies $\mathcal{A}$(i).

Since $g(t, \cdot)$ is a monotone function in each variable separately, so is $g_N(t, \cdot)$. Using the integration by parts formula, we have

$$\int_{\mathbb{R}^d} |\partial_i g_N(t, x+a)| \frac{e^{-|x|^2/u}}{u^{(d-1)/2}} \mathrm{d}x = \Big| \int_{\mathbb{R}^{d-1}} \mathrm{d}x_1 \ldots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \ldots \mathrm{d}x_d \int_{\mathbb{R}} \mathrm{d}x_i \partial_i g_N(t, x+a) \frac{e^{-|x|^2/u}}{u^{(d-1)/2}} \Big|$$
$$\le \int_{\mathbb{R}^d} |g_N(t, x+a)| \frac{2|x_i|}{u} \frac{e^{-|x|^2/u}}{u^{(d-1)/2}} \mathrm{d}x \le \int_{\mathbb{R}^d} \frac{2\|g\|_\infty |x|}{\sqrt{u}} \frac{e^{-|x|^2/u}}{u^{d/2}} \mathrm{d}x = 2\|g\|_\infty \int_{\mathbb{R}^d} |y| e^{-|y|^2} \mathrm{d}y,$$

where we use the change of variable $y = x/\sqrt{u}$ in the last equation. This concludes $(g_N)_{N\in\mathbb{N}}$ satisfies $\mathcal{A}$(iii).

(iii). Let (g_N) be defined as in (ii). For each $L > 0$, since g is Lipschitz continuous, we have

$$\int_{|x|\le L} |g_N(t,x) - g(t,x)|\mathrm{d}x \le \int_{|x|\le L} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y |g(t,y) - g(t,x)| \rho_N(x-y)$$

$$\begin{aligned}&\le C\int_{|x|\le L}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}y|y-x|\rho_N(x-y)\\&=C\int_{|x|\le L}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}z\frac{|z|e^{-\frac{z^2}{2}}}{N}\le\frac{C}{N}\to 0,\end{aligned}$$

as $N\to\infty$. This implies (g_N) satisfying $\mathcal{A}$(i). It is straightforward to verify (g_N) satisfying $\mathcal{A}$(ii).

To check $\mathcal{A}$(iii), we note that from the fact $\partial_i\rho_N(x)=-N^{d+2}x_i\rho(Nx)$ and Lipschitz property of g,

$$\begin{aligned}|\partial_i g_N(t,x)|&=\Big|\int_{\mathbb{R}^d}g(t,y)\partial_i\rho_N(x-y)\mathrm{d}y\Big|=\Big|\int_{\mathbb{R}^d}\{g(t,y)-g(t,x)\}\partial_i\rho_N(x-y)\mathrm{d}y\Big|\\&\le\int_{\mathbb{R}^d}N^{d+2}|y-x||y_i-x_i|\frac{e^{-\frac{N^2|y-x|^2}{2}}}{(2\pi)^{d/2}}\mathrm{d}y.\end{aligned}$$

The change of variable $x=y+z/N$ implies that

$$\int_{\mathbb{R}^d}N^{d+2}|y-x||y_i-x_i|\frac{e^{-\frac{N^2|y-x|^2}{2}}}{(2\pi)^{d/2}}\mathrm{d}y\le\int_{\mathbb{R}^d}|z|\frac{e^{-|z|^2/2}}{(2\pi)^{d/2}}\mathrm{d}z=C<\infty.$$

Hence for any $a\in\mathbb{R}^d$ and $u>0$,

$$\sup_{N\in\mathbb{N}}\sum_{i=1}^d\int_{\mathbb{R}^d}|\partial_i g_N(t,x+a)|\frac{e^{-\frac{|x|^2}{u}}}{u^{(d-1)/2}}\mathrm{d}x\le C\int_{\mathbb{R}^d}\frac{e^{-\frac{|x|^2}{u}}}{u^{(d-1)/2}}\mathrm{d}x\le C\sqrt{u}$$

holds with constant C which is independent of a and u. This concludes $(g_N)_{N\in\mathbb{N}}$ satisfying $\mathcal{A}$(iii). $\square$

Using Proposition 2.4.2 one can easily verify that the class $\mathcal{A}$ contains the functions $\zeta(x)=|x-a|\wedge 1$ or $\zeta(x)=I_{b<x<c}$ for any $a\in\mathbb{R}^d$ and $b,c\in[-\infty,\infty]$. On the other hand, the following proposition shows that there is an example of function which does not belong to $\mathcal{A}$.

Proposition 2.4.3. *Let*

$$f(x):=\begin{cases}\sqrt{|x|}\sin\dfrac{1}{x} & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

which is a bounded and continuous function on $\mathbb{R}$. Then $f\notin\mathcal{A}$.

To prove Proposition 2.4.3, we need to show the following Lemma.

Lemma 2.4.4. *Suppose that $g:\mathbb{R}\to\mathbb{R}$ and $g\in\mathcal{A}$. Then there exists a sequence of functions $(g_N)_{N\in\mathbb{N}}\subset C^\infty(\mathbb{R};\mathbb{R})$ such that*

(i) For any $L>0$, $\displaystyle\int_{|x|\le L}|g_N(x)-g(x)|\mathrm{d}x\to 0$ as $N\to\infty$.

(ii) There exists a positive constant K such that for any $x\in\mathbb{R}$,

$$\sup_{N\in\mathbb{N}}|g_N(t,x)|\le K.$$

(iii) There exists a positive constant K such that for any $a \in \mathbb{R}$ and $u > 0$,

$$\sup_{N\in\mathbb{N}} \int_{\mathbb{R}} |g_N'(x+a)|\, e^{-\frac{x^2}{u}} \mathrm{d}x \leq K(1+\sqrt{u}).$$

Remark 2.4.5. Lemma 2.4.4 implies that we can approximate a function $f \in \mathcal{A}$ by a sequence of C^∞ functions instead of C^1.

Proof of Lemma 2.4.4. Let $\rho(x) := e^{-x^2/2}/\sqrt{2\pi}$. Suppose that $(\zeta_N)_{N\in\mathbb{N}}$ is the $\mathcal{A}$-approximation sequence of g. We denote $g_N(x) := \int_{\mathbb{R}} \zeta_N(x - z/N)\rho(z)\mathrm{d}z$. Then we will show that $(g_N)_{N\in\mathbb{N}} \subset C^\infty(\mathbb{R};\mathbb{R})$ satisfies (i), (ii) and (iii). Since g is a bounded and ζ_N satisfies $\mathcal{A}$(ii), there exists K such that for any $x \in \mathbb{R}$, $|g(x)| \vee \sup_{N\in\mathbb{N}} |\zeta_N(x)| \leq K$. So since $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$, we have $|g_N(x)| \leq K$. Thus $(g_N)_{N\in\mathbb{N}}$ satisfies (ii).

Note that $g_N(x) - g(x) = \int_{\mathbb{R}}(\zeta_N(x - z/N) - g(z))\rho(z)\mathrm{d}z$. For any $L > 0$,

$$\begin{aligned}
&\int_{|x|\leq L} |g_N(x) - g(x)|\mathrm{d}x \\
&\leq \int_{|x|\leq L} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}z \left|\zeta_N\left(x - \frac{z}{N}\right) - g\left(x - \frac{z}{N}\right)\right| \rho(z) + \int_{|x|\leq L} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}z \left|g\left(x - \frac{z}{N}\right) - g(x)\right| \rho(z) \\
&=: J_1 + J_2.
\end{aligned}$$

By Fubini's theorem and the change of variable, we have

$$\begin{aligned}
J_1 &\leq \int_{|z|\leq N} \mathrm{d}z \int_{|x|\leq L+1} \mathrm{d}x\, |\zeta_N(x) - g(x)|\, \rho(z) + \int_{|z|>N} \mathrm{d}z \int_{|x|\leq L} \mathrm{d}x \left|\zeta_N\left(x - \frac{z}{N}\right) - g\left(x - \frac{z}{N}\right)\right| \rho(z) \\
&\leq \int_{|x|\leq L+1} |\zeta_N(x) - g(x)|\, \mathrm{d}x + 4KL \int_{|z|>N} \rho(z)\mathrm{d}z \to 0,
\end{aligned}$$

as $N \to \infty$. Since g is bounded and continuous function on $\mathbb{R}$, by the dominated convergence theorem, $F_N(z) := \int_{|x|\leq L} |g(x - z/N) - g(x)|\mathrm{d}x \to 0$ and $|F_N(z)| \leq 4KL$. Therefore, since $\int_{\mathbb{R}} \rho(z)\mathrm{d}z = 1$, from Fubini's theorem and the dominated convergence theorem, we have

$$J_2 \leq \int_{\mathbb{R}} \mathrm{d}z \int_{|x|\leq L} \mathrm{d}x \left|g\left(x - \frac{z}{N}\right) - g(x)\right| \rho(z) \to 0,$$

as $N \to \infty$. Thus $(g_N)_{N\in\mathbb{N}}$ satisfies (i).

Since $(\zeta_N)_{N\in\mathbb{N}}$ satisfies $\mathcal{A}$(iii), there exists K such that for any $a \in \mathbb{R}$ and $u > 0$,

$$\int_{\mathbb{R}} |\zeta_N'(x+a)|\, e^{-\frac{x^2}{u}} \mathrm{d}x < K(1+\sqrt{u}).$$

Therefore by Fubini's theorem, we have

$$\int_{\mathbb{R}} |g_N'(x+a)|\, e^{-\frac{x^2}{u}} \mathrm{d}x \leq \int_{\mathbb{R}} \mathrm{d}z \rho(z) \int_{\mathbb{R}} \mathrm{d}x \left|\zeta_N'\left(x - \frac{z}{N}\right)\right| e^{-\frac{x^2}{u}} \leq K(1+\sqrt{u}).$$

This concludes $(g_N)_{N\in\mathbb{N}}$ satisfies (iii). ☐

To prove Proposition 2.4.3, we introduce the following lemma.

Lemma 2.4.6 ([3], Proposition 3.6 and Theorem 3.9)**.** *Let $\Omega_0 \subset \mathbb{R}$ be a open set and $h \in L^1(\Omega_0)$. Then $h : \Omega_0 \to \mathbb{R}$ is a bounded variation function (in this case this means that $\int_{\Omega_0} |h'(x)|\mathrm{d}x < \infty$, see p.119) if and only if there exists a sequence $(h_N)_{N\in\mathbb{N}} \subset C^\infty(\Omega_0)$ converging to h in $L^1(\Omega_0)$ and satisfying $\lim_{N\to\infty} \int_{\Omega_0} |h'_N(x)|\mathrm{d}x < \infty$.*

Proof of Proposition 2.4.3. Assume that $f \in \mathcal{A}$. Let $(f_N)_{N\in\mathbb{N}}$ be a $\mathcal{A}$-approximation sequence of f. It follows from Lemma 2.4.4 that we can suppose $(f_N)_{N\in\mathbb{N}} \subset C^\infty(\mathbb{R};\mathbb{R})$. We choose $\Omega_0 := (0,1)$, $h_N := f_N|_{\Omega_0}$ and $h := f|_{\Omega_0}$. Since $(f_N)_{N\in\mathbb{N}}$ satisfies $\mathcal{A}$(i), $(h_N)_{N\in\mathbb{N}}$ converges to h in $L^1(\Omega_0)$.

Since $(f_N)_{N\in\mathbb{N}}$ satisfies $\mathcal{A}$(iii), by choosing $a = 0$ and $u = 1$, we have

$$2K = K(1+\sqrt{1}) \geq \sup_{N\in\mathbb{N}} \int_{\mathbb{R}} |f'_N(x)|e^{-x^2}\mathrm{d}x \geq \sup_{N\in\mathbb{N}} \int_{\Omega_0} |h'_N(x)|e^{-x^2}\mathrm{d}x \geq e^{-1} \sup_{N\in\mathbb{N}} \int_{\Omega_0} |h'_N(x)|\mathrm{d}x.$$

It means

$$\sup_{N\in\mathbb{N}} \int_{\Omega_0} |h'_N(x)|\mathrm{d}x < \infty.$$

Hence we can choose an increasing subsequence $(N_k)_{k\in\mathbb{N}}$ such that there exists the limit

$$\lim_{N_k\to\infty} \int_{\Omega_0} |h'_{N_k}(x)|\mathrm{d}x < \infty.$$

Thanks to Lemma 2.4.6, this convergence together with the fact that $h_{N_k} \xrightarrow{L^1(\Omega_0)} h$ implies h has a bounded variation on Ω_0. However, since

$$\int_{\Omega_0} |h'(x)|\mathrm{d}x = \infty,$$

the variation of h on Ω_0 is infinite. This contradiction concludes that $f \notin \mathcal{A}$. □

Class of irregular functions $\mathcal{L}$

Next, we define the class of one-sided Lipschitz functions.

Definition 2.4.7. *A function $f : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is called* one-sided Lipschitz function *in space if there exists a positive constant K such that for any $(t,x,y) \in [0,T]\times\mathbb{R}^{2d}$,*

$$\langle x-y, f(t,x)-f(t,y)\rangle_{\mathbb{R}^d} \leq K|x-y|^2. \tag{2.19}$$

Let $\mathcal{L}$ be the class of all one-sided Lipschitz functions.

Remark 2.4.8. By the definition of the class $\mathcal{L}$, if $f,g \in \mathcal{L}$ and $\alpha \geq 0$, then $f+g$, $\alpha f \in \mathcal{L}$. The one-sided Lipschitz property is closely related to the monotonicity condition introduced in [38] and the class $\mathcal{L}$ obviously contains all functions which are the sum of a Lipschitz function and a monotone decreasing γ-Hölder continuous function considered in [40]. Another example of one-sided Lipschitz function is $f(x) = \sum_{k\in\mathbb{Z}}(x-k)\mathbf{1}_{[k,k+1)}(x)$.

Many properties and applications of SDEs with one-sided Lipschitz drift can be found in [107].

Proposition 2.4.9. *Let $g : [0,t] \times \mathbb{R} \to \mathbb{R}$ be a bounded and measurable function. Assume that for any $x \in \mathbb{R}$, $g(\cdot, x)$ is continuous; for any $t \in [0,T]$, the number of discontinuous points of $g(t,\cdot)$ is countable and $g \in \mathcal{L}$. Then $g \in \mathcal{A}$.*

Proof. Let $(g_N)_{N\in\mathbb{N}}$ be defined as in (ii) of the previous section. Since $g \in \mathcal{L}$, there exists a positive constant K such that for any $(t,x,y) \in [0,T] \times \mathbb{R}^2$,

$$(x-y)(g(t,x) - g(t,y)) \le K|x-y|^2.$$

Then g_N is also a one-sided Lipschitz function. Indeed,

$$\begin{aligned}(x-y)(g_N(t,x) - g_N(t,y)) &= \int_{\mathbb{R}} (x-y)(g(t,x-z) - g(t,y-z))\rho_N(z)\mathrm{d}z \\ = \int_{\mathbb{R}} \{(x-z)-(y-z)\}&(g(t,x-z) - g(t,y-z))\rho_N(z)\mathrm{d}z \le K|x-y|^2.\end{aligned}$$

Since for any $t \in [0,T]$ and $L > 0$,

$$\int_{|x|\le L} |g_N(t,x) - g(t,x)|\mathrm{d}x \le \int_{\mathbb{R}} \rho(z) \int_{|x|\le L} \left| g\left(t, x - \frac{z}{N}\right) - g(t,x) \right| \mathrm{d}x\mathrm{d}z$$

and $\sup_{t\in[0,T]} |g(t, x - z/N) - g(t,x)| \to 0$ almost every x, from the dominated convergence theorem, we have

$$\sup_{t\in[0,T]} \int_{|x|\le L} |g_N(t,x) - g(t,x)|\mathrm{d}x \to 0.$$

Thus $(g_N)_{N\in\mathbb{N}}$ satisfies $\mathcal{A}(i)$.

Let $a \in \mathbb{R}$, $u > 0$ and $t \in [0,T]$. Since $\mathbb{R} = \bigcup_{\ell=-\infty}^{\infty} [\ell\sqrt{u}, (\ell+1)\sqrt{u})$, we have

$$\begin{aligned}\int_{\mathbb{R}} |g_N'(t,x+a)| e^{-\frac{x^2}{u}} \mathrm{d}x &= \sum_{\ell=-\infty}^{\infty} \int_{\ell\sqrt{u}}^{(\ell+1)\sqrt{u}} |g_N'(t,x+a)| e^{-\frac{x^2}{u}} \mathrm{d}x \\ &\le \sum_{\ell=-\infty}^{\infty} e^{-\ell^2} \int_{\ell\sqrt{u}}^{(\ell+1)\sqrt{u}} |g_N'(t,x+a)|\mathrm{d}x.\end{aligned}$$

We prove that for any $\ell \in \mathbb{Z}$,

$$I_\ell(a,u) := \int_{\ell\sqrt{u}}^{(\ell+1)\sqrt{u}} |g_N'(t,x+a)|\mathrm{d}x \le 2(\|g\|_\infty + K\sqrt{u}).$$

We fix $t \in [0,T]$ and write $[\ell\sqrt{u}, (\ell+1)\sqrt{u}) = \bigcup_{i\in\mathbb{N}} [x_{2i}, x_{2i+1}) \cup \bigcup_{j\in\mathbb{N}} [y_{2j}, y_{2j+1})$ where

$$g_N'(t,x+a) \ge 0 \textit{ if } x+a \in [x_{2i}, x_{2i+1}) \text{ and } g_N'(t,x+a) < 0 \textit{ if } x+a \in [y_{2j}, y_{2j+1}).$$

Then we have

$$I_\ell(a,u) = \sum_{i\in\mathbb{N}} (g_N(t, x_{2i+1}+a) - g_N(t,x_{2i}+a)) + \sum_{j\in\mathbb{N}} (g_N(t,y_{2j}+a)) - g_N(t, y_{2j+1}+a))$$

$$=: I_\ell^1(a,u) + I_\ell^2(a,u).$$

Since g_N is a one-sided Lipschitz function, we have

$$((x_{2i+1}+a)-(x_{2i}+a))(g_N(t,x_{2i+1}+a)-g_N(t,x_{2i}+a)) \le K|x_{2i+1}-x_{2i}|^2.$$

Therefore, we have $(g_N(t,x_{2i+1}+a)-g_N(t,x_{2i}+a)) \le K(x_{2i+1}-x_{2i})$. Hence we obtain

$$I_\ell^1(a,u) = \sum_{i\in\mathbb{N}}(g_N(t,x_{2i+1}+a)-g_N(t,x_{2i}+a)) \le K\sqrt{u}.$$

Finally, we estimate $I_\ell^2(a,u)$. Since g_N is bounded, $|g_N(t,(\ell+1)\sqrt{u}+a)-g_N(t,\ell\sqrt{u}+a)| \le 2||g||_\infty$. Hence, we have

$$\begin{aligned} I_\ell^2(a,u) &= \sum_{i\in\mathbb{N}}(g_N(t,x_{2i+1}+a)) - g_N(t,x_{2i}+a)) - (g_N(t,(\ell+1)\sqrt{u}+a)-g_N(t,\ell\sqrt{u}+a)) \\ &\le K\sqrt{u} + 2||g||_\infty. \end{aligned}$$

Therefore, $I_\ell(a,u) \le 2(||g||_\infty + K\sqrt{u})$. This concludes $(g_N)_{N\in\mathbb{N}}$ satisfying $\mathcal{A}$(iii). □

Key lemma

From now, we derive a key estimation (Lemma 2.4.12) for proving the rate of convergence for the Euler-Maruyama scheme with irregular coefficients.

Lemma 2.4.10. *Let $\zeta : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function and $(\zeta_N)_{N\in\mathbb{N}}$ be a sequence of functions satisfying $\mathcal{A}$(i) and $\mathcal{A}$(ii) for ζ. Let $(Y_t)_{0\le t\le T}$ be a d-dimensional stochastic process with $Y_0 = y_0 \in \mathbb{R}^d$. Suppose that Y_t satisfies the Gaussian bound condition on $[\kappa, T]$ for some $\kappa \in (0,T]$, i.e., there exist positive constants C_1 and c_1 such that*

$$p_t(y) \le C_1 \frac{e^{\frac{-c_1|y-y_0|^2}{t}}}{t^{d/2}}, \quad t\in[\kappa,T], \tag{2.20}$$

where p_t is the density function of Y_t. Then

$$\int_\kappa^T \mathbb{E}\left[|\zeta_N(t,Y_t)-\zeta(t,Y_t)|\right]\mathrm{d}t \to 0,\ N\to\infty \tag{2.21}$$

and if $T/n \ge \kappa$

$$\int_{\frac{T}{n}}^T \mathbb{E}\left[\left|\zeta_N(t,Y_{\eta_n(t)})-\zeta(t,Y_{\eta_n(t)})\right|\right]\mathrm{d}t \to 0,\ N\to\infty. \tag{2.22}$$

Proof. For given $\varepsilon > 0$, there exists $M \equiv M(\varepsilon, y_0, c_1) > 0$ such that for any $|y| \ge M$,

$$e^{-\frac{c_1|y-y_0|^2}{2t}} \le e^{-\frac{c_1|y-y_0|^2}{2T}} < \varepsilon. \tag{2.23}$$

From $\mathcal{A}$(i), there exists $N' \equiv N'(\varepsilon)$ such that for any $N \geq N'$,

$$\int_{|y|<M} |\zeta_N(t,y) - \zeta(t,y)|\mathrm{d}y < \varepsilon.$$

Therefore for any $N \geq N'$, using the Gaussian bound condition (2.20), the uniform boundedness of ζ_N and ζ, and (2.23), we get

$$\begin{aligned}
&\int_\kappa^T |\mathbb{E}[\zeta_N(t,Y_t) - \zeta(t,Y_t)]|\mathrm{d}t \leq \int_\kappa^T \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}y|\zeta_N(t,y) - \zeta(t,y)|p_t(y) \\
&\leq C_1 \int_\kappa^T \mathrm{d}t \left(\int_{|y|<M} \mathrm{d}y + \int_{|y|\geq M} \mathrm{d}y \right) |\zeta_N(t,y) - \zeta(t,y)| \frac{e^{\frac{-c_1|y-y_0|^2}{t}}}{t^{d/2}} \\
&\leq C_1 \int_\kappa^T \frac{1}{t^{d/2}}\mathrm{d}t \int_{|y|<M} \mathrm{d}y|\zeta_N(t,y) - \zeta(t,y)| + C_1 \int_\kappa^T \mathrm{d}t \int_{|y|\geq M} \mathrm{d}y \frac{e^{\frac{-c_1|y-y_0|^2}{t}}}{t^{d/2}} \\
&\leq C_1 T \frac{1}{\kappa^{d/2}}\varepsilon + C_1 T\varepsilon \int_{\mathbb{R}^d} \frac{e^{\frac{-c|y-y_0|^2}{2t}}}{t^{d/2}}\mathrm{d}y \leq C_{T,\kappa}\varepsilon,
\end{aligned}$$

where the constant $C_{T,\kappa}$ depends only on T and κ. Hence by letting ε go to 0, we conclude (2.21). In the same way, we can show (2.22). $\square$

Corollary 2.4.11. *Let (UE) and (SB) hold, and let $\zeta \in \mathcal{A}$ with an $\mathcal{A}$-approximation sequence $(\zeta_N)_{N\in\mathbb{N}}$. Then for any $n \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \int_{\frac{T}{n}}^T \mathbb{E}[|\zeta_N(s, X_s^{(n)}) - \zeta_N(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s = \int_{\frac{T}{n}}^T \mathbb{E}[|\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s < +\infty.$$

Proof. It follows from Lemma 2.2.5 that the densities of $X_s^{(n)}$ and $X_{\eta_n(s)}^{(n)}$ satisfy the Gaussian bound condition for $s \geq \frac{T}{n}$. Hence using Lemma 2.4.10 with $\kappa = T/n$ and the simple inequality $||a-b|-|a'-b'|| \leq |a-a'| + |b-b'|$, we have

$$\begin{aligned}
&\left| \int_{\frac{T}{n}}^T \mathbb{E}[|\zeta_N(s, X_s^{(n)}) - \zeta_N(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s - \int_{\frac{T}{n}}^T \mathbb{E}[|\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s \right| \\
&\leq \int_{\frac{T}{n}}^T \mathbb{E}\left[\left| |\zeta_N(s, X_s^{(n)}) - \zeta_N(s, X_{\eta_n(s)}^{(n)})| - |\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})| \right| \right] \mathrm{d}s \\
&\leq \int_{\frac{T}{n}}^T \mathbb{E}\left[|\zeta_N(s, X_s^{(n)}) - \zeta(s, X_s^{(n)})| \right] \mathrm{d}s + \int_{\frac{T}{n}}^T \mathbb{E}\left[|\zeta_N(s, X_{\eta_n(s))}^{(n)}) - \zeta(s, X_{\eta_n(s))}^{(n)})| \right] \mathrm{d}s \\
&\to 0, \text{ as } N \to \infty,
\end{aligned}$$

which implies the desired result. $\square$

The above corollary is useful for proving the following key estimate.

Lemma 2.4.12. *Let $\zeta \in \mathcal{A}$. Under Assumption (UE) and (SB) for the coefficients b and σ, for any $q \geq 1$, there exists $C \equiv C(K, T, \lambda_0, x_0, d, q)$ such that*

$$\int_0^T \mathbb{E}[|\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})|^q]\mathrm{d}s \leq \frac{C}{\sqrt{n}}. \tag{2.24}$$

In particular, if $b^{(i)} \in \mathcal{A}$ for each $i = 1, \ldots, d$, then it holds that

$$\sum_{i=1}^d \int_0^T \mathbb{E}[|b^{(i)}(s, X_s^{(n)}) - b^{(i)}(s, X_{\eta_n(s)}^{(n)})|^q]\mathrm{d}s \leq \frac{C}{\sqrt{n}}. \tag{2.25}$$

Remark 2.4.13. We note that if $d = 1$ then, for $\zeta : \mathbb{R} \to \mathbb{R}$ with $V(\zeta) < \infty$ (see Definition 5.2.14), by using Lemma 5.2.17 (Theorem 4.3 in [8]), Corollary 2.2.6 and Lemma 2.2.2, for every $p \geq 1$, we have

$$\begin{aligned}
\int_0^T \mathbb{E}[|\zeta(X_s^{(n)}) - \zeta(X_{\eta_n(s)}^{(n)})|^q]\mathrm{d}s &\leq 3^{q+1} V(\zeta)^q \int_0^T \left(\sup_{x \in \mathbb{R}} p_s^{(n)}(x)\right)^{\frac{p}{p+1}} \mathbb{E}[|X_s^{(n)} - X_{\eta_n(s)}^{(n)}|^p]^{1/(p+1)}\mathrm{d}s \\
&\leq \frac{C}{n^{\frac{p}{2(p+1)}}} \int_0^T \frac{1}{s^{\frac{p}{2(p+1)}}}\mathrm{d}s = \frac{2(p+2)T^{\frac{p+2}{2(p+1)}}}{p+2} \frac{C}{n^{\frac{p}{2(p+1)}}},
\end{aligned}$$

for some $C > 0$. Thus for any $\varepsilon \in (0, 1/6]$, we choose $p = \frac{1-2\varepsilon}{4\varepsilon} \geq 1$, it holds that

$$\int_0^T \mathbb{E}[|\zeta(X_s^{(n)}) - \zeta(X_{\eta_n(s)}^{(n)})|^q]\mathrm{d}s \leq \frac{C_\varepsilon}{n^{\frac{1}{2}-\varepsilon}},$$

for some $C_\varepsilon > 0$. This approach can be used for one-dimensional setting and the rate of convergence is almost 1/2 but not 1/2.

Proof of Lemma 2.4.12. Since ζ is bounded, it is sufficient to prove (2.24) for $q = 1$. Let $(b_N^{(i)})$ be an $\mathcal{A}$-approximation sequence of ζ. From Corollary 2.4.11, we have

$$\begin{aligned}
&\int_0^T \mathbb{E}[|\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s \\
&= \int_0^{\frac{T}{n}} \mathbb{E}[|\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s + \int_{\frac{T}{n}}^T \mathbb{E}[|\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s \\
&\leq \frac{C_1}{n} + \lim_{N \to \infty} \int_{\frac{T}{n}}^T \mathbb{E}[|\zeta_N(s, X_s^{(n)}) - \zeta_N(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s.
\end{aligned} \tag{2.26}$$

So we estimate the second part of (2.26). Since $W_s - W_{\eta_n(s)}$ and $X_{\eta_n(s)}^{(n)}$ are independent, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\zeta_N(s, X_s^{(n)}) - \zeta_N(s, X_{\eta_n(s)}^{(n)})\right|\right] \\
&= \mathbb{E}\left[\left|\zeta_N\left(s, X_{\eta_n(s)}^{(n)} + (s - \eta_n(s))b(\eta_n(s), X_{\eta_n(s)}^{(n)}) + \sigma(\eta_n(s), X_{\eta_n(s)}^{(n)})(W_s - W_{\eta_n(s)})\right) - \zeta_N(s, X_{\eta_n(s)}^{(n)})\right|\right] \\
&= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \left|\zeta_N\left(s, x + (s - \eta_n(s))b(\eta_n(s), x) + \sigma(\eta_n(s), x)y\right) - \zeta_N(s, x)\right|
\end{aligned}$$

$$\times\, p^{(n)}_{\eta_n(s)}(x) \left(\frac{1}{2\pi(s-\eta_n(s))}\right)^{d/2} \exp\left(-\frac{|y|^2}{2(s-\eta_n(s))}\right). \tag{2.27}$$

From the Gaussian bound condition for $p^{(n)}_{\eta_n(s)}$, there exists positive constants $C_2 \geq 1$ and $c_2 > 0$ such that the last term of (2.27) is less than

$$\begin{aligned}
&C_2 \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \left|\zeta_N\left(s, x + (s-\eta_n(s))b(\eta_n(s),x) + \sigma(\eta_n(s),x)y\right) - \zeta_N(s,x)\right| \\
&\times \left(\frac{1}{\eta_n(s)}\right)^{d/2} \exp\left(-c_2\frac{|x-x_0|^2}{2\eta_n(s)}\right) \left(\frac{1}{s-\eta_n(s)}\right)^{d/2} \exp\left(-\frac{|y|^2}{2(s-\eta_n(s))}\right).
\end{aligned} \tag{2.28}$$

Applying the change of variables $z = (s-\eta_n(s))b(\eta_n(s),x) + \sigma(\eta_n(s),x)y$, (2.28) is bounded by

$$\begin{aligned}
&C_2 \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}z \frac{|\zeta_N\left(s, x+z\right) - \zeta_N(s,x)|}{|\det(\sigma(\eta_n(s),x))|} \left(\frac{1}{\eta_n(s)}\right)^{d/2} \exp\left(-c_2\frac{|x-x_0|^2}{2\eta_n(s)}\right) \\
&\times \left(\frac{1}{s-\eta_n(s)}\right)^{d/2} \exp\left(-\frac{|\sigma^{-1}(\eta_n(s),x)(z-(s-\eta_n(s))b(\eta_n(s),x))|^2}{2(s-\eta_n(s))}\right).
\end{aligned} \tag{2.29}$$

Since a^{-1} is uniformly elliptic,

$$\begin{aligned}
&|\sigma^{-1}(\eta_n(s),x)(z-(s-\eta_n(s))b(\eta_n(s),x))|^2 \\
&= \langle a^{-1}(\eta_n(s),x)(z-(s-\eta_n(s))b(\eta_n(s),x)), z-(s-\eta_n(s))b(\eta_n(s),x)\rangle_{\mathbb{R}^d} \\
&\geq \lambda_0^{-1}|z-(s-\eta_n(s))b(\eta_n(s),x)|^2.
\end{aligned}$$

By the inequality $|x-y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$ for any $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned}
&-\frac{|\sigma^{-1}(\eta_n(s),x)(z-(s-\eta_n(s))b(\eta_n(s),x))|^2}{2(s-\eta_n(s))} \\
&\leq -\frac{\lambda_0^{-1}|z|^2}{4(s-\eta_n(s))} + \frac{\lambda_0^{-1}(s-\eta_n(s))^2|b(\eta_n(s),x)|^2}{2(s-\eta_n(s))} \leq -c_3\frac{|z|^2}{2(s-\eta_n(s))} + C_3.
\end{aligned}$$

Using this estimate and Fubini's theorem, (2.29) is less than

$$C_4 \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} \mathrm{d}x \left|\zeta_N\left(s, x+z\right) - \zeta_N(s,x)\right| \frac{\exp\left(-c_4\frac{|x-x_0|^2}{2\eta_n(s)}\right)}{(\eta_n(s))^{d/2}} \frac{\exp\left(-c_4\frac{|z|^2}{2(s-\eta_n(s))}\right)}{(s-\eta_n(s))^{d/2}}. \tag{2.30}$$

Since $\zeta_N\left(s, x+z\right) - \zeta_N(s,x) = \int_0^1 \langle z, \nabla\zeta_N(s, x+\theta z)\rangle_{\mathbb{R}^d} \mathrm{d}\theta$, (2.30) is less than

$$\begin{aligned}
&C_4 \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} \mathrm{d}x \int_0^1 \mathrm{d}\theta \left|\langle z, \nabla\zeta_N(s, x+\theta z)\rangle_{\mathbb{R}^d}\right| \frac{\exp\left(-c_4\frac{|x-x_0|^2}{2\eta_n(s)}\right)}{(\eta_n(s))^{d/2}} \frac{\exp\left(-c_4\frac{|z|^2}{2(s-\eta_n(s))}\right)}{(s-\eta_n(s))^{d/2}} \\
&\leq C_4 \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} \mathrm{d}x \int_0^1 \mathrm{d}\theta |z||\nabla\zeta_N(s, x+\theta z)| \frac{\exp\left(-c_4\frac{|x-x_0|^2}{2\eta_n(s)}\right)}{(\eta_n(s))^{d/2}} \frac{\exp\left(-c_4\frac{|z|^2}{2(s-\eta_n(s))}\right)}{(s-\eta_n(s))^{d/2}}
\end{aligned}$$

$$\leq C_4 \sum_{j=1}^{d} \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} \mathrm{d}y \int_0^1 \mathrm{d}\theta |z||\partial_j \zeta_N(s, y + x_0 + \theta z)| \frac{\exp\left(-c_4 \frac{|y|^2}{2\eta_n(s)}\right)}{(\eta_n(s))^{d/2}} \frac{\exp\left(-c_4 \frac{|z|^2}{2(s-\eta_n(s))}\right)}{(s - \eta_n(s))^{d/2}}, \tag{2.31}$$

where we use the change of variable $y = x - x_0$ in the last inequality. It follows from Fubini's theorem and condition $\mathcal{A}$(iii) that (2.31) is bounded by

$$\begin{aligned}
&\frac{C_5}{\sqrt{\eta_n(s)}} \int_{\mathbb{R}^d} \mathrm{d}z \int_0^1 \mathrm{d}\theta |z| \frac{1}{(s - \eta_n(s))^{d/2}} \exp\left(-c_4 \frac{|z|^2}{2(s - \eta_n(s))}\right) \\
&= \frac{C_5}{\sqrt{\eta_n(s)}} \int_{\mathbb{R}^d} \frac{|z|}{(s - \eta_n(s))^{d/2}} \exp\left(-\frac{c_4}{2} \frac{|z|^2}{2(s - \eta_n(s))}\right) \exp\left(-\frac{c_4}{2} \frac{|z|^2}{2(s - \eta_n(s))}\right) \mathrm{d}z.
\end{aligned} \tag{2.32}$$

Since $|z| \exp\left(-\frac{c_4}{2} \frac{|z|^2}{2(s-\eta_n(s))}\right) \leq \sqrt{\frac{2}{ec_4}} \sqrt{s - \eta_n(s)}$ for any $z \in \mathbb{R}^d$, (2.32) is less than

$$C_6 \sqrt{\frac{s - \eta_n(s)}{\eta_n(s)}} \leq \frac{C_7}{\sqrt{n\eta_n(s)}}.$$

Therefore we have

$$\int_0^T \mathbb{E}[|\zeta(s, X_s^{(n)}) - \zeta(s, X_{\eta_n(s)}^{(n)})|]\mathrm{d}s \leq \frac{C_1}{n} + \frac{C_7}{\sqrt{n}} \int_{\frac{T}{n}}^T \frac{1}{\sqrt{\eta_n(s)}} \mathrm{d}s \leq \frac{C_1}{n} + \frac{C_7}{\sqrt{n}} \int_0^T \frac{1}{\sqrt{s}} \mathrm{d}s \leq \frac{C_8}{\sqrt{n}},$$

which concludes the proof of Lemma 2.4.12. □

If the drift coefficient $b^{(i)} : \mathbb{R}^d \to \mathbb{R}$ are Lipschitz continuous for each $i = 1, \ldots, d$, then using Lemma 2.2.2, for any $q \geq 1$, we have

$$\sum_{i=1}^{d} \int_0^T \mathbb{E}[|b^{(i)}(s, X_s^{(n)}) - b^{(i)}(s, X_{\eta_n(s)}^{(n)})|^q]\mathrm{d}s \leq \frac{C}{n^{q/2}}.$$

Thus, the convergence rate depends on q. However, the following proposition shows that if the drift coefficient is discontinuous, then the bound (2.25) is tight.

Proposition 2.4.14. *Let $d = 1$, $x_0 = 0$, $\sigma = 1$ and $b(x) = \mathbf{1}_{(-\infty,0]}(x) - \mathbf{1}_{(0,+\infty)}(x)$. Then, there exists $C > 0$ such that for any $n \geq \max\{T, 2\}$,*

$$\int_0^T \mathbb{E}[|b(X_s^{(n)}) - b(X_{\eta_n(s)}^{(n)})|^q]\mathrm{d}s \geq \frac{C}{\sqrt{n}}. \tag{2.33}$$

Proof. For any $s \geq T/n$, since $X_{\eta_n(s)}^{(n)}$ and $W_s - W_{\eta_n(s)}$ are independent,

$$\begin{aligned}
&\mathbb{E}[|b(X_s^{(n)}) - b(X_{\eta_n(s)}^{(n)})|^q] = 2^{q-1}\mathbb{E}[|b(X_s^{(n)}) - b(X_{\eta_n(s)}^{(n)})|] \\
&= 2^q \mathbb{P}\left(\left(X_{\eta_n(s)}^{(n)} + (s - \eta_n(x))\, b(X_{\eta_n(s)}^{(n)}) + W_s - W_{\eta_n(s)}\right) X_{\eta_n(s)}^{(n)} \leq 0\right) \\
&= 2^q \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y \mathbf{1}\left(x^2 + (s - \eta_n(s))b(x)x + yx \leq 0\right) p_{\eta_n(s)}^{(n)}(x) \frac{e^{-\frac{y^2}{2(s-\eta_n(s))}}}{\sqrt{2\pi(s - \eta_n(s))}}
\end{aligned}$$

$$\geq \int_0^\infty \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y \mathbf{1}(x - (s - \eta_n(s)) + y \leq 0) p^{(n)}_{\eta_n(s)}(x) \frac{e^{-\frac{y^2}{2(s-\eta_n(s))}}}{\sqrt{2\pi(s-\eta_n(s))}}.$$

Let $\Phi(u) := \int_{-\infty}^u \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} dv$. Then by the change of variable $z = y/\sqrt{s-\eta_n(s)}$, we have

$$\mathbb{E}[|b(X_s^{(n)}) - b(X^{(n)}_{\eta_n(s)})|^q] \geq \int_0^\infty p^{(n)}_{\eta_n(s)}(x) \Phi\left(-\frac{x - (s-\eta_n(s))}{\sqrt{s-\eta_n(s)}}\right) \mathrm{d}x.$$

Recall that $x_0 = 0$. It follows from the lower bound of (2.6) that there exist constants C_1 and c_1 such that

$$\begin{aligned}
\mathbb{E}[|b(X_s^{(n)}) - b(X^{(n)}_{\eta_n(s)})|^q] &\geq \frac{1}{C_1} \int_0^\infty \frac{e^{-\frac{x^2}{2c_1\eta_n(s)}}}{\sqrt{2\pi c_1 \eta_n(s)}} \Phi\left(-\frac{x - (s-\eta_n(s))}{\sqrt{s-\eta_n(s)}}\right) \mathrm{d}x \\
&\geq \frac{1}{C_1} \int_0^{\sqrt{s-\eta_n(s)}} \frac{e^{-\frac{x^2}{2c_1\eta_n(s)}}}{\sqrt{2\pi c_1 \eta_n(s)}} \Phi\left(-\frac{x - (s-\eta_n(s))}{\sqrt{s-\eta_n(s)}}\right) \mathrm{d}x \\
&\geq \frac{1}{C_1} \int_0^{\sqrt{s-\eta_n(s)}} \frac{e^{-\frac{s-\eta_n(s)}{2c_1\eta_n(s)}}}{\sqrt{2\pi c_1 \eta_n(s)}} \Phi\left(-\frac{\sqrt{s-\eta_n(s)} - (s-\eta_n(s))}{\sqrt{s-\eta_n(s)}}\right) \mathrm{d}x \\
&= \frac{\sqrt{s-\eta_n(s)}}{C_1} \frac{e^{-\frac{s-\eta_n(s)}{2c_1\eta_n(s)}}}{\sqrt{2\pi c_1 \eta_n(s)}} \Phi\left(-\left(1 - \sqrt{s-\eta_n(s)}\right)\right).
\end{aligned}$$

Moreover, using the Komatsu's inequality (see [55] page 17 Problem 1),

$$\Phi(-|x|) \geq \frac{2e^{-\frac{x^2}{2}}}{\sqrt{2\pi}(|x| + \sqrt{x^2+4})},$$

we get for any $n \geq T$,

$$\mathbb{E}[|b(X_s^{(n)}) - b(X^{(n)}_{\eta_n(s)})|^q] \geq \frac{C_{T,c_1}}{\pi C_1 \sqrt{c_1}} \frac{\sqrt{s-\eta_n(s)}}{\sqrt{\eta_n(s)}}, \quad \text{for any } s \geq \frac{T}{n},$$

where the constant C_{T,c_1} is a constant depending only on T and c_1. Therefore, we have

$$\int_0^T \mathbb{E}[|b(X_s^{(n)}) - b(X^{(n)}_{\eta_n(s)})|^q] \mathrm{d}s \geq C_2 \int_{T/n}^T \frac{\sqrt{s-\eta_n(s)}}{\sqrt{\eta_n(s)}} \mathrm{d}s \geq \frac{C_3}{\sqrt{n}},$$

for $n \geq \max\{T, 2\}$. This concludes (2.33). □

2.4.2 Error estimates for the Euler-Maruyama approximation

We now provide our main statements of this chapter. We need to suppose the following assumptions for the coefficients b and σ.

Assumption 2.4.15. *We assume that the coefficients b and σ are measurable functions and satisfy the following conditions:*

(i) $b \in \mathcal{L}$ *and* $b^{(i)} \in \mathcal{A}$ *for any* $i = 1, \ldots, d$ *and there exists* $K > 0$ *such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |b(t,x)| \leq K.$$

(ii) $a = \sigma\sigma^*$ *is bounded and uniformly elliptic, i.e., there exists* $\lambda_0 \geq 1$ *such that for any* $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$,

$$\lambda_0^{-1}|\xi|^2 \leq \langle a(t,x)\xi, \xi\rangle_{\mathbb{R}^d} \leq \lambda_0|\xi|^2.$$

(iii) σ *is a* $(1/2 + \alpha)$*-Hölder continuous with* $\alpha \in [0, 1/2]$ *in space, i.e., there exists* $K > 0$ *such that*

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}^{2d}, x\neq y} \frac{|\sigma(t,x) - \sigma(t,y)|}{|x-y|^{1/2+\alpha}} \leq K.$$

Remark 2.4.16. Many functions satisfy Assumption 2.4.15 (i). For example, any non-increasing function or Lipschitz continuous function does. In particular, for $x \in \mathbb{R}$, the function $\mathbf{1}_{(-\infty,0]}(x) - \mathbf{1}_{(0,+\infty)}(x)$ satisfies Assumption 2.4.15 (i). This function is the optimal drift coefficient for some stochastic control problem (see [14] or [60] page 437). From Proposition 2.4.2 and Remark 2.4.8, we know that if f and g satisfy Assumption 2.4.15 (i) and $\alpha, \beta \geq 0$, then $\alpha f + \beta g$ also satisfies this condition. Assumption 2.4.15 (ii) implies that the diffusion coefficient σ is bounded i.e., for any $(t, x) \in [0, T] \times \mathbb{R}^d$, $|\sigma(t, x)| = \{\sum_{i,j} \sigma_{i,j}^2(t, x)\}^{1/2} \leq \sqrt{d\lambda_0}$.

We will assume the following Hölder continuity in time variable.

Assumption 2.4.17. *The coefficients b and σ are β-Hölder continuous with $\beta \geq 1/2$ in time i.e., there exist $K > 0$ such that for all $t, s \in [0, T]$ and $x \in \mathbb{R}^d$,*

$$|b(t,x) - b(s,x)| + |\sigma(t,x) - \sigma(s,x)| \leq K|t-s|^\beta.$$

We obtain the following results on the rate of the Euler-Maruyama approximation in L^1-norm.

Theorem 2.4.18. *Let Assumptions 2.4.15 and 2.4.17 hold. Then there exists a constant C which depends on $K, T, \lambda_0, x_0, d, \alpha$ and β such that for $d = 1$,*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \begin{cases} \dfrac{C}{\log n} & \text{if } \alpha = 0, \\ \dfrac{C}{n^\alpha} & \text{if } \alpha \in (0, 1/2], \end{cases} \tag{2.34}$$

and for $d \geq 2$,

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \frac{C}{n^{1/2}} \text{ if } \alpha = 1/2,$$

where $\mathcal{T}$ is the set of all stopping times $\tau \leq T$.

Proof. By using Itô's formula, (2.8) and (2.10) we have

$$|Y_t^{(n)}| \leq \varepsilon + \int_0^t I_s^{\delta,\varepsilon,n} \mathrm{d}s + M_t^{\delta,\varepsilon,n} + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_i \partial_j \Phi_{\delta,\varepsilon}(Y_s^{(n)}) \mathrm{d}\langle Y^{(n,i)}, Y^{(n,j)}\rangle_s, \tag{2.35}$$

where

$$I_s^{\delta,\varepsilon,n} := \sum_{i=1}^d \partial_i \Phi_{\delta,\varepsilon}(Y_s^{(n)}) \left\{ b^{(i)}(s, X_s) - b^{(i)}(\eta_n(s), X_{\eta_n(s)}^{(n)}) \right\}$$

and

$$M_t^{\delta,\varepsilon,n} := \sum_{k=1}^d \sum_{i=1}^d \int_0^t \partial_i \Phi_{\delta,\varepsilon}(Y_s^{(n)}) \left\{ \sigma_{i,k}(s, X_s) - \sigma_{i,k}(\eta_n(s), X_{\eta_n(s)}^{(n)}) \right\} \mathrm{d}W_s^k. \tag{2.36}$$

Since $\partial_i \Phi_{\delta,\varepsilon}$ and σ are bounded, $M_t^{\delta,\varepsilon,n}$ is a martingale. Therefore the expectation of $M_t^{\delta,\varepsilon,n}$ equals 0, so we only estimate the second and fourth part of (2.35). First we consider the second part. From Assumption 2.4.17, (2.8), (2.9) and partial differentiations of $\Phi_{\delta,\varepsilon}$, we have

$$\begin{aligned}
&\int_0^t I_s^{\delta,\varepsilon,n} \mathrm{d}s \\
&= \sum_{i=1}^d \int_0^t \phi'_{\delta\varepsilon}(|Y_s^{(n)}|) \frac{Y_s^{(n,i)}}{|Y_s^{(n)}|} \left\{ \left(b^{(i)}(s, X_s) - b^{(i)}(s, X_{\eta_n(s)}^{(n)}) \right) + \left(b^{(i)}(s, X_{\eta_n(s)}^{(n)}) - b^{(i)}(\eta_n(s), X_{\eta_n(s)}^{(n)}) \right) \right\} \mathrm{d}s \\
&\leq \sum_{i=1}^d \int_0^t \phi'_{\delta\varepsilon}(|Y_s^{(n)}|) \frac{Y_s^{(n,i)}}{|Y_s^{(n)}|} \left(b^{(i)}(s, X_s) - b^{(i)}(s, X_{\eta_n(s)}^{(n)}) \right) \mathrm{d}s + \frac{C_1}{n^\beta} \\
&= \int_0^t \sum_{i=1}^d \phi'_{\delta\varepsilon}(|Y_s^{(n)}|) \frac{Y_s^{(n,i)}}{|Y_s^{(n)}|} \left(b^{(i)}(s, X_s) - b^{(i)}(s, X_s^{(n)}) \right) \mathrm{d}s \\
&\quad + \int_0^t \sum_{i=1}^d \phi'_{\delta\varepsilon}(|Y_s^{(n)}|) \frac{Y_s^{(n,i)}}{|Y_s^{(n)}|} \left(b^{(i)}(s, X_s^{(n)}) - b^{(i)}(s, X_{\eta_n(s)}^{(n)}) \right) \mathrm{d}s + \frac{C_1}{n^\beta} \\
&\leq \int_0^t \langle X_s - X_s^{(n)}, b(s, X_s) - b(s, X_s^{(n)}) \rangle_{\mathbb{R}^d} \frac{\phi'_{\delta\varepsilon}(|Y_s^{(n)}|)}{|Y_s^{(n)}|} \mathrm{d}s \\
&\quad + \sum_{i=1}^d \int_0^t \left| b^{(i)}(s, X_s^{(n)}) - b^{(i)}(s, X_{\eta_n(s)}^{(n)}) \right| \mathrm{d}s + \frac{C_1}{n^\beta}.
\end{aligned}$$

By using the one-sided Lipschitz condition (2.19), we have

$$\int_0^t I_s^{\delta,\varepsilon,n} \mathrm{d}s \leq K \int_0^t |Y_s^{(n)}| \mathrm{d}s + \sum_{i=1}^d \int_0^T \left| b^{(i)}(s, X_s^{(n)}) - b^{(i)}(s, X_{\eta_n(s)}^{(n)}) \right| \mathrm{d}s + \frac{C_1}{n^\beta}. \tag{2.37}$$

Next we estimate the fourth part of (2.35). Using partial differentiations of $\Phi_{\delta,\varepsilon}$, the fourth part of (2.35) can be expressed by

$$\frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_i \partial_j \Phi_{\delta,\varepsilon}(Y_s^{(n)}) \mathrm{d}\langle Y^{(n,i)}, Y^{(n,j)}\rangle_s = A_t^{1,\delta,\varepsilon,n} + A_t^{2,\delta,\varepsilon,n},$$

where

$$A_t^{1,\delta,\varepsilon,n} := \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \phi''_{\delta,\varepsilon}(|Y_s^{(n)}|)\frac{Y_s^{(n,i)}Y_s^{(n,j)}}{|Y_s^{(n)}|^2}\mathrm{d}\langle Y^{(n,i)}, Y^{(n,j)}\rangle_s$$

and

$$\begin{aligned}A_t^{2,\delta,\varepsilon,n} :=& \frac{1}{2}\sum_{i=1}^{d}\int_0^t \phi'_{\delta,\varepsilon}(|Y_s^{(n)}|)\left(\frac{|Y_s^{(n)}|^2-|Y_s^{(n,i)}|^2}{|Y_s^{(n)}|^3}\right)\mathrm{d}\langle Y^{(n,i)}, Y^{(n,i)}\rangle_s\\ &+\sum_{1\leq i<j\leq d}\int_0^t\left\{-\phi'_{\delta,\varepsilon}(|Y_s^{(n)}|)\frac{Y_s^{(n,i)}Y_s^{(n,j)}}{|Y_s^{(n)}|^3}\right\}\mathrm{d}\langle Y^{(n,i)}, Y^{(n,j)}\rangle_s.\end{aligned}$$

Here we remark that $A_t^{2,\delta,\varepsilon,n}=0$ for $d=1$. So we should estimate $A_t^{1,\delta,\varepsilon,n}$ and $A_t^{2,\delta,\varepsilon,n}$. By the definition of quadratic variation of $Y_t^{(n)}$,

$$\begin{aligned}A_t^{1,\delta,\varepsilon,n} \leq& \frac{1}{2}\sum_{k=1}^{d}\sum_{i,j=1}^{d}\int_0^t \phi''_{\delta,\varepsilon}(|Y_s^{(n)}|)\frac{|Y_s^{(n,i)}||Y_s^{(n,j)}|}{|Y_s^{(n)}|^2}\left|\sigma_{i,k}(s,X_s)-\sigma_{i,k}(\eta_n(s),X_{\eta_n(s)}^{(n)})\right|\\ &\times\left|\sigma_{j,k}(s,X_s)-\sigma_{j,k}(\eta_n(s),X_{\eta_n(s)}^{(n)})\right|\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}A_t^{2,\delta,\varepsilon,n} \leq& \frac{1}{2}\sum_{k,i=1}^{d}\int_0^t \phi'_{\delta,\varepsilon}(|Y_s^{(n)}|)\left(\frac{|Y_s^{(n)}|^2-|Y_s^{(n,i)}|^2}{|Y_s^{(n)}|^3}\right)\left|\sigma_{i,k}(s,X_s)-\sigma_{i,k}(\eta_n(s),X_{\eta_n(s)}^{(n)})\right|^2\mathrm{d}s\\ &+\sum_{k=1}^{d}\sum_{1\leq i<j\leq d}\int_0^t \phi'_{\delta,\varepsilon}(|Y_s^{(n)}|)\frac{|Y_s^{(n,i)}||Y_s^{(n,j)}|}{|Y_s^{(n)}|^3}\left|\sigma_{i,k}(s,X_s)-\sigma_{i,k}(\eta_n(s),X_{\eta_n(s)}^{(n)})\right|\\ &\times\left|\sigma_{j,k}(s,X_s)-\sigma_{j,k}(\eta_n(s),X_{\eta_n(s)}^{(n)})\right|\mathrm{d}s.\end{aligned}$$

Since σ is $(1/2+\alpha)$-Hölder continuous in space and β-Hölder continuous in time, we have

$$\begin{aligned}A_t^{1,\delta,\varepsilon,n} &\leq C_2\sum_{i,j=1}^{d}\int_0^t \phi''_{\delta,\varepsilon}(|Y_s^{(n)}|)\frac{|Y_s^{(n,i)}||Y_s^{(n,j)}|}{|Y_s^{(n)}|^2}\left\{\left|X_s-X_{\eta_n(s)}^{(n)}\right|^{1+2\alpha}+\frac{1}{n^{2\beta}}\right\}\mathrm{d}s\\ &\leq C_3\int_0^t \phi''_{\delta,\varepsilon}(|Y_s^{(n)}|)\left\{\left|X_s-X_{\eta_n(s)}^{(n)}\right|^{1+2\alpha}+\frac{1}{n^{2\beta}}\right\}\mathrm{d}s\\ &\leq C_4\int_0^t \phi''_{\delta,\varepsilon}(|Y_s^{(n)}|)\left\{\left|Y_s^{(n)}\right|^{1+2\alpha}+\left|U_s^{(n)}\right|^{1+2\alpha}+\frac{1}{n^{2\beta}}\right\}\mathrm{d}s.\end{aligned}$$

Similarly, we obtain

$$A_t^{2,\delta,\varepsilon,n} \leq C_5\int_0^t \frac{\phi'_{\delta,\varepsilon}(|Y_s^{(n)}|)}{|Y_s^{(n)}|}\left\{\left|Y_s^{(n)}\right|^{1+2\alpha}+\left|U_s^{(n)}\right|^{1+2\alpha}+\frac{1}{n^{2\beta}}\right\}\mathrm{d}s.$$

It follows from (2.12) that

$$\begin{aligned} A_t^{1,\delta,\varepsilon,n} &\leq C_6 \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s^{(n)}|)}{|Y_s^{(n)}|\log\delta} \left\{ \left|Y_s^{(n)}\right|^{1+2\alpha} + \left|U_s^{(n)}\right|^{1+2\alpha} + \frac{1}{n^{2\beta}} \right\} \mathrm{d}s \\ &\leq \frac{C_6\varepsilon^{2\alpha}}{\log\delta} + \frac{C_6\delta}{\varepsilon\log\delta} \int_0^T |U_s^{(n)}|^{1+2\alpha} \mathrm{d}s + \frac{C_6\delta}{\varepsilon(\log\delta)n^{2\beta}}, \end{aligned} \tag{2.38}$$

and from (2.9) and (2.11) that

$$A_t^{2,\delta,\varepsilon,n} \leq C_5 \int_0^t \left|Y_s^{(n)}\right|^{2\alpha} \mathrm{d}s + \frac{C_5\delta}{\varepsilon} \int_0^T \left|U_s^{(n)}\right|^{1+2\alpha} \mathrm{d}s + \frac{C_5\delta}{\varepsilon n^{2\beta}}.$$

Let τ be a stopping time with $\tau \leq T$. Define $Z_t^{(n)} := |Y_{t\wedge\tau}^{(n)}|$ and for any $\alpha \in [0, 1/2]$,

$$R(\alpha,\delta,\varepsilon,n) := \varepsilon + \frac{C_6\varepsilon^{2\alpha}}{\log\delta} + \frac{C_6\delta}{\varepsilon\log\delta} \int_0^T \left|U_s^{(n)}\right|^{1+2\alpha} \mathrm{d}s + \frac{C_6\delta}{\varepsilon(\log\delta)n^{2\beta}},$$

and

$$S(\alpha,\delta,\varepsilon,n) := \frac{C_5\delta}{\varepsilon} \int_0^T \left|U_s^{(n)}\right|^{1+2\alpha} \mathrm{d}s + \frac{C_5\delta}{\varepsilon n^{2\beta}}.$$

Then we consider the following two cases.

Case 1 ($d \geq 2$ and $\alpha = 1/2$): In this case, gathering the above estimates, we have

$$\begin{aligned} Z_t^{(n)} \leq& C_7 \int_0^t Z_s^{(n)} \mathrm{d}s + \sum_{i=1}^d \int_0^T \left| b^{(i)}(s, X_s^{(n)}) - b^{(i)}(s, X_{\eta_n(s)}^{(n)}) \right| \mathrm{d}s \\ &+ \frac{C_7}{n^\beta} + M_{t\wedge\tau}^{\delta,\varepsilon,n} + R(1/2,\delta,\varepsilon,n) + S(1/2,\delta,\varepsilon,n). \end{aligned} \tag{2.39}$$

We choose $\delta = 2$ and $\varepsilon = n^{-1/2}$. Then for any $\alpha \in (0, 1/2]$, we obtain

$$R(\alpha, 2, n^{-1/2}, n) \leq \frac{C_8}{n^\alpha} + C_8\sqrt{n} \int_0^T |U_s^{(n)}|^{1+2\alpha} \mathrm{d}s + \frac{C_8}{n^{2\beta-1/2}},$$

and

$$S(\alpha, 2, n^{-1/2}, n) \leq C_9\sqrt{n} \int_0^T |U_s^{(n)}|^{1+2\alpha} \mathrm{d}s + \frac{C_9}{n^{2\beta-1/2}}.$$

Note that $2\beta - 1/2 \geq 1/2$. It follows from Lemma 2.2.2 with $q = 1 + 2\alpha$ that for any $\alpha \in (0, 1/2]$,

$$\mathbb{E}[R(\alpha, 2, n^{-1/2}, n)],\ \mathbb{E}[S(\alpha, 2, n^{-1/2}, n)] \leq \frac{C_{10}}{n^\alpha}. \tag{2.40}$$

Recall $\alpha = 1/2$. By using the above estimate and Lemma 2.4.12, we obtain

$$\mathbb{E}[Z_t^{(n)}] \leq C_7 \int_0^t \mathbb{E}[Z_s^{(n)}] \mathrm{d}s + \frac{C_{11}}{\sqrt{n}}.$$

By Gronwall's inequality, we have

$$\mathbb{E}[Z_t^{(n)}] \leq \frac{C_{12}}{\sqrt{n}}.$$

Therefore, from the dominated convergence theorem, we complete the statement taking $t \to T$.

Case 2 ($d = 1$): As remarked before, $A_t^{2,\delta,\varepsilon,n} = 0$. From (2.37) and (2.38), we have

$$Z_t^{(n)} \leq C_6 \int_0^t Z_s^{(n)} \mathrm{d}s + \int_0^T \left| b(s, X_s^{(n)}) - b(s, X_{\eta_n(s)}^{(n)}) \right| \mathrm{d}s + \frac{C_6}{n^\beta} + M_{t\wedge\tau}^{\delta,\varepsilon,n} + R(\alpha, \delta, \varepsilon, n). \tag{2.41}$$

For $\alpha \in (0, 1/2]$, we can prove the statement in (2.34) in the same way as Case 1 by taking $\delta = 2$ and $\varepsilon = n^{-1/2}$. For $\alpha = 0$, we choose $\delta = n^{1/3}$ and $\varepsilon = (\log n)^{-1}$. Then we have

$$R(0, n^{1/3}, (\log n)^{-1}, n) \leq \frac{C_{13}}{\log n} + C_{13} n^{1/3} \int_0^T |U_s^{(n)}| \mathrm{d}s + \frac{C_{13}}{n^{2\beta - 1/3}},$$

and so we get

$$\mathbb{E}[R(0, n^{1/3}, (\log n)^{-1}, n)] \leq \frac{C_{14}}{\log n}. \tag{2.42}$$

From Lemma 2.4.12, 2.2.2 and (2.42), we have

$$\mathbb{E}[Z_t^{(n)}] \leq C_6 \int_0^t \mathbb{E}[Z_s^{(n)}] \mathrm{d}s + \frac{C_{15}}{\log n}.$$

Hence by Gronwall's inequality we see that

$$\mathbb{E}[Z_t^{(n)}] \leq \frac{C_{16}}{\log n}.$$

Therefore, from the dominated convergence theorem, we obtain (2.34) for $\alpha = 0$ as taking $t \to T$. □

We obtain the following results on the rate of the Euler-Maruyama approximation in L^1-sup norm.

Theorem 2.4.19. *Under Assumptions 2.4.15 and 2.4.17, there exists a constant C which depends on $K, T, \lambda_0, x_0, d, \alpha$ and β such that for $d = 1$,*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t - X_t^{(n)}|\right] \leq \begin{cases} \dfrac{C}{\sqrt{\log n}} & \text{if } \alpha = 0, \\ \dfrac{C}{n^{2\alpha^2}} & \text{if } \alpha \in (0, 1/2], \end{cases} \tag{2.43}$$

and for $d \geq 2$,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t - X_t^{(n)}|\right] \leq \frac{C}{n^{1/2}} \text{ if } \alpha = 1/2.$$

Proof. Recalling (2.5), we define $V_t^{(n)} := \sup_{0\le s\le t}|Y_s^{(n)}|$. To estimate the expectation of $V_t^{(n)}$, we use (2.39) and therefore we need to calculate the expectation of $\sup_{0\le s\le t}|M_s^{\delta,\varepsilon,n}|$. We use the notation $\widetilde{C}_i$ for a positive constant instead of C_i. This constant $\widetilde{C}_i$ can depend on K, T, α and β while the constant C_i can be depend on $K, T, \lambda_0, x_0, \beta$ and d. For any $d \in \mathbb{N}$, by using (2.36) and Burkholder-Davis-Gundy's inequality we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left|M_s^{\delta,\varepsilon,n}\right|\right] &\le \widetilde{C}_1\mathbb{E}[\langle M^{\delta,\varepsilon,n}\rangle_t^{1/2}]\\ &= \widetilde{C}_1\mathbb{E}\left[\left(\sum_{k=1}^d\int_0^t\left|\sum_{i=1}^d\partial_i\Phi_{\delta,\varepsilon}(Y_s^{(n)})\left\{\sigma_{i,k}(s,X_s)-\sigma_{i,k}(\eta_n(s),X_{\eta_n(s)}^{(n)})\right\}\right|^2\mathrm{d}s\right)^{1/2}\right].\end{aligned}$$

Since $\partial_i\Phi_{\delta,\varepsilon}$, $(i = 1,\ldots,d)$ are bounded and σ is $1/2+\alpha$-Hölder continuous in space and β-Hölder continuous in time, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left|M_s^{\delta,\varepsilon,n}\right|\right] \le&\widetilde{C}_2\mathbb{E}\Bigg[\Big(\sum_{i,k=1}^d\int_0^t\Big\{\left|\sigma_{i,k}(s,X_s)-\sigma_{i,k}(s,X_s^{(n)})\right|^2+\left|\sigma_{i,k}(s,X_s^{(n)})-\sigma_{i,k}(s,X_{\eta_n(s)}^{(n)})\right|^2\\ &+\left|\sigma_{i,k}(s,X_{\eta_n(s)}^{(n)})-\sigma_{i,k}(\eta_n(s),X_{\eta_n(s)}^{(n)})\right|^2\Big\}\mathrm{d}s\Big)^{1/2}\Bigg]\\ =&\widetilde{C}_2\mathbb{E}\Bigg[\Big(\int_0^t\Big\{\left|\sigma(s,X_s)-\sigma(s,X_s^{(n)})\right|^2+\left|\sigma(s,X_s^{(n)})-\sigma(s,X_{\eta_n(s)}^{(n)})\right|^2\\ &+\left|\sigma(s,X_{\eta_n(s)}^{(n)})-\sigma(\eta_n(s),X_{\eta_n(s)}^{(n)})\right|^2\Big\}\mathrm{d}s\Big)^{1/2}\Bigg]\\ \le&\widetilde{C}_3\mathbb{E}\left[\left(\int_0^t\left\{|X_s-X_s^{(n)}|^{1+2\alpha}+|X_s^{(n)}-X_{\eta_n(s)}^{(n)}|^{1+2\alpha}+|s-\eta_n(s)|^{2\beta}\right\}\mathrm{d}s\right)^{1/2}\right]\\ \le&\widetilde{C_4}\left\{\mathbb{E}[A_t^{(n)}+B_t^{(n)}]+\frac{1}{n^\beta}\right\},\end{aligned}\tag{2.44}$$

where by the definition of $Y^{(n)}$ and $U^{(n)}$ given in (2.5),

$$A_t^{(n)} := \left(\int_0^t|Y_s^{(n)}|^{1+2\alpha}\mathrm{d}s\right)^{1/2} \text{ and } B_t^{(n)} := \left(\int_0^t|U_s^{(n)}|^{1+2\alpha}\mathrm{d}s\right)^{1/2}.$$

From Lemma 2.2.2 with $q = 1+2\alpha$ and using Jensen's inequality, we have

$$\mathbb{E}[B_t^{(n)}] \le \left(\int_0^T\mathbb{E}\left[|U_s^{(n)}|^{1+2\alpha}\right]\mathrm{d}s\right)^{1/2} \le \frac{C_1}{n^{1/4+\alpha/2}}.\tag{2.45}$$

Next we estimate $A_t^{(n)}$ and $V_t^{(n)} = \sup_{0\le s\le t}|Y_s^{(n)}|$ for the following two cases.

Case 1 ($d \ge 2$ and $\alpha = 1/2$): Since $|Y_s^{(n)}| \le V_t^{(n)}$ for all $s \le t$, we have

$$\mathbb{E}[A_t^{(n)}] = \mathbb{E}\left[\left(\int_0^t|Y_s^{(n)}|^{1+2\alpha}\mathrm{d}s\right)^{1/2}\right] \le \mathbb{E}\left[(V_t^{(n)})^{1/2}\left(\int_0^t|Y_s^{(n)}|^{2\alpha}\mathrm{d}s\right)^{1/2}\right].$$

Using Young's inequality $xy \leq \frac{x^2}{2\widetilde{C}_4} + \frac{\widetilde{C}_4 y^2}{2}$, for any $x, y \geq 0$ and Theorem 2.4.18, as $\alpha = 1/2$, we get

$$\mathbb{E}[A_t^{(n)}] \leq \frac{1}{2\widetilde{C}_4}\mathbb{E}[V_t^{(n)}] + \frac{\widetilde{C}_4}{2}\int_0^T \mathbb{E}[|Y_s^{(n)}|]\mathrm{d}s \leq \frac{1}{2\widetilde{C}_4}\mathbb{E}[V_t^{(n)}] + \frac{C_2}{\sqrt{n}}. \tag{2.46}$$

Therefore as $\beta \geq 1/2$, we have, using (2.44), (2.45) and (2.46),

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|M_s^{\delta,\varepsilon,n}\right|\right] \leq \frac{1}{2}\mathbb{E}[V_t^{(n)}] + C_3\left\{\frac{1}{\sqrt{n}} + \frac{1}{n^\beta}\right\} \leq \frac{1}{2}\mathbb{E}[V_t^{(n)}] + \frac{C_4}{\sqrt{n}}. \tag{2.47}$$

Taking supremum in (2.39) with $\tau = T$, we obtain

$$\begin{aligned} V_t^{(n)} \leq & C_5\int_0^t V_s^{(n)}\mathrm{d}s + \sum_{i=1}^d \int_0^T \left|b^{(i)}(s, X_s^{(n)}) - b^{(i)}(s, X_{\eta_n(s)}^{(n)})\right|\mathrm{d}s \\ & + \frac{C_5}{n^\beta} + \sup_{0\leq s\leq t}|M_s^{\delta,\varepsilon,n}| + R(1/2,\delta,\varepsilon,n) + S(1/2,\delta,\varepsilon,n). \end{aligned} \tag{2.48}$$

From (2.40), (2.47) and (2.48), we have

$$\mathbb{E}[V_t^{(n)}] \leq C_6\int_0^t \mathbb{E}[V_s^{(n)}]\mathrm{d}s + \frac{C_6}{\sqrt{n}}.$$

From Gronwall's inequality we have

$$\mathbb{E}[V_t^{(n)}] \leq \frac{C_7}{\sqrt{n}}.$$

Case 2 ($d = 1$): For $\alpha \in (0, 1/2]$, by using the same method as in Case 1, we have that (2.46) becomes

$$\mathbb{E}[A_t^{(n)}] \leq \frac{1}{2\widetilde{C}_4}\mathbb{E}[V_t^{(n)}] + \frac{\widetilde{C}_4}{2}\int_0^T \left(\mathbb{E}[|Y_s^{(n)}|]\right)^{2\alpha}\mathrm{d}s \leq \frac{1}{2\widetilde{C}_4}\mathbb{E}[V_t^{(n)}] + \frac{C_8}{n^{2\alpha^2}}. \tag{2.49}$$

Therefore from (2.44), using (2.45) and (2.49) we obtain

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|M_s^{\delta,\varepsilon,n}\right|\right] \leq \frac{1}{2}\mathbb{E}[V_t^{(n)}] + C_9\left\{\frac{1}{n^{2\alpha^2}} + \frac{1}{n^{1/4+\alpha/2}} + \frac{1}{n^\beta}\right\} \leq \frac{1}{2}\mathbb{E}[V_t^{(n)}] + \frac{C_{10}}{n^{2\alpha^2}}. \tag{2.50}$$

Taking supremum in (2.41) with $\tau = T$, we have

$$V_t^{(n)} \leq C_{11}\int_0^t V_s^{(n)}\mathrm{d}s + \int_0^T |b(s, X_s^{(n)}) - b(s, X_{\eta_n(s)}^{(n)})|\mathrm{d}s + \frac{C_{11}}{n^\beta} + \sup_{0\leq s\leq t}|M_s^{\delta,\varepsilon,n}| + R(\alpha,\delta,\varepsilon,n). \tag{2.51}$$

Therefore by using (2.40), (2.50) and applying Gronwall's inequality we have

$$\mathbb{E}[V_t^{(n)}] \leq \frac{C_{12}}{n^{2\alpha^2}}.$$

For $\alpha = 0$, it follows from Theorem 2.4.18 that we have

$$\mathbb{E}[A_t^{(n)}] \leq \left(\int_0^T \mathbb{E}\left[|Y_s^{(n)}| \right] \mathrm{d}s \right)^{1/2} \leq \frac{C_{13}}{\sqrt{\log n}}.$$

Therefore from (2.51) and applying Gronwall's inequality we have

$$\mathbb{E}[V_t^{(n)}] \leq \frac{C_{14}}{\sqrt{\log n}}.$$

Hence we finish the proof of Theorem 2.4.19. □

The following theorem provides a bound on error in L^p-norm which is useful for designing a Multilevel Monte Carlo approximation scheme.

Theorem 2.4.20. *Let Assumptions 2.4.15 and 2.4.17 hold. Then for any $p \geq 2$, there exists a constant C which depends on $K, T, \lambda_0, x_0, d, \alpha, \beta$ and p such that for $d = 1$,*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| X_t - X_t^{(n)} \right|^p \right] \leq \begin{cases} \dfrac{C}{\log n} & \text{if } \alpha = 0, \\ \dfrac{C}{n^{\alpha}} & \text{if } \alpha \in (0, 1/2], \end{cases} \tag{2.52}$$

and for $d \geq 2$,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| X_t - X_t^{(n)} \right|^p \right] \leq \frac{C}{n^{1/2}} \ \text{if } \alpha = 1/2.$$

Remark 2.4.21. In Theorem 2.4.20, for $\alpha \in [0, 1/2)$, the result is the same as in Gyöngy and Rásonyi [40]. But for $\alpha = 1/2$, every moment bigger than 2 of the error is of the same order. The reason is that we deal with the discontinuous drift coefficients and the estimate of discontinuous part is of order 1/2 for any $q \geq 1$ (see Lemma 2.4.12). The proof of Theorem 2.4.20 does not make use of Theorem 2.4.19, but only of Theorem 2.4.18. On the other hand, for $\alpha \in (0, 1/2)$, using Theorem 2.4.20 and Jensen's inequality, we can obtain that the rate of convergence is $\alpha/2$ in L^1-sup norm. For $\alpha \in [1/4, 1/2)$, this result is better than Theorem 2.4.19 and for $\alpha \in (0, 1/4]$, this result is worse than Theorem 2.4.19.

To prove Theorem 2.4.20, we introduce the following Gronwall type inequality.

Lemma 2.4.22 ([40], Lemma 3.2.)**.** *Let $(Z_t)_{t \geq 0}$ be a nonnegative continuous stochastic process and set $V_t := \sup_{s \leq t} Z_s$. Assume that for some $r > 0$, $q \geq 1$, $\rho \in [1, q]$ and some constants C_0 and $\xi \geq 0$,*

$$\mathbb{E}[V_t^r] \leq C_0 \mathbb{E}\left[\left(\int_0^t V_s \mathrm{d}s \right)^r \right] + C_0 \mathbb{E}\left[\left(\int_0^t Z_s^{\rho} \mathrm{d}s \right)^{r/q} \right] + \xi < \infty$$

for all $t \geq 0$. Then for each $T \geq 0$ the following statements hold.
(i) If $\rho = q$ then there exists a constant C_1 depending on C_0, T, q and r such that

$$\mathbb{E}[V_T^r] \leq C_1 \xi.$$

(ii) If $r \geq q$ or $q+1-\rho < r < q$ hold, then there exists constant C_2 depending on C_0, T, ρ, q and r, such that

$$\mathbb{E}[V_T^r] \leq C_2\xi + C_2\int_0^T \mathbb{E}[Z_s]\mathrm{d}s.$$

Proof of Theorem 2.4.20. Let $p \geq 2$. First we estimate the expectation of $\sup_{0\leq s\leq t}|M_s^{\delta,\varepsilon,n}|^p$. By using (2.36) and Burkholder-Davis-Gundy's inequality, for any $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq t}|M_s^{\delta,\varepsilon,n}|^p\right] &\leq C_1\mathbb{E}[\langle M^{\delta,\varepsilon,n}\rangle_t^{p/2}] \\
&\leq C_2\mathbb{E}\left[\left(\int_0^t |Y_s^{(n)}|^{1+2\alpha} + |U_s^{(n)}|^{1+2\alpha} + |s-\eta_n(s)|^{2\beta}\,\mathrm{d}s\right)^{p/2}\right] \\
&\leq C_3\mathbb{E}\left[\left(\int_0^t |Y_s^{(n)}|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right] + \frac{C_3}{n^{p/4+p\alpha/2}} + \frac{C_3}{n^{p\beta}} \qquad (2.53)\\
&\leq C_4\mathbb{E}\left[\left(\int_0^t |Y_s^{(n)}|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right] + \frac{C_4}{n^{p\alpha}}. \qquad (2.54)
\end{aligned}$$

Now we estimate the expectation of $(V_t^{(n)})^p$.

Case 1 ($d \geq 2$ and $\alpha = 1/2$): We choose $\delta = 2$ and $\varepsilon = n^{-1/2}$. From (2.48), by using the inequality $(\sum_{i=1}^m a_i)^q \leq m^{(q-1)\vee 0}\sum_{i=1}^m a_i^q$ for any $m \in \mathbb{N}$, $a_i \geq 0$ and $q > 0$, we have

$$\begin{aligned}
(V_t^{(n)})^p \leq C_5\Bigg\{ &\left(\int_0^t V_s^{(n)}\mathrm{d}s\right)^p + \sum_{i=1}^d \int_0^T |b^{(i)}(s,X_s^{(n)}) - b^{(i)}(s,X_{\eta_n(s)}^{(n)})|^p\mathrm{d}s \\
&+ \frac{1}{n^{p\beta}} + \sup_{0\leq s\leq t}|M_s^{\delta,\varepsilon,n}|^p + R^p(1/2,\delta,\varepsilon,n) + S^p(1/2,\delta,\varepsilon,n)\Bigg\}. \qquad (2.55)
\end{aligned}$$

In the same way as in (2.40), for any $\alpha \in (0,1/2]$ we have

$$\mathbb{E}[R^p(\alpha,2,n^{-1/2},n)],\ \mathbb{E}[S^p(\alpha,2,n^{-1/2},n)] \leq C_6 n^{-p\alpha}. \qquad (2.56)$$

Using Lemma 2.4.12 with $q = p$, (2.54), (2.55) and (2.56) we have

$$\mathbb{E}[(V_t^{(n)})^p] \leq C_7\mathbb{E}\left[\left(\int_0^t V_s^{(n)}\mathrm{d}s\right)^p\right] + C_7\mathbb{E}\left[\left(\int_0^t |Y_s^{(n)}|^2\mathrm{d}s\right)^{p/2}\right] + \frac{C_7}{\sqrt{n}}.$$

From Lemma 2.4.22 (i) with $r = p$, $\rho = q = 2$ and $\xi = C_7 n^{-1/2}$, we obtain

$$\mathbb{E}[(V_t^{(n)})^p] \leq \frac{C_8}{\sqrt{n}}.$$

Case 2 ($d = 1$): From (2.51), we have

$$(V_t^{(n)})^p \leq C_9\Bigg\{\left(\int_0^t V_s^{(n)}\mathrm{d}s\right)^p + \int_0^T |b(s,X_s^{(n)}) - b(s,X_{\eta_n(s)}^{(n)})|^p\mathrm{d}s$$

$$+\frac{1}{n^{p\beta}}+\sup_{0\le s\le t}|M_s^{\delta,\varepsilon,n}|^p+R^p(\alpha,\delta,\varepsilon,n)\Bigg\}. \tag{2.57}$$

For $\alpha=1/2$, we can show the statement in the same way as in Case 1.
For $\alpha\in(0,1/2)$, we also take $\delta=2$ and $\varepsilon=n^{-1/2}$. By using (2.54), (2.56) and Lemma 2.4.12 with $q=p$ we have

$$\mathbb{E}[(V_t^{(n)})^p]\le C_{10}\mathbb{E}\left[\left(\int_0^t V_s^{(n)}\mathrm{d}s\right)^p\right]+C_{10}\mathbb{E}\left[\left(\int_0^t |Y_s^{(n)}|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right]+\frac{C_{10}}{n^{p\alpha}}+\frac{C_{10}}{\sqrt{n}}.$$

From Lemma 2.4.22 (ii) with $r=p$, $q=2$, $\rho=1+2\alpha$ and $\xi=C_{10}n^{-p\alpha}+C_{10}n^{-1/2}$, we have

$$\mathbb{E}[(V_t^{(n)})^p]\le\frac{C_{11}}{n^{p\alpha}}+\frac{C_{11}}{\sqrt{n}}+C_{11}\int_0^T\mathbb{E}[|Y_s^{(n)}|]\mathrm{d}s\le C_{12}\left\{\frac{1}{n^{p\alpha}}+\frac{1}{\sqrt{n}}+\frac{1}{n^{\alpha}}\right\}\le\frac{C_{13}}{n^{\alpha}}.$$

For $\alpha=0$, we choose $\delta=n^{1/3}$ and $\varepsilon=(\log n)^{-1}$. In the same way as in (2.42), we have

$$\mathbb{E}[R^p(0,n^{1/3},(\log n)^{-1},n)]\le\frac{C_{14}}{(\log n)^p}.$$

Using Lemma 2.4.12 with $q=p$, (2.53),(2.56) and (2.57) we obtain

$$\mathbb{E}[(V_t^{(n)})^p]\le C_{15}\mathbb{E}\left[\left(\int_0^t V_s^{(n)}\mathrm{d}s\right)^p\right]+C_{15}\mathbb{E}\left[\left(\int_0^t |Y_s^{(n)}|\mathrm{d}s\right)^{p/2}\right]+\frac{C_{15}}{(\log n)^p}.$$

From Lemma 2.4.22 (ii) with $r=p$, $q=2$, $\rho=1$ and $\xi=C_{15}(\log n)^{-p}$, we have

$$\mathbb{E}[(V_t^{(n)})^p]\le\frac{C_{16}}{(\log n)^p}+C_{16}\int_0^T\mathbb{E}[|Y_s^{(n)}|]\mathrm{d}s\le\frac{C_{17}}{\log n}.$$

Hence the proof of the theorem is complete. $\square$

Chapter 3

The Euler-Maruyama approximation for one-dimensional SDEs

3.1 Introduction

Let us consider the one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s,\ x_0 \in \mathbb{R},\ t \in [0,T], \tag{3.1}$$

where $W := (W_t)_{0\le t\le T}$ is a standard one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions.

It is well-known that the solution to the SDE (3.1) is related to the Kolmogorov equation. Stroock and Varadhan [110] prove that if the drift coefficient b is bounded, measurable and the diffusion coefficient σ is bounded, uniformly elliptic and continuous, then a solution to the Kolmogorov equation $\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} + \sigma^2\frac{\partial^2 u}{\partial^2 x} = 0$ with the boundary condition $u(T,x) = f(x)$, in the class $W_p^{1,2}$ with $p > 3/2$ admits the stochastic representation (see also Theorem 1 in [122]). Zvonkin [122] studied the existence and uniqueness of solution of the SDE (3.1) under very weak regularity assumption of coefficients b and σ. In particular, he showed that if b is bounded, measurable and σ is bounded, uniformly elliptic and $(\alpha + \frac{1}{2})$-Hölder continuous for some $\alpha \in [0, \frac{1}{2}]$ then equation (3.1) has a unique strong solution (see also Veretennikov [116]).

Since the solution of (3.1) is rarely analytically tractable, one often approximates $X = (X_t)_{0\le t\le T}$ by using the Euler-Maruyama scheme given by

$$X_t^{(n)} = x_0 + \int_0^t b\left(X_{\eta_n(s)}^{(n)}\right)\mathrm{d}s + \int_0^t \sigma\left(X_{\eta_n(s)}^{(n)}\right)\mathrm{d}W_s,\ t \in [0,T],$$

where $\eta_n(s) = kT/n =: t_k^{(n)}$ if $s \in [kT/n, (k+1)T/n)$.

Both the strong and weak rates of convergence of $X^{(n)}$ to X are known when b and σ satisfy globally Lipschitz continuous condition (see [10, 64]). It has been shown recently that there exist SDEs with smooth and bounded coefficients such that neither the EM approximation nor any approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than any given speed of convergence (see [41, 56]). However, there are few results when b is irregular. When b is not continuous, most of the works so far need the assumption that b is one-sided Lipschitz to establish the rate of convergence (see [38, 97]). Note that this one-sided Lipschitz condition also plays an indispensable role to establish the rate of convergence for SDEs with super-linear growth coefficients (see [17],[48]). Outside the framework of one-sided Lipschitz, Halidias and Kloeden [42] showed that $X^{(n)}$ converges to X in L^2-norm if b is increasing, continuous from bellow and σ is Lipschitz continuous. Since their proof uses upper and lower solutions of the SDEs and the Euler-Maruyama approximation, it is hardly possible to get any rate of convergence by using their method. Recently, Leobacher and Szölgyenyi [81] studied the SDE (3.1) under the assumption that b is piecewise Lipschitz, has a finite number of discontinuous points and σ is Lipschitz and uniformly elliptic. They introduced a clever way to transfer equation (3.1) to an equivalent equation whose coefficients are Lipschitz continuous and therefore the new equation can be approximated by an Euler-Maruyama scheme with the standard rate of convergence 1/2.

The strong rates of the Euler-Maruyama approximation for SDEs with Hölder continuous diffusion coefficient were first established in [13, 40, 121]. The main idea in [40] is to use the so-called Yamada-Watanabe approximation method to estimate the error. This remarkable idea has been developed in [11, 96, 97] to obtain strong rate under various assumptions on coefficients b and σ. It is undoubted that Yamada-Watanabe approximation is still a key tool to deal with the Hölder continuity of σ in this chapter. When b is only Hölder continuous of order $\beta \in (0, 1]$ and σ is a non-zero constant, Menoukeu-Pamen and Taguchi [90] have used a PDE technique to show very recently that the strong rate of the Euler-Maruyama approximation is of order $\beta/2$.

In this chapter, we will study the rates of strong convergence of the Euler-Maruyama approximation for SDE (3.1) when the coefficients b and σ may have a very low regularity. In section 3.2, we consider the case that σ is $(\alpha + \frac{1}{2})$-Hölder continuous and $b = b_A + b_H$ where b_A is, roughly speaking, a function of bounded variation on compact sets and b_H is Hölder continuous of some order $\beta \in (0, 1]$. Note that b is not necessary continuous or one-sided Lipschitz function. By introducing a new approach based on the removal drift transformation, we are able to establish the rates of convergence of $X^{(n)}$ to X in L^1, L^1-sup and L^p-sup norm ($p \geq 2$). Our finding partly improves upon recent results in [40, 90, 97] as well as the well-known ones in [38, 42] in the one-dimensional setting (see Remark 3.2.12). It worth noting that SDEs with discontinuous drift appear in many applications such as mathematical finance, optimal control and interacting infinite particle systems [1, 14, 16, 18, 72]. In section 3.3, we consider the case that the σ is discontinuous function. These SDEs appears in many applied domains such as stochastic control and quantitative finance (see [1, 18]). For such SDEs, the existence and uniqueness of solution was studied in [18, 77, 94]; the weak convergence of EM approximation was shown in [121]. To the best of our knowledge, the strong convergence of the EM approximation of SDEs with discontinuous diffusion coefficient has not been considered before in the literature. It is worth noting that the key ingredients to establish the strong rate of convergence of EM approximation for SDEs with discontinuous drift are either the Krylov estimate (see [40, 65]) or the Gaussian bound estimate for the density of the numerical solution ([80, 97, 99]). However, these estimates seem no longer available for SDEs with discontinuous diffusion coefficients. Therefore in section 3.3 we develop another method, which is based on an argument with local time, to overcome this obstacle. In section 3.4, we will see that the arguments of section 3.3 are useful to improve the rate of convergence provided by Gyöngy and Rśasonyi in [40] under the additional assumption that the diffusion coefficient σ is monotone Hölder continuous.

3.2 SDEs with discontinuous drift and Hölder continuous diffusion coefficient

3.2.1 Notations and assumptions

For bounded measurable function f on $\mathbb{R}$, we define $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$. We denote by $L^1(\mathbb{R})$ the space of all integrable functions on $\mathbb{R}$ with semi-norm $\|f\|_{L^1(\mathbb{R})} := \int_{\mathbb{R}} |f(x)|\mathrm{d}x$. For $\beta \in (0,1]$, we denote by H^β the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are bounded and β-Hölder continuous, i.e., a function $f \in H^\beta$ iff

$$\|f\|_\beta := \|f\|_\infty + \sup_{x,y\in\mathbb{R},x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Next, we recall the following class of functions $\mathcal{A}$ for one-dimensional setting (see Chapter 2, section 2.4.1) which is first introduced in [66] (see also [97]). Let $\mathcal{A}$ be a class of all bounded measurable functions $\zeta : \mathbb{R} \to \mathbb{R}$ such that there exist a finite positive constant $K_\mathcal{A}$ and a sequence of functions $(\zeta_N)_{N\in\mathbb{N}} \subset C^1(\mathbb{R})$ satisfying:

$$\begin{cases} \mathcal{A}(i): & \zeta_N \to \zeta \text{ in } L^1_{\ell oc}(\mathbb{R}), \text{ as } N \to \infty, \\ \mathcal{A}(ii): & \sup_{N\in\mathbb{N}} |\zeta_N(x)| + |\zeta(x)| \leq K_\mathcal{A}, \\ \mathcal{A}(iii): & \displaystyle\sup_{N\in\mathbb{N},a\in\mathbb{R}} \int_{\mathbb{R}} |\zeta_N'(x+a)| e^{-|x|^2/u}\mathrm{d}x \leq (1+\sqrt{u})K_\mathcal{A} \quad \text{for all } u>0. \end{cases}$$

We denote $\|\zeta\|_\mathcal{A}$ the smallest constant $K_\mathcal{A}$ satisfying the above conditions. The class $\mathcal{A}$ will be used to model a part of the drift coefficient b.

Recall that the following propositions shows that this class is quite large, (see Proposition 2.4.2). It is worth noting that if a function $\zeta \in \mathcal{A}$ has a compact support then it follows from Theorem 3.9 of [3] that ζ is of bounded variation. Therefore class $\mathcal{A}$ does not contain class of Hölder continuous function H^β for any $\beta \in (0,1)$.

In this section, we need the following assumptions on the coefficients b and σ.

Assumption 3.2.1. *We assume that the coefficients b and σ are measurable functions and satisfy the following conditions:*

(i) $b = b_A + b_H \in L^1(\mathbb{R})$ *where* $b_A \in \mathcal{A}$ *and* $b_H \in H^\beta$ *with* $\beta \in (0,1]$.

(ii) σ *is uniform elliptic, globally bounded and globally Hölder continuous: there exist real numbers* $K_\sigma > 1$ *and* $\alpha \in [0, \frac{1}{2}]$ *such that*

$$\frac{1}{K_\sigma^2} \leq \sigma^2(x) \leq K_\sigma^2 \text{ for any } x \in \mathbb{R},$$

and

$$|\sigma(x)-\sigma(y)| \leq K_\sigma |x-y|^{\frac{1}{2}+\alpha} \text{ for any } x,y \in \mathbb{R}.$$

The following lemma is a one-dimensional version of Lemma 2.4.12.

Lemma 3.2.2. *Assume that b is bounded, measurable and σ satisfies Assumption 3.2.1 (ii). Suppose that $\zeta \in \mathcal{A}$. Then for any $q \geq 1$, there exists $C \equiv C(T, K_\sigma, \|\zeta\|_{\mathcal{A}}, \|b\|_\infty, x_0, q)$ such that*

$$\int_0^T \mathbb{E}[|\zeta(X_s^{(n)}) - \zeta(X_{\eta_n(s)}^{(n)})|^q]\mathrm{d}s \leq \frac{C}{\sqrt{n}}. \tag{3.2}$$

Finally we recall that we have the following standard estimation, (see Lemma 2.2.2).

Lemma 3.2.3. *Suppose that b and σ are bounded, measurable. Then for any $q > 0$, there exist $C \equiv C(q, \|b\|_\infty, \|\sigma\|_\infty, T)$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t^{(n)} - X_{\eta_n(t)}^{(n)}|^q] \leq \frac{C}{n^{q/2}}.$$

3.2.2 The method of removal of drift

The following removal of drift transformation plays a crucial role in our argument. Under the assumption that $b \in L^1(\mathbb{R})$ and σ^2 is uniformly elliptic, the scale function

$$\varphi(x) := \int_0^x \exp\left(-2\int_0^y \frac{b(z)}{\sigma^2(z)}\mathrm{d}z\right)\mathrm{d}y$$

is well-defined. Moreover, since φ'' exists and $\varphi'' = -\frac{2b\varphi'}{\sigma^2}$ almost everywhere, φ satisfies the following ODE

$$b(x)\varphi'(x) + \frac{1}{2}\sigma^2(x)\varphi''(x) = 0.$$

Define $Y_t := \varphi(X_t)$ and $Y_t^{(n)} := \varphi(X_t^{(n)})$. Then by the generalized Itô's formula (see Problem 3.7.3 in [60]), we have

$$Y_t = \varphi(x_0) + \int_0^t \varphi'(X_s)\sigma(X_s)\mathrm{d}W_s,$$

and

$$Y_t^{(n)} = \varphi(x_0) + \int_0^t \left(\varphi'(X_s^{(n)})b(X_{\eta_n(s)}^{(n)}) + \frac{1}{2}\varphi''(X_s^{(n)})\sigma^2(X_{\eta_n(s)}^{(n)})\right)\mathrm{d}s + \int_0^t \varphi'(X_s^{(n)})\sigma(X_{\eta_n(s)}^{(n)})\mathrm{d}W_s.$$

We will make repeated use of the following elementary lemma.

Lemma 3.2.4. *Suppose that $b \in L^1(\mathbb{R})$ and Assumption 3.2.1 (ii) hold. Let $C_0 = e^{2K_\sigma^2\|b\|_{L^1(\mathbb{R})}}$.*

(i) For any $x \in \mathbb{R}$,

$$\frac{1}{C_0} \leq \varphi'(x) = \exp(f(x)) \leq C_0.$$

(ii) For any $x \in \mathbb{R}$,

$$|\varphi''(x)| \leq 2K_\sigma^2 \|b\|_\infty \|\varphi'\|_\infty \leq 2\|b\|_\infty K_\sigma^2 C_0.$$

(iii) For any $z, w \in Dom(\varphi^{-1})$,

$$|\varphi^{-1}(z) - \varphi^{-1}(w)| \leq C_0 |z - w|. \tag{3.3}$$

The proof of Lemma 3.2.4 is trivial and therefore will be omitted.

To deal with the Hölder continuity of the diffusion coefficient σ, we again use Yamada and Watanabe approximation technique (see Section 2.3). For each $\delta \in (1, \infty)$ and $\varepsilon \in (0, 1)$, we define a continuous function $\psi_{\delta,\varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with *supp* $\psi_{\delta,\varepsilon} \subset [\varepsilon/\delta, \varepsilon]$ such that

$$\int_{\varepsilon/\delta}^{\varepsilon} \psi_{\delta,\varepsilon}(z) \mathrm{d}z = 1 \text{ and } 0 \leq \psi_{\delta,\varepsilon}(z) \leq \frac{2}{z \log \delta}, \quad z > 0.$$

Since $\int_{\varepsilon/\delta}^{\varepsilon} \frac{2}{z \log \delta} \mathrm{d}z = 2$, there exists such a function $\psi_{\delta,\varepsilon}$. We define a function $\phi_{\delta,\varepsilon} \in C^2(\mathbb{R}; \mathbb{R})$ by

$$\phi_{\delta,\varepsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta,\varepsilon}(z) \mathrm{d}z \mathrm{d}y.$$

It is easy to verify that $\phi_{\delta,\varepsilon}$ has the following useful properties:

$$|x| \leq \varepsilon + \phi_{\delta,\varepsilon}(x), \text{ for any } x \in \mathbb{R}, \tag{3.4}$$

$$0 \leq |\phi'_{\delta,\varepsilon}(x)| \leq 1, \text{ for any } x \in \mathbb{R}, \tag{3.5}$$

$$\phi''_{\delta,\varepsilon}(\pm|x|) = \psi_{\delta,\varepsilon}(|x|) \leq \frac{2}{|x| \log \delta} \mathbf{1}_{[\varepsilon/\delta, \varepsilon]}(|x|), \text{ for any } x \in \mathbb{R} \setminus \{0\}. \tag{3.6}$$

From (3.3) and (3.4), for any $t \in [0, T]$, we have

$$|X_t - X_t^{(n)}| \leq C_0 |Y_t - Y_t^{(n)}| \leq C_0 \left(\varepsilon + \phi_{\delta,\varepsilon}(Y_t - Y_t^{(n)}) \right). \tag{3.7}$$

Using Itô's formula, we have

$$\phi_{\delta,\varepsilon}(Y_t - Y_t^{(n)}) = M_t^{n,\delta,\varepsilon} + I_t^{(n)} + J_t^{(n)}, \tag{3.8}$$

where

$$M_t^{n,\delta,\varepsilon} := \int_0^t \phi'_{\delta,\varepsilon}(Y_s - Y_s^{(n)}) \left\{ \varphi'(X_s)\sigma(X_s) - \varphi'(X_s^{(n)})\sigma(X_{\eta_n(s)}^{(n)}) \right\} \mathrm{d}W_s,$$

$$I_t^{(n)} := -\int_0^t \phi'_{\delta,\varepsilon}(Y_s - Y_s^{(n)}) \left\{ \varphi'(X_s^{(n)}) b(X_{\eta_n(s)}^{(n)}) + \frac{1}{2}\varphi''(X_s^{(n)})\sigma^2(X_{\eta_n(s)}^{(n)}) \right\} \mathrm{d}s,$$

$$J_t^{(n)} := \frac{1}{2} \int_0^t \phi''_{\delta,\varepsilon}(Y_s - Y_s^{(n)}) \left| \varphi'(X_s)\sigma(X_s) - \varphi'(X_s^{(n)})\sigma(X_{\eta_n(s)}^{(n)}) \right|^2 \mathrm{d}s.$$

3.2.3 Error estimates for the Euler-Maruyama approximation

We obtain the following results on the rate of the Euler-Maruyama approximation in L^1-norm.

Theorem 3.2.5. *Let Assumption 3.2.1 hold. Then there exists a constant C^* which depends on K_σ, $\|b_A\|_{\mathcal{A}}$, $\|b_H\|_\beta$, $\|b\|_{L^1(\mathbb{R})}$, T, x_0, α and β such that*

$$\sup_{\tau\in\mathcal{T}}\mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \begin{cases} \dfrac{C^*}{\log n} & \text{if } \alpha = 0, \\ \dfrac{C^*}{n^{\frac{\beta}{2}\wedge\alpha}} & \text{if } \alpha \in (0,1/2], \end{cases} \tag{3.9}$$

where $\mathcal{T}$ is the set of all stopping times $\tau \leq T$. Moreover, for any $\gamma \in (0,1)$, it holds

$$\mathbb{E}[\sup_{0\leq s\leq T}|X_s - X_s^{(n)}|^\gamma] \leq \begin{cases} \dfrac{2-\gamma}{1-\gamma}\Big(\dfrac{C^*}{\log n}\Big)^\gamma & \text{if } \alpha = 0, \\ \dfrac{2-\gamma}{1-\gamma}\Big(\dfrac{C^*}{n^{\frac{\beta}{2}\wedge\alpha}}\Big)^\gamma & \text{if } \alpha \in (0,1/2]. \end{cases} \tag{3.10}$$

Proof. Recall that by (3.7) and (3.8), we have

$$|X_t - X_t^{(n)}| \leq C_0\left(\varepsilon + M_t^{n,\delta,\varepsilon} + I_t^{(n)} + J_t^{(n)}\right).$$

We first consider $I_t^{(n)}$. Since $\varphi'' = -\frac{2b\varphi'}{\sigma^2}$,

$$|I_t^{(n)}| \leq \int_0^T \left|\phi'_{\delta,\varepsilon}(Y_s - Y_s^{(n)})\varphi'(X_s^{(n)})\right| \left| b(X_{\eta_n(s)}^{(n)}) - \frac{b((X_s^{(n)}))\sigma^2(X_{\eta_n(s)}^{(n)})}{\sigma^2(X_s^{(n)})}\right| \mathrm{d}s.$$

Thanks to Lemma 3.2.4 and estimate (3.5), we have

$$\begin{aligned} |I_t^{(n)}| &\leq K_\sigma^2 C_0 \int_0^T \left| b(X_{\eta_n(s)}^{(n)})\sigma^2(X_s^{(n)}) - b((X_s^{(n)}))\sigma^2(X_{\eta_n(s)}^{(n)})\right| \mathrm{d}s \\ &\leq K_\sigma^2 C_0 \int_0^T \left\{ K_\sigma^2 \left| b(X_s^{(n)}) - b(X_{\eta_n(s)}^{(n)})\right| + \|b\|_\infty \left|\sigma^2(X_s^{(n)}) - \sigma^2(X_{\eta_n(s)}^{(n)})\right|\right\} \mathrm{d}s. \end{aligned}$$

It follows from Assumption 3.2.1 that

$$\begin{aligned} |I_t^{(n)}| \leq & K_\sigma^4 C_0 \int_0^T \left\{ \left| b_A(X_s^{(n)}) - b_A(X_{\eta_n(s)}^{(n)})\right| + \|b_H\|_\beta \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^\beta \right\} \mathrm{d}s \\ &+ 2K_\sigma^3 \|b\|_\infty C_0 \int_0^T \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^{1/2+\alpha} \mathrm{d}s. \end{aligned} \tag{3.11}$$

Now we estimate $J_t^{(n)}$. From (3.6), we have

$$J_t^n \leq \int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}|\log\delta} \left|\varphi'(X_s)\sigma(X_s) - \varphi'(X_s^{(n)})\sigma(X_{\eta_n(s)}^{(n)})\right|^2 \mathrm{d}s$$

$$\leq 3(J_T^{1,n} + J_T^{2,n} + J_T^{3,n}),$$

where

$$J_t^{1,n} := \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} |\sigma(X_s)|^2 \left|\varphi'(X_s) - \varphi'(X_s^{(n)})\right|^2 \mathrm{d}s,$$

$$J_t^{2,n} := \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} |\varphi'(X_s^{(n)})|^2 \left|\sigma(X_s) - \sigma(X_s^{(n)})\right|^2 \mathrm{d}s,$$

$$J_t^{3,n} := \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} |\varphi'(X_s^{(n)})|^2 \left|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})\right|^2 \mathrm{d}s.$$

From Lemma 3.2.4 (ii), φ' is Lipschitz continuous with Lipschitz constant $\|\varphi''\|_\infty$. Hence, we have

$$\begin{aligned} J_T^{1,n} &\leq \frac{K_\sigma^2 \|\varphi''\|_\infty^2}{\log \delta} \int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}|} \left|X_s - X_s^{(n)}\right|^2 \mathrm{d}s \\ &\leq \frac{K_\sigma^2 \|\varphi''\|_\infty^2 C_0^2}{\log \delta} \int_0^T \mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|) \left|Y_s - Y_s^{(n)}\right| \mathrm{d}s \\ &\leq \frac{4K_\sigma^6 C_0^4 \|b\|_\infty^2 T\varepsilon}{\log \delta} \end{aligned} \tag{3.12}$$

and since σ is $(\frac{1}{2} + \alpha)$-Hölder continuous, we have

$$\begin{aligned} J_T^{2,n} &\leq \frac{K_\sigma^2 C_0^2}{\log \delta} \int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}|} \left|X_s - X_s^{(n)}\right|^{1+2\alpha} \mathrm{d}s \\ &\leq \frac{K_\sigma^2 C_0^{3+2\alpha}}{\log \delta} \int_0^T \mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|) \left|Y_s - Y_s^{(n)}\right|^{2\alpha} \mathrm{d}s \\ &\leq \frac{K_\sigma^2 C_0^{3+2\alpha} T \varepsilon^{2\alpha}}{\log \delta}, \end{aligned} \tag{3.13}$$

and

$$J_T^{3,n} \leq \frac{K_\sigma^2 C_0^2 \delta}{\varepsilon \log \delta} \int_0^T \left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^{1+2\alpha} \mathrm{d}s. \tag{3.14}$$

Therefore, from (3.7), (3.8), (3.11), (3.12), (3.13) and (3.14), for any stopping time $\tau \in \mathcal{T}$,

$$\begin{aligned} |X_\tau - X_\tau^{(n)}| \leq & C_0\varepsilon + C_0 M_\tau^{n,\delta,\varepsilon} \\ & + K_\sigma^4 C_0^2 \int_0^T \left\{ \left|b_A(X_s^{(n)}) - b_A(X_{\eta_n(s)}^{(n)})\right| + \|b_H\|_\beta \left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^\beta \right\} \mathrm{d}s \\ & + 2K_\sigma^3 \|b\|_\infty C_0^2 \int_0^T \left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^{1/2+\alpha} \mathrm{d}s \\ & + \frac{12K_\sigma^6 C_0^5 \|b\|_\infty^2 T\varepsilon}{\log \delta} + \frac{3K_\sigma^2 C_0^{4+2\alpha} T \varepsilon^{2\alpha}}{\log \delta} + \frac{3K_\sigma^2 C_0^3 \delta}{\varepsilon \log \delta} \int_0^T \left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^{1+2\alpha} \mathrm{d}s. \end{aligned} \tag{3.15}$$

Note that since ϕ', φ' and σ are bounded, $(M_t^{n,\delta,\varepsilon})_{0\le t\le T}$ is martingale, so expectation of $M_\tau^{n,\delta,\varepsilon}$ equals to zero.

If $\alpha \in (0,1/2]$, then by choosing $\varepsilon = n^{-1/2}$ and $\delta = 2$, the estimate (3.15) becomes

$$\begin{aligned}|X_\tau - X_\tau^{(n)}| \le & \frac{C_0}{n^{1/2}} + C_0 M_\tau^{n,2,n^{-1/2}} \\ & + K_\sigma^4 C_0^2 \int_0^T \left\{ \left| b_A(X_s^{(n)}) - b_A(X_{\eta_n(s)}^{(n)}) \right| + \|b_H\|_\beta \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)} \right|^\beta \right\} \mathrm{d}s \\ & + 2K_\sigma^3 \|b\|_\infty C_0^2 \int_0^T \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)} \right|^{1/2+\alpha} \mathrm{d}s \\ & + \frac{12 K_\sigma^6 C_0^5 \|b\|_\infty^2 T}{n^{1/2}\log 2} + \frac{3K_\sigma^2 C_0^{4+2\alpha} T}{n^\alpha \log 2} + \frac{6K_\sigma^2 C_0^3 n^{1/2}}{\log 2} \int_0^T \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)} \right|^{1+2\alpha} \mathrm{d}s. \end{aligned} \tag{3.16}$$

By taking an expectation in (3.16) it follows from Lemma 3.2.2 and Lemma 3.2.3 with $q=1$ that

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \le \frac{C_1}{n^{\frac{\beta}{2}\wedge\alpha}}, \tag{3.17}$$

where

$$C_1 = K_\sigma^6 C_0^5 \Big(1 + \frac{3T(1+4\|b\|_\infty^2)}{\log 2} + C(1 + T\|b_H\|_\beta + 2\|b\|_\infty T + \frac{6T}{\log 2}) \Big),$$

where C is a constant depending on $\alpha, \beta, T, \|b_A\|_{\mathcal{A}}, K_\sigma, \|b\|_\infty$ and x_0. Note that C doesn't depend on $\|b\|_{L_1(\mathbb{R})}$. This concludes (3.9) for $\alpha \in (0,1/2]$.

If $\alpha = 0$, then by choosing $\varepsilon = (\log n)^{-1}$ and $\delta = n^{1/3}$, the estimate (3.15) becomes

$$\begin{aligned}|X_\tau - X_\tau^{(n)}| \le & \frac{C_0}{\log n} + C_0 M_\tau^{n,n^{1/3},(\log n)^{-1}} \\ & + K_\sigma^4 C_0^2 \int_0^T \left\{ \left| b_A(X_s^{(n)}) - b_A(X_{\eta_n(s)}^{(n)}) \right| + \|b_H\|_\beta \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)} \right|^\beta \right\} \mathrm{d}s \\ & + 2K_\sigma^3 \|b\|_\infty C_0^2 \int_0^T \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)} \right|^{1/2} \mathrm{d}s \\ & + \frac{36 K_\sigma^6 C_0^5 \|b\|_\infty^2 T}{(\log n)^2} + \frac{9K_\sigma^2 C_0^4 T}{\log n} + 9K_\sigma^2 C_0^3 n^{1/3} \int_0^T \left| X_s^{(n)} - X_{\eta_n(s)}^{(n)} \right| \mathrm{d}s. \end{aligned} \tag{3.18}$$

By taking an expectation in (3.18), it follows from Lemma 3.2.2 and 3.2.3 with $q=1$ that

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \le \frac{C_2}{\log n}, \tag{3.19}$$

where

$$C_2 := K_\sigma^6 C_0^5 \Big(1 + 9T + C\big(1 + T\|b_H\|_\beta + 2T\|b\|_\infty + 36T\|b\|_\infty^2 + 9T\big) \Big),$$

where C is defined as in (3.17). This concludes (3.9) for $\alpha = 0$.

Finally, the estimate (3.10) follows directly from (3.9) and Lemma 3.2 in [39]. $\square$

The assumption that $b \in L^1(\mathbb{R})$ in Theorem 3.2.5 is in fact quite restricted since it excludes some simple function such as $b(x) = \mathbf{1}_{x\geq 0}$. Fortunately, removing that assumption does not affect much on the strong rate of convergence as shown in the following theorem.

Theorem 3.2.6. *Suppose that the drift coefficient $b = b_A + b_H$, where $b_A \in \mathcal{A}$ and $b_H \in H^\beta$ with $\beta \in (0,1]$. Let Assumption 3.2.1 (ii) hold. Then there exists a constant C^* which depends on K_σ, $\|b_A\|_{\mathcal{A}}$, $\|b_H\|_\beta$, T, x_0, α and β such that*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \begin{cases} \dfrac{C^* e^{C^*\sqrt{\log(\log n)}}}{\log n} & \text{if } \alpha = 0, \\ \dfrac{C^* e^{C^*\sqrt{\log n}}}{n^{\frac{\beta}{2}\wedge\alpha}} & \text{if } \alpha \in (0,1/2]. \end{cases} \tag{3.20}$$

Moreover, for any $\gamma \in (0,1)$, it holds

$$\mathbb{E}[\sup_{0\leq s\leq T} |X_s - X_s^{(n)}|^\gamma] \leq \begin{cases} \dfrac{2-\gamma}{1-\gamma}\Big(\dfrac{C^* e^{C^*\sqrt{\log(\log n)}}}{\log n}\Big)^\gamma & \text{if } \alpha = 0, \\ \dfrac{2-\gamma}{1-\gamma}\Big(\dfrac{C^* e^{C^*\sqrt{\log n}}}{n^{\frac{\beta}{2}\wedge\alpha}}\Big)^\gamma & \text{if } \alpha \in (0,1/2]. \end{cases} \tag{3.21}$$

Remark 3.2.7. Note that the function $x \mapsto e^{\sqrt{\log x}}$ increases faster than any polynomial function of $\log x$ but slower than any polynomial of x, i.e, for any $\varepsilon > 0$ and $k > 0$,

$$\lim_{x\to+\infty} \frac{e^{\sqrt{\log x}}}{x^\varepsilon} = \lim_{x\to+\infty} \frac{(\log x)^k}{e^{\sqrt{\log x}}} = 0.$$

Proof of Theorem 3.2.6. The main idea of the proof is to approximate b by a sequence of functions $(b_m)_{m\in\mathbb{N}} \subset L^1(\mathbb{R})$ and to apply Theorem 3.2.5 for solution of the SDE with drift coefficient b_m.

For $m \in \mathbb{N}$, we choose a smooth function $g_m \in C^1(\mathbb{R})$ with support $[-(m+2), m+2]$ such that $g_m = 1$ on $[-m,m]$ and $0 \leq g_m(x) \leq 1$ for all $x \in \mathbb{R}$, and $\|g_m'\|_\infty \leq 1$.
We define $b_m := bg_m$. It is easy to verify that

- $\|b_m\|_\infty \leq \|b\|_\infty$;
- $b_m \in L^1(\mathbb{R})$ and $\|b_m\|_{L^1(\mathbb{R})} \leq (2m+2)\|b\|_\infty$;
- $b_A g_m \in \mathcal{A}$ and $\|b_A g_m\|_{\mathcal{A}} \leq 3\|b_A\|_{\mathcal{A}}$;
- $b_H g_m \in H^\beta$ and $\|b_H g_m\|_\beta \leq 2\|b_H\|_\beta$ for all m.

Let $\overline{X}^m$ and $\overline{X}^{m,n}$ be a unique solution of SDE (3.1) with drift b_m and its Euler-Maruyama approximation, respectively. Then it holds that

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \sup_{\tau\in\mathcal{T}} \Big\{ \mathbb{E}[|X_\tau - \overline{X}_\tau^m|] + \mathbb{E}[|\overline{X}_\tau^{m,n} - X_\tau^{(n)}|] + \mathbb{E}[|\overline{X}_\tau^m - \overline{X}_\tau^{m,n}|] \Big\}. \tag{3.22}$$

From (3.17) and (3.19), it holds that

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|\overline{X}^m_\tau - \overline{X}^{m,n}_\tau|] \leq \begin{cases} \dfrac{C_3 e^{20K_\sigma^2\|b\|_\infty m}}{\log n} & \text{if } \alpha = 0, \\ \dfrac{C_3 e^{20K_\sigma^2\|b\|_\infty m}}{n^{\frac{\beta}{2}\wedge\alpha}} & \text{if } \alpha \in (0, 1/2], \end{cases} \tag{3.23}$$

where $C_3 \equiv C_3(x_0, K_\sigma, \|b\|_\infty, T, \alpha, \beta)$ is a finite constant which depends neither on n nor on m. On the other hand, for any stopping time $\tau \in \mathcal{T}$, it holds that

$$\{X_\tau \neq \overline{X}^m_\tau\} \subset \{\sup_{0\leq t\leq\tau} |X_t| \geq m\} \subset \left\{\sup_{0\leq t\leq\tau}\left|\int_0^t \sigma(X_s)\mathrm{d}W_s\right| \geq m - |x_0| - \|b\|_\infty T\right\}.$$

Since $\langle\int_0^\cdot \sigma(X_s)\mathrm{d}W_s\rangle_\tau \leq K_\sigma^2 T$ almost surely, from Proposition 6.8 in [108], we have

$$\begin{aligned} \mathbb{P}(\sup_{0\leq t\leq\tau} |X_t| \geq m) &\leq 2\exp\left(-\frac{(m - |x_0| - \|b\|_\infty T)^2}{2K_\sigma^2 T}\right) \\ &\leq 2\exp\left(\frac{(|x_0| + \|b\|_\infty T)^2}{2K_\sigma^2 T}\right)\exp\left(-\frac{m^2}{4K_\sigma^2 T}\right). \end{aligned} \tag{3.24}$$

Since $\mathbb{E}[\sup_{0\leq t\leq T} |X_t|^2] \vee \mathbb{E}[\sup_{0\leq t\leq T} |\overline{X}^m_t|^2] \leq C_4 := 3(x_0^2 + T^2\|b\|_\infty^2 + 12K_\sigma^2 T)$, we have

$$\begin{aligned} \left(\mathbb{E}[|X_\tau - \overline{X}^m_\tau|]\right)^2 &= \left(\mathbb{E}[|X_\tau - \overline{X}^m_\tau|\mathbf{1}_{\{X_\tau\neq\overline{X}^m_\tau\}}]\right)^2 \\ &\leq \mathbb{E}[|X_\tau - \overline{X}^m_\tau|^2]\mathbb{P}(X_\tau \neq \overline{X}^m_\tau) \\ &\leq C_5^2 \exp\left(-\frac{m^2}{4K_\sigma^2 T}\right), \end{aligned} \tag{3.25}$$

where $C_5^2 = 4C_4 \exp\left(\frac{(|x_0|+\|b\|_\infty T)^2}{2K_\sigma^2 T}\right)$. In the same way, we have

$$\left(\mathbb{E}[|X^{(n)}_\tau - \overline{X}^{m,n}_\tau|]\right)^2 \leq C_5^2 \exp\left(-\frac{m^2}{4K_\sigma^2 T}\right). \tag{3.26}$$

If $\alpha \in (0, \frac{1}{2}]$, from (3.22), (3.23), (3.25) and (3.26), we have

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X^{(n)}_\tau|] \leq C_3\frac{e^{20K_\sigma^2\|b\|_\infty m}}{n^{\frac{\beta}{2}\wedge\alpha}} + 2C_5\exp\Big(-\frac{m^2}{8K_\sigma^2 T}\Big).$$

Choose $m^2 = 8\big(\frac{\beta}{2}\wedge\alpha\big)K_\sigma^2 T\log n$, we obtain

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[|X_\tau - X^{(n)}_\tau|] \leq \frac{C_3 e^{C_6\sqrt{\log n}}}{n^{\frac{\beta}{2}\wedge\alpha}} + \frac{2C_5}{n^{\frac{\beta}{2}\wedge\alpha}},$$

where $C_6 = 40K_\sigma^3\|b\|_\infty\sqrt{2\big(\frac{\beta}{2}\wedge\alpha\big)T}$. This concludes (3.20) for $\alpha \in (0, 1/2]$.

If $\alpha = 0$, from (3.22), (3.23), (3.25) and (3.26), we have

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \frac{C_3 e^{20K_\sigma^2 \|b\|_\infty m}}{\log n} + 2C_5 \exp\Big(- \frac{m^2}{8K_\sigma^2 T}\Big).$$

Choose $m^2 = 8K_\sigma^2 T \log(\log n)$, we obtain

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \frac{C_3 e^{C_7 \sqrt{\log(\log n)}}}{\log n} + \frac{2C_5}{\log n},$$

where $C_7 = 40\sqrt{2T} K_\sigma^3 \|b\|_\infty$. This concludes(3.20) for $\alpha = 0$.

Finally, the estimate (3.21) follows directly from (3.20) and Lemma 3.2 in [39]. □

The estimates (3.10) and (3.21) become worst when $\gamma \uparrow 1$. Fortunately, we have the following bounds for the L^1-sup norm.

Theorem 3.2.8. *Let Assumption 3.2.1 hold. Then there exists a constant C which depends on K_σ, $\|b_A\|_{\mathcal{A}}$, $\|b_H\|_\beta$, $\|b\|_{L^1(\mathbb{R})}$, T, x_0, α and β such that*

$$\mathbb{E}[\sup_{0 \leq s \leq T} |X_s - X_s^{(n)}|] \leq \begin{cases} \dfrac{C}{\sqrt{\log n}} & \text{if } \alpha = 0, \\ \dfrac{C}{n^{\alpha(\beta \wedge 2\alpha)}} & \text{if } \alpha \in (0, 1/2]. \end{cases}$$

Proof. Define $V_t := \sup_{0 \leq s \leq t} |X_s - X_s^{(n)}|$. To estimate the expectation of V_T, we need to estimate the expectation of $\sup_{0 \leq s \leq T} |M_s^{n,\delta,\varepsilon}|$. Using Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leq s \leq T} |M_s^{n,\delta,\varepsilon}|\right] &\leq \sqrt{32}\mathbb{E}\left[\langle M^{n,\delta,\varepsilon}\rangle_T^{1/2}\right] \\ &= \sqrt{32}\mathbb{E}\left[\left(\int_0^T \left|\phi'_{\delta,\varepsilon}(Y_t - Y_t^{(n)})\right|^2 \left|\varphi'(X_s)\sigma(X_s) - \varphi'(X_s^{(n)})\sigma(X_{\eta_n(s)}^{(n)})\right|^2 \mathrm{d}s\right)^{1/2}\right]. \end{aligned}$$

Using the fact that $\|\phi'\|_\infty \leq 1$, we have

$$\begin{aligned} &\mathbb{E}\left[\sup_{0 \leq s \leq T} |M_s^{n,\delta,\varepsilon}|\right] \\ &\leq \sqrt{96}K_\sigma \mathbb{E}\left[\left(\int_0^T \left|\varphi'(X_s) - \varphi'(X_s^{(n)})\right|^2 \mathrm{d}s\right)^{1/2}\right] + \sqrt{96}C_0\mathbb{E}\left[\left(\int_0^T \left|\sigma(X_s) - \sigma(X_s^{(n)})\right|^2 \mathrm{d}s\right)^{1/2}\right] \\ &+ \sqrt{96}C_0\mathbb{E}\left[\left(\int_0^T \left|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})\right|^2 \mathrm{d}s\right)^{1/2}\right]. \end{aligned}$$

Since σ is Hölder continuous and φ' is Lipschitz, $\mathbb{E}\left[\sup_{0\le s\le T}|M_s^{n,\delta,\varepsilon}|\right]$ is bounded by

$$
\begin{aligned}
&\sqrt{96}K_\sigma\|\varphi''\|_\infty\mathbb{E}\left[\left(\int_0^T\left|X_s-X_s^{(n)}\right|^2\mathrm{d}s\right)^{1/2}\right]+\sqrt{96}C_0K_\sigma\mathbb{E}\left[\left(\int_0^T\left|X_s-X_s^{(n)}\right|^{1+2\alpha}\mathrm{d}s\right)^{1/2}\right]\\
&+\sqrt{96}C_0K_\sigma\left(\int_0^T\mathbb{E}\big[|X_s^{(n)}-X_{\eta_n(s)}^{(n)}|^{1+2\alpha}\big]\mathrm{d}s\right)^{1/2}\\
&\le\tilde{c}_1\mathbb{E}\left[A_T^{(n)}\right]+\tilde{c}_2\mathbb{E}\left[B_T^{(n)}\right]+\frac{\tilde{c}_3}{n^{1/4+\alpha/2}},
\end{aligned}
\tag{3.27}
$$

where $\tilde{c}_1:=\sqrt{96}K_\sigma\|\varphi''\|_\infty$, $\tilde{c}_2:=\sqrt{96}C_0K_\sigma$, $\tilde{c}_3:=\sqrt{96CT}C_0K_\sigma$,

$$
A_T^{(n)}:=\left(\int_0^T\left|X_s-X_s^{(n)}\right|^2\mathrm{d}s\right)^{1/2}\text{ and }B_T^{(n)}:=\left(\int_0^T\left|X_s-X_s^{(n)}\right|^{1+2\alpha}\mathrm{d}s\right)^{1/2}.
$$

Since $|X_s-X_s^{(n)}|\le V_T^{(n)}$ for any $s\in[0,T]$, by using Young's inequality $xy\le\frac{x^2}{4\tilde{c}_1C_0}+\tilde{c}_1C_0y^2$, we have

$$
A_T^{(n)}\le\left(V_T^{(n)}\right)^{1/2}\left(\int_0^T\left|X_s-X_s^{(n)}\right|\mathrm{d}s\right)^{1/2}\le\frac{V_T^{(n)}}{4\tilde{c}_1C_0}+\tilde{c}_1C_0\int_0^T|X_s-X_s^{(n)}|\mathrm{d}s.
$$

Hence it holds that

$$
\mathbb{E}[A_T^{(n)}]\le\frac{\mathbb{E}[V_T^{(n)}]}{4\tilde{c}_1C_0}+\tilde{c}_1C_0\int_0^T\mathbb{E}[|X_s-X_s^{(n)}|]\mathrm{d}s.\tag{3.28}
$$

Next we estimate the expectation of $B_T^{(n)}$ and $V_T^{(n)}$.

For $\alpha\in(0,1/2]$, by using Young's inequality $xy\le\frac{x^2}{4\tilde{c}_2C_0}+\tilde{c}_2C_0y^2$, we have

$$
B_T^{(n)}\le\left(V_T^{(n)}\right)^{1/2}\left(\int_0^T\left|X_s-X_s^{(n)}\right|^{2\alpha}\mathrm{d}s\right)^{1/2}\le\frac{V_T^{(n)}}{4\tilde{c}_2C_0}+\tilde{c}_2C_0\int_0^T|X_s-X_s^{(n)}|^{2\alpha}\mathrm{d}s.
$$

Hence it holds from Jensen's inequality that

$$
\mathbb{E}[B_T^{(n)}]\le\frac{\mathbb{E}[V_T^{(n)}]}{4\tilde{c}_2C_0}+\tilde{c}_2C_0\int_0^T\mathbb{E}[|X_s-X_s^{(n)}|]^{2\alpha}\mathrm{d}s.\tag{3.29}
$$

Therefore from (3.27), (3.28) and (3.29), we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le T}|M_s^{n,2,n^{-1/2}}|\right]\le&\frac{\mathbb{E}[V_T^{(n)}]}{2C_0}+\tilde{c}_1^2C_0\int_0^T\mathbb{E}[|X_s-X_s^{(n)}|]\mathrm{d}s\\
&+\tilde{c}_2^2C_0\int_0^T\mathbb{E}[|X_s-X_s^{(n)}|]^{2\alpha}\mathrm{d}s+\frac{\tilde{c}_3}{n^{1/4+\alpha/2}}.
\end{aligned}
$$

It follows from (3.17) that

$$\mathbb{E}\left[\sup_{0\le s\le T}|M_s^{n,2,n^{-1/2}}|\right]\le \frac{\mathbb{E}[V_T^{(n)}]}{2C_0}+\frac{(\tilde{c}_1^2+\tilde{c}_2^2)TC_0C_1}{n^{\alpha(\beta\wedge 2\alpha)}}+\frac{\tilde{c}_3}{n^{1/4+\alpha/2}}.$$

This fact together with (3.16) and (3.17) implies that

$$\mathbb{E}[V_T^{(n)}]\le \frac{2(\tilde{c}_1^2+\tilde{c}_2^2)TC_0^2C_1}{n^{\alpha(\beta\wedge 2\alpha)}}+\frac{2\tilde{c}_3C_0}{n^{1/4+\alpha/2}}+\frac{2C_1}{n^{\frac{\beta}{2}\wedge\alpha}}.$$

Since $1/4+\alpha/2\ge 2\alpha^2$, we have

$$\mathbb{E}[V_T^{(n)}]\le \frac{C_8}{n^{\alpha(\beta\wedge 2\alpha)}}, \tag{3.30}$$

where $C_8:=2(\tilde{c}_1^2+\tilde{c}_2^2)TC_0^2C_1+2\tilde{c}_3C_0+2C_1$.

For $\alpha=0$, by Jensen's inequality we have

$$\mathbb{E}[B_T^{(n)}]\le\left(\int_0^T\mathbb{E}\left[\left|X_s-X_s^{(n)}\right|\right]\mathrm{d}s\right)^{1/2}\le\frac{\sqrt{C_2T}}{\sqrt{\log n}}. \tag{3.31}$$

Therefore from (3.27), (3.28) and (3.31), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le T}\left|M_s^{n,n^{1/3},(\log n)^{-1}}\right|\right]&\le\frac{\mathbb{E}[V_T]}{4C_0}+\tilde{c}_1^2C_0\int_0^T\mathbb{E}[|X_s-X_s^{(n)}|]\mathrm{d}s+\frac{\tilde{c}_2\sqrt{C_2T}}{\sqrt{\log n}}+\frac{\tilde{c}_3}{n^{1/4+\alpha/2}}\\&\le\frac{\mathbb{E}[V_T]}{4C_0}+\frac{\tilde{c}_1^2C_0C_2T}{\log n}+\frac{\tilde{c}_2\sqrt{C_2T}}{\sqrt{\log n}}+\frac{\tilde{c}_3}{n^{1/4+\alpha/2}}.\end{aligned}$$

This fact together with estimates (3.18) and (3.19) implies that

$$\mathbb{E}[V_T^{(n)}]\le\frac{4\tilde{c}_1^2C_0^2C_2T}{3\log n}+\frac{4C_0\tilde{c}_2\sqrt{C_2T}}{3\sqrt{\log n}}+\frac{4C_0\tilde{c}_3}{3n^{1/4+\alpha/2}}+\frac{4C_2}{3\log n}\le\frac{C_9}{\sqrt{\log n}}, \tag{3.32}$$

where $C_9:=\frac{4}{3}(\tilde{c}_1^2C_0^2C_2T+C_0\tilde{c}_2\sqrt{C_2T}+C_0\tilde{c}_3+C_2)$. Hence we finish the proof of Theorem 3.2.8. □

In the similar arguments of Theorem 3.2.6, we can remove the assumption that $b\in L^1(\mathbb{R})$.

Theorem 3.2.9. *Under the assumption of Theorem 3.2.6, there exists a constant C which depends on $K_\sigma,\|b_A\|_{\mathcal{A}},\|b_H\|_\beta,T,x_0,\alpha$ and β such that*

$$\mathbb{E}[\sup_{0\le s\le T}|X_s-X_s^{(n)}|]\le\begin{cases}\dfrac{Ce^{C\sqrt{\log(\log n)}}}{\sqrt{\log n}} & \textit{if } \alpha=0,\\[2mm] \dfrac{Ce^{C\sqrt{\log n}}}{n^{\alpha(\beta\wedge 2\alpha)}} & \textit{if } \alpha\in(0,1/2].\end{cases}$$

Proof. We denote $b_m, \overline{X}^m$ and $\overline{X}^{m,n}$ as in the proof of Theorem 3.2.6. Then it holds that

$$\begin{aligned}&\mathbb{E}[\sup_{0\le t\le T}|X_t - X_t^{(n)}|]\\ &\le \left\{\mathbb{E}[\sup_{0\le t\le T}|X_t - \overline{X}_t^m|] + \mathbb{E}[\sup_{0\le t\le T}|\overline{X}_t^{m,n} - X_t^{(n)}|] + \mathbb{E}[\sup_{0\le t\le T}|\overline{X}_t^m - \overline{X}_t^{m,n}|]\right\}. \end{aligned}\tag{3.33}$$

From (3.30) and (3.32), it holds that

$$\mathbb{E}[\sup_{0\le t\le T}|\overline{X}_t^m - \overline{X}_t^{m,n}|] \le \begin{cases} \dfrac{C_{10}e^{72K_\sigma^2\|b\|_\infty m}}{\sqrt{\log n}} & \text{if } \alpha = 0,\\ \dfrac{C_{10}e^{72K_\sigma^2\|b\|_\infty m}}{n^{\alpha(\beta\wedge 2\alpha)}} & \text{if } \alpha \in (0, 1/2],\end{cases}\tag{3.34}$$

where $C_{10} \equiv C_{10}(x_0, K_\sigma, \|b\|_\infty, T, \alpha, \beta)$ is a finite constant which depends neither on n nor on m.

Since $\{\sup_{0\le t\le T}|X_t - \overline{X}_t^m| \neq 0\} \subset \{\sup_{0\le t\le T}|X_t| \ge m\}$, it follows from (3.24) with $\tau = T$ that

$$\begin{aligned}\left(\mathbb{E}[\sup_{0\le t\le T}|X_t - \overline{X}_t^m|]\right)^2 &= \left(\mathbb{E}[\sup_{0\le t\le T}|X_t - \overline{X}_t^m|\mathbf{1}_{\{\sup_{0\le t\le T}|X_t - \overline{X}_t^m|\neq 0\}}]\right)^2\\ &\le \mathbb{E}[\sup_{0\le t\le T}|X_t - \overline{X}_t^m|^2]\mathbb{P}(\sup_{0\le t\le T}|X_t| \ge m)\\ &\le C_5^2\exp\left(-\frac{m^2}{4K_\sigma^2 T}\right).\end{aligned}\tag{3.35}$$

In the same way, we have

$$\left(\mathbb{E}[\sup_{0\le t\le T}|X_t^{(n)} - \overline{X}_t^{m,n}|]\right)^2 \le C_5^2\exp\left(-\frac{m^2}{4K_\sigma^2 T}\right).\tag{3.36}$$

If $\alpha \in (0, \frac{1}{2}]$, from (3.33), (3.34), (3.35) and (3.36), we have

$$\mathbb{E}[\sup_{0\le t\le T}|X_t - X_t^{(n)}|] \le C_{10}\frac{e^{72K_\sigma^2\|b\|_\infty m}}{n^{\alpha(\beta\wedge 2\alpha)}} + 2C_5\exp\Big(-\frac{m^2}{8K_\sigma^2 T}\Big).$$

Choose $m^2 = 8\alpha(\beta\wedge 2\alpha)K_\sigma^2 T\log n$, we obtain

$$\mathbb{E}[\sup_{0\le t\le T}|X_t - X_t^{(n)}|] \le \frac{C_{10}e^{C_{11}\sqrt{\log n}}}{n^{\frac{\beta}{2}\wedge\alpha}} + \frac{2C_5}{n^{\frac{\beta}{2}\wedge\alpha}},$$

where $C_{11} = 144K_\sigma^3\|b\|_\infty\sqrt{2T\alpha(\beta\wedge 2\alpha)}$. This concludes the statement for $\alpha \in (0, 1/2]$.

If $\alpha = 0$, from (3.33), (3.34), (3.35) and (3.36), we have

$$\mathbb{E}[\sup_{0\le t\le T}|X_t - X_t^{(n)}|] \le \frac{C_{10}e^{72K_\sigma^2\|b\|_\infty m}}{\sqrt{\log n}} + 2C_5\exp\Big(-\frac{m^2}{8K_\sigma^2 T}\Big).$$

Choose $m^2 = 8K_\sigma^2 T\log(\log n)$, we obtain

$$\mathbb{E}[\sup_{0\le t\le T}|X_t - X_t^{(n)}|] \le \frac{C_3 e^{C_{12}\sqrt{\log(\log n)}}}{\sqrt{\log n}} + \frac{2C_5}{\sqrt{\log n}},$$

where $C_{12} = 144\sqrt{2T}K_\sigma^3\|b\|_\infty$. This concludes the statement for $\alpha = 0$. $\square$

The following L^p-norm estimation is useful to construct a Multi-level Monte Carlo simulation for X (see [32]).

Theorem 3.2.10. *Let Assumption 3.2.1 hold. For any $p \geq 2$, then there exists a constant C which may depend on $K_\sigma, \|b_A\|_{\mathcal{A}}, \|b_H\|_\beta, \|b\|_{L^1(\mathbb{R})}, T, x_0, \alpha, \beta$ and p such that*

$$\mathbb{E}[\sup_{0\leq s\leq T} |X_s - X_s^{(n)}|^p] \leq \begin{cases} \dfrac{C}{\log n} & \text{if } \alpha = 0, \\ \dfrac{C}{n^{\frac{\beta}{2}\wedge\alpha}} & \text{if } \alpha \in (0, 1/2), \\ \dfrac{C}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}}} & \text{if } \alpha = 1/2. \end{cases}$$

If we suppose that σ is Lipschitz continuous and $b \in H^\beta$, i.e. $b_{\mathcal{A}} \equiv 0$, then Theorem 3.2.10 implies the following result which improves the one in [90] for SDEs with non-constant diffusion.

Corollary 3.2.11. *Assume that $b \in L^1(\mathbb{R}) \bigcap H^\beta$ for some $\beta \in (0, 1]$ and the diffusion coefficient σ is Lipschitz continuous and uniformly elliptic. Then for any $p \geq 1$, there exists positive constant C which depends on $K_\sigma, \|b\|_\beta, \|b\|_{L^1(\mathbb{R})}, T, x_0, \alpha, \beta$ and p such that*

$$\mathbb{E}[\sup_{0\leq s\leq T} |X_s - X_s^{(n)}|^p] \leq \frac{C}{n^{p\beta/2}}.$$

Remark 3.2.12.

1. Gyöngy [38] studied the rate of convergence in the almost sure sense of the Euler-Maruyama approximation for SDEs with irregular drift. He showed that the rate is $1/4$ when σ is locally Lipschitz and b satisfies an one-sided Lipschitz type condition.
2. In the case that $\beta = 1$, the results of Theorems 3.2.5, 3.2.8 and 3.2.10 were proven in [97] under a further assumption that b is one-sided Lipschitz. In this chapter, thanks to the method of removal drift we are able to get rid of this assumption. Note that if σ is Lipschitz function, the strong rate of the Euler-Maruyama approximation mentioned in Theorem 3.2.5 is $1/2$.

To prove Theorem 3.2.10, we need the following Gronwall type inequality.

Lemma 3.2.13 ([40], Lemma 3.2.)**.** *Let $(Z_t)_{t\geq 0}$ be a nonnegative continuous stochastic process and set $V_t := \sup_{s\leq t} Z_s$. Assume that for some $r > 0$, $q \geq 1$, $\rho \in [1, q]$ and some constants $\overline{C}_0$ and $\xi \geq 0$,*

$$\mathbb{E}[V_t^r] \leq \overline{C}_0 \mathbb{E}\left[\left(\int_0^t V_s \mathrm{d}s\right)^r\right] + \overline{C}_0 \mathbb{E}\left[\left(\int_0^t Z_s^\rho \mathrm{d}s\right)^{r/q}\right] + \xi < \infty$$

for all $t \geq 0$. Then for each $T \geq 0$ the following statements hold.
(i) If $\rho = q$ then there exists a constant $\overline{C}_1$ depending on $\overline{C}_0, T, q$ and r such that

$$\mathbb{E}[V_T^r] \leq \overline{C}_1 \xi.$$

(ii) If $r \geq q$ or $q + 1 - \rho < r < q$ hold, then there exists constant $\overline{C}_2$ depending on $\overline{C}_0, T, \rho, q$ and r, such that

$$\mathbb{E}[V_T^r] \leq \overline{C}_2 \xi + \overline{C}_2 \int_0^T \mathbb{E}[Z_s] \mathrm{d}s.$$

Proof of Theorem 3.2.10. Throughout this proof, the letter K denotes some positive constant whose value can change from line to line. The constant K may depend on $K_\sigma, \|b_A\|_{\mathcal{A}}, \|b_H\|_\beta, \|b\|_\infty, \|b\|_{L^1(\mathbb{R})}, T, x_0, \alpha$ and β but it does not depend on n. We will use again the estimates (3.7) and (3.8). Let us first consider the expectation of $\sup_{0\leq s\leq T}|M_s^{n,\delta,\varepsilon}|^p$. Using Burkholder-Davis-Gundy's inequality, for any $t\in[0,T]$, $\delta\in(1,\infty)$ and $\varepsilon\in(0,1)$, we have

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|M_s^{n,\delta,\varepsilon}|^p\right]\leq K\mathbb{E}\left[\left(\int_0^t\left|\varphi'(X_s)\sigma(X_s)-\varphi'(X_s^{(n)})\sigma(X_{\eta_n(s)}^{(n)})\right|^2\mathrm{d}s\right)^{p/2}\right].$$

Note that for any bounded Lipschitz continuous function g with Lipschitz constant $L>0$, it holds that

$$\sup_{x,y\in\mathbb{R},x\neq y}\frac{|g(x)-g(y)|}{|x-y|^\gamma}\leq(2\|g\|_\infty)^{1-\gamma}L^\gamma,\ \text{for any }\gamma\in(0,1]. \tag{3.37}$$

Since σ is bounded and $1/2+\alpha$-Hölder continuous, ϕ' is bounded by 1 and φ',φ'' are bounded, by using (3.37) for $g=\varphi'$ with $\gamma=1/2+\alpha$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}|M_s^{n,\delta,\varepsilon}|^p\right]\leq&K\mathbb{E}\left[\left(\int_0^t\left|\varphi'(X_s)-\varphi'(X_s^{(n)})\right|^2\mathrm{d}s\right)^{p/2}\right]+K\mathbb{E}\left[\left(\int_0^t\left|\sigma(X_s)-\sigma(X_s^{(n)})\right|^2\mathrm{d}s\right)^{p/2}\right]\\&+K\mathbb{E}\left[\left(\int_0^t\left|\sigma(X_s^{(n)})-\sigma(X_{\eta_n(s)}^{(n)})\right|^2\mathrm{d}s\right)^{p/2}\right]\\\leq&K\mathbb{E}\left[\left(\int_0^t\left|X_s-X_s^{(n)}\right|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right]+K\mathbb{E}\left[\left(\int_0^T\left|X_s^{(n)}-X_{\eta_n(s)}^{(n)}\right|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right]\\\leq&K\mathbb{E}\left[\left(\int_0^t\left|X_s-X_s^{(n)}\right|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right]+\frac{K}{n^{p/4+p\alpha/2}}.\end{aligned} \tag{3.38}$$

For $\alpha\in(0,1/2]$, it follows from (3.16) that for any $t\in[0,T]$, we have

$$\begin{aligned}\left|V_t^{(n)}\right|^p\leq&K\Bigg\{\frac{1}{n^{p/2}}+\sup_{0\leq s\leq t}\left|M_s^{n,2,n^{-1/2}}\right|^p+\int_0^T\left\{\left|b_A(X_s^{(n)})-b_A(X_{\eta_n(s)}^{(n)})\right|^p+K\left|X_s^{(n)}-X_{\eta_n(s)}^{(n)}\right|^{p\beta}\right\}\mathrm{d}s\\&+\int_0^T\left|X_s^{(n)}-X_{\eta_n(s)}^{(n)}\right|^{p/2+p\alpha}\mathrm{d}s+\frac{1}{n^{p/2}}+\frac{1}{n^{p\alpha}}+n^{p/2}\int_0^T\left|X_s^{(n)}-X_{\eta_n(s)}^{(n)}\right|^{p+2p\alpha}\mathrm{d}s\Bigg\}.\end{aligned} \tag{3.39}$$

By taking the expectation on (3.39), from Lemma 3.2.2 and 3.2.3, we have

$$\mathbb{E}\left[\left|V_t^{(n)}\right|^p\right]\leq K\mathbb{E}\left[\sup_{0\leq s\leq t}\left|M_s^{n,2,n^{-1/2}}\right|^p\right]+\frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}\wedge p\alpha}}. \tag{3.40}$$

Since $\alpha\leq 1/4+\alpha/2$, from (3.38) and (3.40), we obtain

$$\mathbb{E}\left[\left|V_s^{(n)}\right|^p\right]\leq K\mathbb{E}\left[\left(\int_0^t\left|X_s-X_s^{(n)}\right|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right]+\frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}\wedge p\alpha}}. \tag{3.41}$$

If $\alpha = 1/2$, from Jensen's inequality, we have

$$\mathbb{E}\left[\left|V_t^{(n)}\right|^p\right] \leq K \int_0^t \mathbb{E}\left[\left|X_s - X_s^{(n)}\right|^p\right] \mathrm{d}s + \frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}}} \leq K \int_0^t \mathbb{E}\left[\left|V_s^{(n)}\right|^p\right] \mathrm{d}s + \frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}}}.$$

Using Gronwall's inequality, we have

$$\mathbb{E}\left[\left|V_s^{(n)}\right|^p\right] \leq \frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}}}. \tag{3.42}$$

If $\alpha \in (0, 1/2)$, then from (3.41), for any $t \in [0, T]$ we have

$$\mathbb{E}\left[\left|V_s^{(n)}\right|^p\right] \leq K\mathbb{E}\left[\left(\int_0^t V_s^{(n)} \mathrm{d}s\right)^p\right] + K\mathbb{E}\left[\left(\int_0^t \left|X_s - X_s^{(n)}\right|^{1+2\alpha} \mathrm{d}s\right)^{p/2}\right] + \frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}\wedge p\alpha}}.$$

Using Lemma 3.2.13 (ii) with $r = p$, $q = 2$, $\rho = 1 + 2\alpha$, $\xi = Kn^{-(\frac{1}{2}\wedge\frac{p\beta}{2}\wedge p\alpha)}$ and Theorem 3.2.5, we have

$$\mathbb{E}\left[\left|V_t^{(n)}\right|^p\right] \leq \frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}\wedge p\alpha}} + K \int_0^t \left[\left|X_s - X_s^{(n)}\right|\right] \mathrm{d}s \leq \frac{K}{n^{\frac{1}{2}\wedge\frac{p\beta}{2}\wedge p\alpha}} + \frac{K}{n^{\frac{\beta}{2}\wedge\alpha}} \leq \frac{K}{n^{\frac{\beta}{2}\wedge\alpha}}. \tag{3.43}$$

This concludes the case of $\alpha \in (0, 1/2]$.

For $\alpha = 0$, it follows from (3.18) that for any $t \in [0, T]$, we have

$$\begin{aligned}\left|V_t^{(n)}\right|^p \leq & K\Bigg\{\frac{1}{(\log n)^p} + \sup_{0\leq s\leq t}\left|M_s^{n,n^{1/3},(\log n)^{-1}}\right|^p \\ & + \int_0^t \left\{\left|b_A(X_s^{(n)}) - b_A(X_{\eta_n(s)}^{(n)})\right|^p + K^p\left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^{p\beta}\right\} \mathrm{d}s \\ & + \int_0^t \left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^{p/2} \mathrm{d}s + \frac{1}{(\log n)^{2p}} \\ & + \frac{1}{(\log n)^p} + n^{p/3} \int_0^T \left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^p \mathrm{d}s\Bigg\}.\end{aligned} \tag{3.44}$$

By taking the expectation on (3.44), from Lemma 3.2.2 and 3.2.3, we have

$$\mathbb{E}\left[\left|V_t^{(n)}\right|^p\right] \leq K\mathbb{E}\left[\sup_{0\leq s\leq t}\left|M_s^{n,n^{1/3},(\log n)^{-1}}\right|^p\right] + \frac{K}{(\log n)^p}. \tag{3.45}$$

From (3.38) and (3.45), we obtain

$$\mathbb{E}\left[\left|V_t^{(n)}\right|^p\right] \leq K\mathbb{E}\left[\left(\int_0^t \left|X_s - X_s^{(n)}\right| \mathrm{d}s\right)^{p/2}\right] + \frac{K}{(\log n)^p}.$$

Using Lemma 3.2.13 (ii) with $r = p$, $q = 2$, $\rho = 1$, $\xi = K(\log n)^{-p}$ and Theorem 3.2.5, we have

$$\mathbb{E}\left[\left|V_T^{(n)}\right|^p\right] \leq \frac{K}{(\log n)^p} + K \int_0^T \mathbb{E}\left[\left|X_s - X_s^{(n)}\right|\right] \mathrm{d}s \leq \frac{K}{\log n}. \tag{3.46}$$

This concludes the case for $\alpha = 0$. □

Remark 3.2.14. Note that it is hardly possible to obtain a L^p bound for the error if we remove the condition $b \in L^1(\mathbb{R})$ by following the method used in the proofs of Theorem 3.2.6 and 3.2.9 since a careful tracking of constants K will show that the constants K in (3.42), (3.43) and (3.46) increase with the order of double exponent with respect to $\|b\|_{L^1(\mathbb{R})}$. This makes the localization technique for b not applicable.

3.3 SDEs with discontinuous diffusion coefficient

3.3.1 Notations and assumptions

Throughout this section the following notations are used. For any continuous semimartingale Y, we denote $L_t^x(Y)$ the symmetric local time of Y up to time t at the level $x \in \mathbb{R}$ (see [77]). For bounded measurable function f on $\mathbb{R}$, we define $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$. We denote by $L^1(\mathbb{R})$ the space of all integrable functions with respect to Lebesgue measure on $\mathbb{R}$ with semi-norm $\|f\|_{L^1(\mathbb{R})} := \int_{\mathbb{R}} |f(x)| \mathrm{d}x$. For each $\beta \in (0,1]$ and $\kappa > 0$, we denote by $H^{\beta,\kappa}$ the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists a measurable subset $S(f)$ of $\mathbb{R}$ satisfying

(i) $\|f\|_\beta := \|f\|_\infty + \sup_{x<y;[x,y]\cap S(f)=\emptyset} \dfrac{|f(x)-f(y)|}{|x-y|^\beta} < \infty$; and

(ii) $C_{\beta,\kappa} := \sup_{K\geq 1} \sup_{\varepsilon>0} \dfrac{\lambda(S(f)^\varepsilon \cap [-K,K])}{K\varepsilon^\kappa} < +\infty$ where λ denotes the Lebesgue measure on $\mathbb{R}$ and $S(f)^\varepsilon$ is the ε-neighbourhood of $S(f)$, i.e., $S(f)^\varepsilon = \{y \in \mathbb{R} : \text{ there exists } x \in S(f) \text{ such that } |x-y| \leq \varepsilon\}$.

Here are some remarks on the class $H^{\beta,\kappa}$.

Remark 3.3.1. (i) $H^{\beta,\kappa}$ is a vector space on $\mathbb{R}$, i.e., if $a, b \in \mathbb{R}$ and $f, g \in H^{\beta,\kappa}$ then $af + bg \in H^{\beta,\kappa}$.

(ii) A bounded function f is called piecewise β-Hölder if there exist a positive constant L and a sequence $-\infty = s_0 < s_1 < s_2 < \ldots < s_m < s_{m+1} = \infty$ such that $|f(u) - f(v)| \leq L|u-v|^\beta$ for any u, v satisfying $s_k < u < v < s_{k+1}$. It is easy to verify that such function $f \in H^{\beta,1}$, $S(f) = \{s_1, \ldots, s_m\}$ and $C_{\beta,1} \leq 2m$.

(iii) The following ζ is a non-trivial example of function of $H^{\beta,\kappa}$ with $\kappa < 1$. For each $\hat{\beta}, \kappa \in (0,1)$, we denote

$$\zeta(x) = \begin{cases} \frac{x-1}{2x-1} & \text{if } x \leq 0, \\ 1 + \frac{\log 2}{\log(n+1)} x^{\hat{\beta}} & \text{if } (n+1)^{-1/(1-\kappa)} \leq x < n^{-1/(1-\kappa)} \text{ and } n \in \mathbb{N}, \\ \frac{3x+1}{x+1} & \text{if } x \geq 1. \end{cases} \tag{3.47}$$

It can be shown that ζ is a strictly increasing function with an infinite number of discontinuous points which are cumulative at 0, $\frac{1}{2} < \zeta < 3$, and $\zeta \in H^{\beta,\kappa}$ with $\beta = \frac{1+\hat{\beta}-\kappa}{2-\kappa}$, $S(\zeta) = \{n^{-1/(1-\kappa)}, n = 1, 2, \ldots\}$ and $C_{\beta,\kappa} \leq 3$.

We need the following assumptions on the diffusion coefficient σ.

Assumption 3.3.2. *(i) There exists a bounded and strictly increasing function f_σ such that for any $x, y \in \mathbb{R}$,*

$$|\sigma(x) - \sigma(y)|^2 \leq |f_\sigma(x) - f_\sigma(y)|.$$

(ii) σ is bounded and uniformly positive, i.e. there exist positive constants $\overline{\sigma}$ and $\underline{\sigma}$ such that for any $x \in \mathbb{R}$,

$$\underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}.$$

Le Gall [77] has shown that if b is bounded measurable, and σ satisfies Assumption 3.3.2, then there exists a unique strong solution to SDE (3.1) (see also [94]). We now give some remarks on the Assumption 3.3.2.

Remark 3.3.3. (i) The function $\sigma(x) = 1 + \mathbf{1}_{x \geq 0}$ satisfies Assumption 3.3.2 and belongs to $H^{1,1}$.

(ii) The function ζ defined in (3.47) also satisfies Assumption 3.3.2.

(iii) If $a, b > 0$ and σ_1, σ_2 satisfies Assumption 3.3.2, then $a\sigma_1 + b\sigma_2$ also satisfies Assumption 3.3.2.

(iv) Let f_1, f_2 be two strictly increasing, piecewise 1-Hölder functions. Let $\widetilde{\rho}$ be a 1/2-Hölder continuous function satisfying $0 < \inf_{x\in\mathbb{R}} \widetilde{\rho}(x) \leq \sup_{x\in\mathbb{R}} \widetilde{\rho}(x) < \infty$. Then $\sigma := \widetilde{\rho} \circ (f_1 - f_2)$ is piecewise 1/2-Hölder and it satisfies Assumption 3.3.2 with $f_\sigma = C(f_1 + f_2)$ for some positive constant C.

3.3.2 Error estimates for the Euler-Maruyama approximation

We obtain the following results on the rate of the Euler-Maruyama approximation with discontinuous coefficients in L^1-norm.

Theorem 3.3.4. *Let Assumption 3.3.2 hold, and $b, \sigma \in H^{\beta,\kappa}$ for some $\beta \in (0, 1]$ and $\kappa > 0$.*

(i) There exists a constant C such that for all $n \geq 3$,

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t - X_t^{(n)}|] \leq \frac{Ce^{C\sqrt{\log\log n}}}{\log n}. \tag{3.48}$$

(ii) Moreover, if $b \in L^1(\mathbb{R})$, then there exists a constant C such that for all $n \geq 3$,

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t - X_t^{(n)}|] \leq \frac{C}{\log n}. \tag{3.49}$$

The estimates (3.48) and (3.49) were obtained in [40, 97, 99] under a stronger assumption that σ is 1/2-Hölder continuous on $\mathbb{R}$.

For proving Theorem 3.3.4, we derive a key estimation (Lemma 3.3.8). The next estimation is a uniform L^2-bounded of the local time of solution of SDE (3.1) and its EM approximation.

Lemma 3.3.5. *Suppose that b is bounded, measurable and σ is measurable and satisfies Assumption 3.3.2 (ii). For each $\theta \in [0,1]$, define*

$$\begin{aligned} V_t^{(n)}(\theta) &:= (1-\theta)X_t + \theta X_t^{(n)}. \\ &= x_0 + \int_0^t \left\{(1-\theta)b(X_s) + \theta b(X_{\eta_n(s)}^{(n)})\right\} \mathrm{d}s + \int_0^t \left\{(1-\theta)\sigma(X_s) + \theta\sigma(X_{\eta_n(s)}^{(n)})\right\} \mathrm{d}W_s. \end{aligned}$$

Then it holds that

$$\sup_{\theta\in[0,1],x\in\mathbb{R}} \mathbb{E}[|L_T^x(V^{(n)}(\theta))|^2] \leq 12\|b\|_\infty^2 T^2 + 6\overline{\sigma}^2 T. \tag{3.50}$$

Proof. By using the symmetric Itô-Tanaka formula, we have

$$\begin{aligned} L_T^x(V^{(n)}(\theta)) =& |V_T^{(n)}(\theta) - x| - |x_0 - x| - \int_0^T \Big(\mathbf{1}(V_s^{(n)}(\theta) > x) - \mathbf{1}(V_s^{(n)}(\theta) < x)\Big) \mathrm{d}V_s^{(n)}(\theta) \\ \leq& |V_T^{(n)}(\theta) - x_0| + \left|\int_0^T \Big(\mathbf{1}(V_s^{(n)}(\theta) > x) - \mathbf{1}(V_s^{(n)}(\theta) < x)\Big) \mathrm{d}V_s^{(n)}(\theta)\right| \\ \leq& 2\int_0^T \left|(1-\theta)b(X_s) + \theta b(X_{\eta_n(s)}^{(n)})\right| \mathrm{d}s + \left|\int_0^T \left\{(1-\theta)\sigma(X_s) + \theta\sigma(X_{\eta_n(s)}^{(n)})\right\} \mathrm{d}W_s\right| \\ &+ \left|\int_0^T \Big(\mathbf{1}(V_s^{(n)}(\theta) > x) - \mathbf{1}(V_s^{(n)}(\theta) < x)\Big) \left\{(1-\theta)\sigma(X_s) + \theta\sigma(X_{\eta_n(s)}^{(n)})\right\} \mathrm{d}W_s\right|. \end{aligned}$$

Since b and σ are bounded, it follows from inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and the L^2-isometry that,

$$\begin{aligned} \sup_{\theta\in[0,1],x\in\mathbb{R}} \mathbb{E}[|L_T^x(V^{(n)}(\theta))|^2] &\leq 12\|b\|_\infty^2 T^2 + 6 \sup_{\theta\in[0,1],x\in\mathbb{R}} \int_0^T \mathbb{E}\Big[\left|(1-\theta)\sigma(X_s) + \theta\sigma(X_{\eta(s)}^{(n)})\right|^2\Big] \mathrm{d}s \\ &\leq 12\|b\|_\infty^2 T^2 + 6\overline{\sigma}^2 T. \end{aligned}$$

This concludes the statement. □

The following lemma, which is similar to Lemma 2.2 in [121], plays a crucial role in our argument.

Lemma 3.3.6. *Assume that b and σ are bounded measurable. For any $\varepsilon, \chi > 0$ such that $\delta := \frac{\chi\varepsilon^4}{8(T\|b\|_\infty^4 + 2^7\overline{\sigma}^4)} \leq T$, it holds that for any $t \geq 0$ and $n \in \mathbb{N}$, $\mathbb{P}(\sup_{t\leq r\leq t+\delta} |X_r^{(n)} - X_t^{(n)}| \geq \varepsilon) \leq \delta\chi$.*

Proof. Let $t \in [0,T]$ be fixed. We define $Z_s^{(n)} := X_{t+s}^{(n)} - X_t^{(n)}$. Then using Burkholder-Davis-Gundy's inequality, it holds that for any $\delta \in [0,T]$,

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq\delta} |Z_s^{(n)}|^4\right] &\leq 8\mathbb{E}\left[\sup_{0\leq s\leq\delta}\left|\int_t^{t+s} b(X_{\eta_n(r)}^{(n)}) dr\right|^4\right] + 8\mathbb{E}\left[\sup_{0\leq s\leq\delta}\left|\int_t^{t+s} \sigma(X_{\eta_n(r)}^{(n)}) \mathrm{d}W_r\right|^4\right] \\ &\leq 8\delta^3\mathbb{E}\left[\int_t^{t+\delta} \left|b(X_{\eta_n(r)}^{(n)})\right|^4 dr\right] + 2^{10}\delta\mathbb{E}\left[\int_t^{t+\delta} \left|\sigma(X_{\eta_n(r)}^{(n)})\right|^4 dr\right] \end{aligned}$$

$$\leq 8\|b\|_\infty^4 \delta^4 + 2^{10}\overline{\sigma}^4\delta^2 \leq 8\left(\|b\|_\infty^4 T^2 + 2^7\overline{\sigma}^4\right)\delta^2.$$

Hence, for any $\varepsilon, \chi > 0$ such that $\delta := \frac{\chi\varepsilon^4}{8(T^2\|b\|_\infty^4 + 2^7\overline{\sigma}^4)} \leq T$, from Markov's inequality, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{t\leq s\leq t+\delta}|X_s^{(n)} - X_t^{(n)}| \geq \varepsilon\right) &\leq \frac{1}{\varepsilon^4}\mathbb{E}\left[\sup_{t\leq s\leq t+\delta}|X_s^{(n)} - X_t^{(n)}|^4\right] = \frac{1}{\varepsilon^4}\mathbb{E}\left[\sup_{0\leq s\leq \delta}|Z_s^{(n)}|^4\right] \\ &\leq \frac{8\left(\|b\|_\infty^4 T^2 + 2^7\overline{\sigma}^4\right)\delta^2}{\varepsilon^4} = \delta\chi,\end{aligned}$$

which concludes the statement. □

Lemma 3.3.6 directly implies the following result.

Lemma 3.3.7. *Assume that b and σ are bounded measurable. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $\gamma_n \in (0,1]$ and $\gamma_n \downarrow 0$ and $\gamma_n n^2 \to \infty$ as $n \to \infty$. Denote $\varepsilon_n := \frac{\widetilde{c}}{\gamma_n^{1/4} n^{1/2}}$, $\widetilde{c} := 2^{3/4}T^{1/2}\{T^2\|b\|_\infty^4 + 2^7\overline{\sigma}^4\}^{1/4}$, $\chi_n := \frac{\gamma_n n}{T}$, $\delta_n := \frac{\chi_n\varepsilon_n^4}{8(T^2\|b\|_\infty^4 + 2^7\overline{\sigma}^4)} = \frac{T}{n}$. For each $k = 0, \ldots, n-1$, we define*

$$\Omega_{k,n,\varepsilon_n} := \left\{\omega \in \Omega \middle| \sup_{t_k^{(n)} \leq s \leq t_{k+1}^{(n)}} |X_s^{(n)}(\omega) - X_{t_k^{(n)}}^{(n)}(\omega)| \geq \varepsilon_n\right\}.$$

Then it holds that $\mathbb{P}(\Omega_{k,n,\varepsilon_n}) \leq \delta_n\chi_n = \gamma_n$.

Now we state the key lemma of our demonstration.

Lemma 3.3.8. *Let Assumption 3.3.2 (ii) hold and the drift coefficient b be bounded and measurable. Let $f \in H^{\beta,\kappa}$ for some $\beta \in (0,1]$. Then for any $p \geq 1$ and $0 < \alpha < \frac{p\beta}{2} \wedge \frac{2\kappa}{\kappa+4}$, there exists a positive constant $C_p^*(f) = C^*(p, \alpha, \beta, \kappa, T, x_0, \|f\|_\beta, C_{\beta,\kappa}, \|b\|_\infty, \overline{\sigma}, \underline{\sigma})$ which does not depend on n such that for each $n \geq 3$,*

$$\int_0^T \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^p\right] \mathrm{d}s \leq \frac{C_p^*(f)}{n^\alpha \log n}. \tag{3.51}$$

Proof. From Lemma 3.3.7 and the boundedness of f, it holds that

$$\begin{aligned}&\int_0^T \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^p\right] \mathrm{d}s \\ &= \sum_{k=0}^{n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)})\right|^p \left(\mathbf{1}_{\Omega_{k,n,\varepsilon_n}} + \mathbf{1}_{\Omega_{k,n,\varepsilon_n}^c}\right)\right] \mathrm{d}s \\ &\leq 2^p\|f\|_\infty^p T\gamma_n + \sum_{k=0}^{n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)})\right|^p \mathbf{1}_{\Omega_{k,n,\varepsilon_n}^c}\right] \mathrm{d}s.\end{aligned} \tag{3.52}$$

We estimate the second term of (3.52) as follows

$$\sum_{k=0}^{n-1}\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)})\right|^p \mathbf{1}_{\Omega_{k,n,\varepsilon_n}^c}\right] \mathrm{d}s$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E}\left[\left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p \mathbf{1}_{\Omega^c_{k,n,\varepsilon_n}} \mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(f)} \right] \mathrm{d}s \\
&\quad + \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E}\left[\left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p \mathbf{1}_{\Omega^c_{k,n,\varepsilon_n}} \mathbf{1}_{X_s^{(n)} \notin S^{\varepsilon_n}(f)} \right] \mathrm{d}s. \qquad (3.53)
\end{aligned}
$$

On the set $\Omega^c_{k,n,\varepsilon_n} \cap \{X_s^{(n)} \notin S^{\varepsilon_n}(f)\}$, it holds that $S(f) \cap [X_s^{(n)} \wedge X_{t_k^{(n)}}^{(n)}, X_s^{(n)} \vee X_{t_k^{(n)}}^{(n)}] = \emptyset$, thus,

$$
\left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p \mathbf{1}_{\Omega^c_{k,n,\varepsilon_n}} \mathbf{1}_{X_s^{(n)} \notin S^{\varepsilon_n}(f)} \leq \|f\|_\beta^p \left| X_s^{(n)} - X_{t_k^{(n)}}^{(n)} \right|^{p\beta}.
$$

This implies the second term of (3.53) is bounded by

$$
\|f\|_\beta^p \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E}\left[\left| X_s^{(n)} - X_{t_k^{(n)}}^{(n)} \right|^{p\beta} \right] \mathrm{d}s \leq \|f\|_\beta^p T C_{p\beta} n^{-p\beta/2}, \qquad (3.54)
$$

where the last inequality follows from Lemma 3.2.3. For each constant $K_n \geq 1 \vee (|x_0| + T\|b\|_\infty)$, the first term of (3.53) is bounded by

$$
\begin{aligned}
&2^p \|f\|_\infty^p \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \Big(\mathbb{E}\left[\mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(f) \cap [-K_n, K_n]} \right] + \mathbb{E}\left[\mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(f) \setminus [-K_n, K_n]} \right] \Big) \mathrm{d}s \\
&\leq 2^p \|f\|_\infty^p \int_0^T \mathbb{E}\left[\mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(f) \cap [-K_n, K_n]} \right] \mathrm{d}s + 2^p \|f\|_\infty^p \int_0^T \mathbb{E}\left[\mathbf{1}_{|X_s^{(n)}| \geq K_n} \right] \mathrm{d}s. \qquad (3.55)
\end{aligned}
$$

Since σ is uniformly elliptic, $\langle X^{(n)} \rangle_t \geq \underline{\sigma}^2 t$, we obtain

$$
\begin{aligned}
\int_0^T \mathbb{E}\left[\mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(f) \cap [-K_n, K_n]} \right] \mathrm{d}s &\leq \underline{\sigma}^{-2} \mathbb{E}\left[\int_0^T \mathbf{1}_{X_s^{(n)} \in S^{\varepsilon_n}(f) \cap [-K_n, K_n]} \mathrm{d}\langle X^{(n)} \rangle_s \right] \\
&= \underline{\sigma}^{-2} \mathbb{E}\left[\int_{\mathbb{R}} \mathbf{1}_{S^{\varepsilon_n}(f) \cap [-K_n, K_n]}(x) L_T^x(X^{(n)}) \mathrm{d}x \right],
\end{aligned}
$$

where the last equation follows from the occupation time formula. Moreover, it follows from Lemma 3.3.5 that

$$
\begin{aligned}
\mathbb{E}\left[\int_{\mathbb{R}} \mathbf{1}_{S^{\varepsilon_n}(f) \cap [-K_n, K_n]}(x) L_T^x(X^{(n)}) \mathrm{d}x \right] &\leq \int_{\mathbb{R}} \mathbf{1}_{S^{\varepsilon_n}(f) \cap [-K_n, K_n]}(x) \mathbb{E}[L_T^x(X^{(n)})] \mathrm{d}x \\
&\leq \sup_{x \in \mathbb{R}} \mathbb{E}[L_T^x(X^{(n)})] \lambda\Big(S^{\varepsilon_n}(f) \cap [-K_n, K_n] \Big) \\
&\leq \{12\|b\|_\infty^2 T^2 + 6\overline{\sigma}^2 T\}^{1/2} C_{\beta,\kappa} K_n \varepsilon_n^\kappa.
\end{aligned}
$$

Now we consider the second term of (3.55). For each $s \in [0, T]$,

$$
\mathbb{E}\left[\mathbf{1}_{|X_s^{(n)}| \geq K_n} \right] \leq \mathbb{P}\Big(\Big| \int_0^s \sigma(X_{\eta_n(u)}^{(n)}) \mathrm{d}W_u \Big| \geq K_n - \Big| x_0 + \int_0^s b(X_{\eta_n(u)}^{(n)}) \mathrm{d}u \Big| \Big)
$$

$$\leq \mathbb{P}\Big(\Big|\int_0^s \sigma(X^{(n)}_{\eta_n(u)})\mathrm{d}W_u\Big| \geq K_n - \|b\|_\infty T - |x_0|\Big).$$

Since $\langle\int_0^\cdot \sigma(X^{(n)}_{\eta_n(s)})\mathrm{d}W_s\rangle_t \leq \overline{\sigma}^2 T$ almost surely, from Proposition 6.8 of [108] and the inequality $(a-b)^2 \geq a^2/2 - b^2$ for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq t\leq T}\left|\int_0^t \sigma(X^{(n)}_{\eta_n(s)})\mathrm{d}W_s\right| \geq K_n - \|b\|_\infty T - |x_0|\right)\\
&\leq 2\exp\left(-\frac{(K_n - |x_0| - \|b\|_\infty T)^2}{2\overline{\sigma}^2 T}\right) \leq 2\exp\left(\frac{(|x_0| + \|b\|_\infty T)^2}{2\overline{\sigma}^2 T}\right)\exp\left(-\frac{K_n^2}{4\overline{\sigma}^2 T}\right).
\end{aligned}$$

This implies

$$\int_0^T \mathbb{E}\left[\mathbf{1}_{|X^{(n)}_s|\geq K_n}\right]\mathrm{d}s \leq 2T\exp\left(\frac{(|x_0| + \|b\|_\infty T)^2}{2\overline{\sigma}^2 T}\right)\exp\left(-\frac{K_n^2}{4\overline{\sigma}^2 T}\right). \tag{3.56}$$

Gathering together the estimates (3.52) –(3.56), we get

$$\begin{aligned}
\int_0^T \mathbb{E}\left[\left|f(X^{(n)}_s) - f(X^{(n)}_{\eta_n(s)})\right|^p\right]\mathrm{d}s \leq\, &2^p\|f\|^p_\infty T\gamma_n + \|f\|^p_\beta T C_{p\beta} n^{-p\beta/2}\\
&+ 2^p\|f\|^p_\infty \underline{\sigma}^{-2}\{12\|b\|^2_\infty T^2 + 6\overline{\sigma}^2 T\}^{1/2} C_{\beta,\kappa} K_n \varepsilon_n^\kappa\\
&+ 2^{p+1}\|f\|^p_\infty T\exp\left(\frac{(|x_0| + \|b\|_\infty T)^2}{2\overline{\sigma}^2 T}\right)\exp\left(-\frac{K_n^2}{4\overline{\sigma}^2 T}\right).
\end{aligned} \tag{3.57}$$

For each $0 < \alpha < \frac{p\beta}{2} \wedge \frac{2\kappa}{\kappa+4}$, by choosing $K_n = (1 + |x_0| + T\|b\|_\infty + 2\overline{\sigma}\sqrt{T\alpha})\sqrt{\log n}$ and $\gamma_n = \frac{1}{n^\alpha \log n}$, we obtain (3.51) from (3.57). ☐

Now we prove Theorem 3.3.4 by using the key Lemma 3.3.8.

Proof of Theorem 3.3.4. We will only present the detail proof for the case that $b \in L^1(\mathbb{R})$. The proof for the case $b \notin L^1(\mathbb{R})$ is based on the localization technique given in proof of Theorem 3.2.6 and it will be omitted.

We fix $n \geq 3$ and a constant $0 < \alpha < \frac{\beta}{2} \wedge \frac{2\kappa}{\kappa+4}$. Recall that by (3.7) and (3.8), we have

$$|X_t - X^{(n)}_t| \leq C_0\left(\varepsilon + M^{n,\delta,\varepsilon}_t + I^{(n)}_t + J^{(n)}_t\right). \tag{3.58}$$

We first consider $I^{(n)}_t$. Since $\varphi'' = -\frac{2b\varphi'}{\sigma^2}$,

$$|I^{(n)}_t| \leq \int_0^T \left|\phi'_{\delta,\varepsilon}(Y_t - Y^{(n)}_t)\varphi'(X^{(n)}_s)\right|\left|b(X^{(n)}_{\eta_n(s)}) - \frac{b(X^{(n)}_s)\sigma^2(X^{(n)}_{\eta_n(s)})}{\sigma^2(X^{(n)}_s)}\right|\mathrm{d}s.$$

Thanks to Lemma 3.2.4 and estimate (3.5), we have

$$|I^{(n)}_t| \leq K_\sigma^2 C_0 \int_0^T \left|b(X^{(n)}_{\eta_n(s)})\sigma^2(X^{(n)}_s) - b(X^{(n)}_s)\sigma^2(X^{(n)}_{\eta_n(s)})\right|\mathrm{d}s$$

$$\leq K_\sigma^2 C_0 \int_0^T \left\{ K_\sigma^2 \left| b(X_s^{(n)}) - b(X_{\eta_n(s)}^{(n)}) \right| + \|b\|_\infty \left| \sigma^2(X_s^{(n)}) - \sigma^2(X_{\eta_n(s)}^{(n)}) \right| \right\} \mathrm{d}s.$$

It follows from Lemma 3.3.8 that

$$\mathbb{E}[|I_t^{(n)}|] \leq \frac{C_I}{n^\alpha \log n}, \tag{3.59}$$

where $C_I := K_\sigma^2 C_0 \{ K_\sigma^2 C_1^*(b) + 2\|b\|_\infty \overline{\sigma} C_1^*(\sigma)\}$. Now we estimate $J_t^{(n)}$. From (3.6), we have

$$\begin{aligned} J_t^{(n)} &\leq \int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} \left| \varphi'(X_s)\sigma(X_s) - \varphi'(X_s^{(n)})\sigma(X_{\eta_n(s)}^{(n)}) \right|^2 \mathrm{d}s \\ &\leq 3(J_T^{1,n} + J_T^{2,n} + J_T^{3,n}), \end{aligned}$$

where

$$\begin{aligned} J_t^{1,n} &:= \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} |\sigma(X_s)|^2 \left| \varphi'(X_s) - \varphi'(X_s^{(n)}) \right|^2 \mathrm{d}s, \\ J_t^{2,n} &:= \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} |\varphi'(X_s^{(n)})|^2 \left| \sigma(X_s) - \sigma(X_s^{(n)}) \right|^2 \mathrm{d}s, \\ J_t^{3,n} &:= \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} |\varphi'(X_s^{(n)})|^2 \left| \sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)}) \right|^2 \mathrm{d}s. \end{aligned}$$

From Lemma 3.2.4 (ii), φ' is Lipschitz continuous with Lipschitz constant $\|\varphi''\|_\infty$. Hence, we have

$$\begin{aligned} J_T^{1,n} &\leq \frac{K_\sigma^2 \|\varphi''\|_\infty^2}{\log \delta} \int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}|} \left| X_s - X_s^{(n)} \right|^2 \mathrm{d}s \\ &\leq \frac{K_\sigma^2 \|\varphi''\|_\infty^2 C_0^2}{\log \delta} \int_0^T \mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|) \left| Y_s - Y_s^{(n)} \right| \mathrm{d}s \\ &\leq \frac{C_{J,1}\varepsilon}{\log \delta}, \end{aligned} \tag{3.60}$$

where $C_{J,1} := 4K_\sigma^6 C_0^4 \|b\|_\infty^2 T$. Next we consider $J_T^{2,n}$. We first note that by (3.3),

$$J_T^{2,n} \leq \frac{C_0^3}{\log \delta} \int_0^T \frac{\left| \sigma(X_s) - \sigma(X_s^{(n)}) \right|^2}{|X_s - X_s^{(n)}|} \mathbf{1}_{|X_s - X_s^{(n)}| \geq \varepsilon/(C_0\delta)} \mathrm{d}s.$$

Recall that by Assumption 3.3.2 (i), there exists a bounded and strictly increasing function $f_\sigma : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)|^2 \leq |f_\sigma(x) - f_\sigma(y)|.$$

We consider approximation $f_{\sigma,\ell} \in C^1(\mathbb{R})$ of f_σ which is also strictly increasing function and satisfies $\|f_{\sigma,\ell}\|_\infty \leq \|f_\sigma\|_\infty$ and $f_{\sigma,\ell} \uparrow f_\sigma$ as $\ell \to \infty$ on $\mathbb{R}$. Then by using Fatou's lemma and the mean value theorem, we have

$$J_T^{2,n} \leq \frac{C_0^3}{\log \delta} \int_0^T \frac{|f_\sigma(X_s) - f_\sigma(X_s^{(n)})|}{|X_s - X_s^{(n)}|} \mathbf{1}_{|X_s - X_s^{(n)}| > \varepsilon/(C_0\delta)} \mathrm{d}s$$

$$\begin{aligned}
&\leq \liminf_{\ell\to\infty} \frac{C_0^3}{\log\delta}\int_0^T \frac{|f_{\sigma,\ell}(X_s)-f_{\sigma,\ell}(X_s^{(n)})|}{|X_s-X_s^{(n)}|}\mathbf{1}_{|X_s-X_s^{(n)}|>\varepsilon/(C_0\delta)}\mathrm{d}s\\
&\leq \liminf_{\ell\to\infty} \frac{C_0^3}{\log\delta}\int_0^T \mathrm{d}s\int_0^1 \mathrm{d}\theta f'_{\sigma,\ell}(V_s^{(n)}(\theta)),
\end{aligned} \tag{3.61}$$

where $V^{(n)}(\theta)=(V_t^{(n)}(\theta))_{0\leq t\leq T}$ is defined in Lemma 3.3.5. Since $\sigma\geq\underline{\sigma}$, the quadratic variation of $V^{(n)}(\theta)$ satisfies

$$\langle V^{(n)}(\theta)\rangle_t=\int_0^t\Big\{(1-\theta)\sigma(X_s)+\theta\sigma(X_{\eta_n(s)}^{(n)})\Big\}^2\mathrm{d}s\geq\underline{\sigma}^2 t,$$

which implies

$$\begin{aligned}
\int_0^T \mathrm{d}s\int_0^1 \mathrm{d}\theta f'_{\sigma,\ell}(V_s^{(n)}(\theta)) &\leq \underline{\sigma}^{-2}\int_0^1\mathrm{d}\theta\int_0^T \mathrm{d}\langle V^{(n)}(\theta)\rangle_s f'_{\sigma,\ell}(V_s^{(n)}(\theta))\\
&=\underline{\sigma}^{-2}\int_{\mathbb{R}}\mathrm{d}x f'_{\sigma,\ell}(x)\int_0^1\mathrm{d}\theta L_T^x(V^{(n)}(\theta)),
\end{aligned}$$

where the last equality is implied from the occupation time formula. Using Lemma 3.3.5 and the estimate $\|f'_{\sigma,\ell}\|_{L^1(\mathbb{R})}\leq 2\|f_{\sigma,\ell}\|_\infty\leq 2\|f_\sigma\|_\infty$ we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T\mathrm{d}s\int_0^1\mathrm{d}\theta f'_{\sigma,\ell}(V_s^{(n)}(\theta))\right] &\leq \underline{\sigma}^{-2}\int_{\mathbb{R}}\mathrm{d}x f'_{\sigma,\ell}(x)\int_0^1\mathrm{d}\theta\mathbb{E}[L_T^x(V^{(n)}(\theta))]\\
&\leq \underline{\sigma}^{-2}\|f'_{\sigma,\ell}\|_{L^1(\mathbb{R})}\sup_{\theta\in[0,1],x\in\mathbb{R}}\mathbb{E}[|L_T^x(V^{(n)}(\theta))|^2]^{1/2}\\
&\leq 2\underline{\sigma}^{-2}\|f_\sigma\|_\infty\{12\|b\|_\infty^2T^2+6\overline{\sigma}^2T\}^{1/2}.
\end{aligned}$$

By plugging this estimate to (3.61) and using Fatou's lemma, we get the following estimate for the expectation of $J_T^{2,n}$,

$$\mathbb{E}[J_T^{2,n}]\leq\frac{C_{J,2}}{\log\delta}, \tag{3.62}$$

where $C_{J,2}:=2C_0^3\underline{\sigma}^{-2}\|f_\sigma\|_\infty\{12\|b\|_\infty^2T^2+6\overline{\sigma}^2T\}^{1/2}$. Finally, we estimate $J_T^{3,n}$ as follows

$$\mathbb{E}[J_T^{3,n}]\leq\frac{C_0^2\delta}{\varepsilon\log\delta}\int_0^T\mathbb{E}\Big[\Big|\sigma(X_s^n)-\sigma(X_{\eta_n(s)}^{(n)})\Big|^2\Big]\mathrm{d}s.$$

Applying Lemma 3.3.8, we get

$$\mathbb{E}[J_t^{3,n}]\leq\frac{\delta}{\varepsilon\log\delta}\frac{C_{J,3}}{n^\alpha\log n}, \tag{3.63}$$

where $C_{J,3}:=C_0^2C_2^*(\sigma)$. Since $\mathbb{E}[M_t^{n,\delta,\varepsilon}]=0$, it follows from (3.58) - (3.63) that there exists a positive constant C which do not depend on n such that

$$\sup_{0\leq t\leq T}\mathbb{E}[|X_t-X_t^{(n)}|]\leq C\Big(\varepsilon+\frac{1}{n^\alpha\log n}+\frac{\varepsilon}{\log\delta}+\frac{1}{\log\delta}+\frac{\delta}{\varepsilon\log\delta}\frac{1}{n^\alpha\log n}\Big).$$

By choosing $\varepsilon=\frac{1}{\log n}$ and $\delta=n^\alpha$, we obtain the desired result. □

3.3.3 Application to skew diffusion processes

In this subsection, we apply result in section 3.3 to a skew diffusion process which is the unique solution to the following SDE with symmetric local time:

$$X_t = x_0 + \int_0^t \sigma(X_s)\mathrm{d}W_s + (2\alpha - 1)L_t^0(X), \ t \in [0,T], \alpha \in (0,1). \tag{3.64}$$

The skew diffusion (3.64) has a unique (weak or strong) solution by using an equivalent SDE without reflection. To this end, fix $\alpha \in (0,1)$ and we define the following functions:

$$\begin{aligned} s_\alpha(x) &:= (1-\alpha)x\mathbf{1}(x \geq 0) + \alpha x \mathbf{1}(x<0), \\ r_\alpha(x) &:= s_\alpha^{-1}(x) = \frac{x}{(1-\alpha)}\mathbf{1}(x \geq 0) + \frac{x}{\alpha}\mathbf{1}(x<0), \\ f_\alpha(x) &:= \frac{D_- s_\alpha(x) + D_+ s_\alpha(x)}{2} = (1-\alpha)\mathbf{1}(x>0) + \alpha\mathbf{1}(x<0) + \frac{1}{2}\mathbf{1}(x=0). \end{aligned}$$

Here D_- and D_+ denote the left and right derivatives, respectively. Note that $f_\alpha \circ r_\alpha(x) = f_\alpha \circ s_\alpha(x) = f_\alpha(x)$. Then $Z_t := s_\alpha(X_t)$ is a solution of the equation

$$Z_t = z_0 + \int_0^t \rho(Z_s)\mathrm{d}W_s, \ z_0 = s_\alpha(x_0), \tag{3.65}$$

where

$$\rho(z) := f_\alpha(z)\sigma(r_\alpha(z)) = (1-\alpha)\sigma\left(\frac{z}{1-\alpha}\right)\mathbf{1}(z>0) + \alpha\sigma\left(\frac{z}{\alpha}\right)\mathbf{1}(z<0) + \frac{\sigma(0)}{2}\mathbf{1}(z=0),$$

(see [43] and [77]). Indeed, by using the symmetric Itô-Tanaka formula (see e.g. (32) of [78]), we have

$$\begin{aligned} Z_t =& s_\alpha(x_0) + \int_0^t f_\alpha(X_s)\mathrm{d}X_s + \frac{1-2\alpha}{2}L_t^0(X) \\ =& s_\alpha(x_0) + \int_0^t f_\alpha(X_s)\sigma(X_s)\mathrm{d}W_s + (2\alpha-1)\int_0^t f_\alpha(X_s)\mathrm{d}L_s^0(X) + \frac{1-2\alpha}{2}L_t^0(X) \\ =& s_\alpha(x_0) + \int_0^t f_\alpha \circ r_\alpha \circ s_\alpha(X_s)\sigma(r_\alpha(Z_s))\mathrm{d}W_s + (2\alpha-1)f_\alpha(0)L_t^0(X) - \frac{2\alpha-1}{2}L_t^0(X) \\ =& s_\alpha(x_0) + \int_0^t f_\alpha(Z_s)\sigma(r_\alpha(Z_s))\mathrm{d}W_s = z_0 + \int_0^t \rho(Z_s)\mathrm{d}W_s. \end{aligned}$$

Therefore, since $r_\alpha = s_\alpha^{-1}$, Z is a solution of (3.65) if and only if X is a solution of (3.64).

Remark 3.3.9. (i) If σ satisfies Assumption 3.3.2, then ρ also satisfies it. Indeed, for any $x, y \in \mathbb{R}$,

$$\begin{aligned} |\rho(x) - \rho(y)|^2 &\leq 2|f_\alpha(x)|^2|\sigma(r_\alpha(x)) - \sigma(r_\alpha(y))|^2 + 2|\sigma(r_\alpha(y))|^2|f_\alpha(x) - f_\alpha(y)|^2 \\ &\leq 2|f_\sigma \circ r_\alpha(x) - f_\sigma \circ r_\alpha(y)| + 4\|\sigma\|_\infty^2|f_\alpha(x) - f_\alpha(y)| \\ &\leq |f_\rho(x) - f_\rho(y)|, \end{aligned}$$

for some bounded, strictly increasing function f_ρ. Thus, the SDE (3.65) has unique strong solution.

(ii) We consider $\sigma = \widetilde{\rho} \circ (f_1 - f_2)$ where f_1, f_2 are two strictly increasing, piecewise 1-Hölder function, and $\widetilde{\rho}$ be a 1/2-Hölder continuous function satisfying $0 < \inf_{x\in\mathbb{R}} \widetilde{\rho}(x) \leq \sup_{x\in\mathbb{R}} \widetilde{\rho}(x) < \infty$. Then since r_α is strictly increasing function, $\rho \in H^{1/2,1}$ and satisfies Assumption 3.3.2 (see Remark 3.3.1 (ii) and 3.3.3 (iv)).

Let us define a transformed Euler-Maruyama scheme for SDE (3.64). Let $Z^{(n)}$ be the Euler-Maruyama approximation for SDE (3.65), that is

$$Z_t^{(n)} = z_0 + \int_0^t \rho(Z_{\eta_n(s)}^{(n)}) \mathrm{d}W_s.$$

We define a transformed Euler-Maruyama scheme $\widetilde{X}^{(n)}$ for SDE (3.64) by $\widetilde{X}_t^{(n)} := r_\alpha(Z_t^{(n)})$. Since r_α is Lipschitz continuous, from Theorem 3.3.4, we have the following Corollary.

Corollary 3.3.10. *Suppose that $\sigma = \widetilde{\rho} \circ (f_1 - f_2)$ where f_1, f_2 are two strictly increasing, piecewise 1-Hölder function, and $\widetilde{\rho}$ be a 1/2-Hölder continuous function satisfying $0 < \inf_{x\in\mathbb{R}} \widetilde{\rho}(x) \leq \sup_{x\in\mathbb{R}} \widetilde{\rho}(x) < \infty$. Then there exists a constant C such that for all $n \geq 3$,*

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t - \widetilde{X}_t^{(n)}|] \leq \max\left\{\frac{1}{1-\alpha}, \frac{1}{\alpha}\right\} \sup_{0\leq t\leq T} \mathbb{E}[|Z_t - Z_t^{(n)}|] \leq \frac{C}{\log n}.$$

3.3.4 Numerical experiment

In this subsection, we give some simulation results for the Euler-Maruyama approximation for SDE with some simple discontinuous coefficient. We will estimate the weak convergence rate by using statistical analysis approach.

We first consider a skew Brownian motion. A skew Brownian motion $X = (X_t)_{0\leq t\leq T}$ is the unique solution of the following one-dimensional stochastic differential equation:

$$X_t = x_0 + \sigma W_t + (2\alpha - 1)L_t^0(X),\ x_0 \in \mathbb{R},\ t \in [0,T] \text{ and } |2\alpha - 1| \in (0,1). \tag{3.66}$$

Then, the stochastic process $Z_t := s_\alpha(X_t)$ is the unique strong solution of the SDE

$$Z_t = z_0 + \int_0^t \rho(Z_s) \mathrm{d}W_s,\ z_0 = s_\alpha(x_0), \tag{3.67}$$

where $\rho(x) := \sigma f_\alpha(x)$. Since for any bounded measurable function f,

$$\mathbb{E}[f(Z_t)] = \mathbb{E}[f \circ s_\alpha(X_t)], \tag{3.68}$$

by using Proposition 6.4.1, Z_t has the explicit density function

$$q_t(z_0, z) = \frac{p_t(r_\alpha(z_0), r_\alpha(z))}{1-\alpha} \mathbf{1}(z \geq 0) + \frac{p_t(r_\alpha(z_0), r_\alpha(z))}{\alpha} \mathbf{1}(z < 0),$$

where $p_t(x_0, \cdot)$ is the density function of skew Brownian motion (3.66).

We consider the Euler-Maruyama scheme $Z^{(n)}$ for Z. From Theorem 2.2 in [121], for any bounded measurable function $f:\mathbb{R}\to\mathbb{R}$, it holds that

$$\lim_{n\to\infty}|\mathbb{E}[f(Z_T)]-\mathbb{E}[f(Z_T^{(n)})]|=0. \tag{3.69}$$

Note that in this case, by Theorem 3.3.4, there exists $C>0$ such that for any $n\geq 3$,

$$\mathbb{E}[|Z_T-Z_T^{(n)}|]\leq \frac{C}{\log n}.$$

In this subsection, we estimate the weak rate of convergence for (3.69). For given f, we assume the following hypothesis: there exists $\beta>0$ and positive constant C such that for any $n\in\mathbb{N}$,

$$Err(f,n):=|\mathbb{E}[f(Z_T)]-\mathbb{E}[f(Z_T^{(n)})]|\leq \frac{C}{n^{\beta}}. \tag{3.70}$$

Then it holds that

$$\log Err(f,n)\leq -\beta\log n+\log C. \tag{3.71}$$

Thus, by using the regression line of the pair $(\log n, \log Err(f,n))$, we can estimate the rate β.

For the expectation $\mathbb{E}[f(Z_T)]$, using the density function $q_T(z_0,\cdot)$, we compute the benchmark as a Riemann sum, that is, for large $M\in\mathbb{N}$ and $K>0$

$$\begin{aligned}\mathbb{E}[f(Z_T)]&=\int_{\mathbb{R}} f(z)q_T(z_0,z)\mathrm{d}z\approx\int_{z_0-K}^{z_0+K} f(z)q_T(z_0,z)\mathrm{d}z\\&\approx\sum_{m=1}^{M} f\left(z_0+\frac{mK}{M}\right)q_T\left(z_0,z_0+\frac{mK}{M}\right)\frac{K}{M}+\sum_{m=1}^{M} f\left(z_0-\frac{mK}{M}\right)q_T\left(z_0,z_0-\frac{mK}{M}\right)\frac{K}{M}.\end{aligned}$$

Numerical results

We choose the parameters as $z_0=0.1$, $\sigma=1$ and $T=1$. The parameters for Riemann sum are defined by $M=10^9$ and $K=30$. For $n=500,600,\ldots,3000$, we compute the expectation $\mathbb{E}[f(Z_T^{(n)})]$ by using n^2 times Monte Carlo simulations, that is,

$$\mathbb{E}[f(Z_T^{(n)})]\approx\frac{1}{n^2}\sum_{k=1}^{n^2} f(Z_T^{(n,k)}),$$

where $Z_T^{(n,k)}$ has the same law of $Z_T^{(n)}$ and are independent.

These are the tables of a convergence rate β assumed in the hypothesis (3.70) estimated by the regression line of the pair $(\log n, \log Err(f,n))$ for $n=500,600,\ldots,3000$.

Table of numerical results 1

α \ $f(x)$	$\sin(x)$	$\cos(x)$	x	$\lvert x\rvert$	x^2
0.1	0.491987	0.504169	1.039288	0.494289	0.504194
0.2	0.432466	0.510081	1.585548	0.498085	0.511158
0.3	0.568442	0.528448	1.439056	0.509959	0.537020
0.4	1.157831	0.474950	1.357145	0.495725	0.482682
0.6	0.990556	0.515444	1.273079	0.447568	0.554261
0.7	0.925687	0.504754	1.460240	0.480132	0.511856
0.8	0.352484	0.514813	1.699516	0.487289	0.521909
0.9	0.278208	0.531746	1.678396	0.499855	0.531106

Table of numerical results 2

α \ $f(x)$	$\mathbf{1}_{[0,\infty)}(x)$	$\mathbf{1}_{(-\infty,0]}(x)$	$\mathbf{1}_{[-1,1]}(x)$	$x \cdot \mathbf{1}_{[-1,1]}(x)$	$x \cdot \mathbf{1}_{[0,\infty)}(x)$
0.1	0.493563	0.493565	0.487793	0.491943	0.492575
0.2	0.490348	0.490349	0.488123	0.490565	0.496419
0.3	0.507297	0.507299	0.428196	0.407625	0.499839
0.4	0.476116	0.476120	0.141331	0.827703	0.331620
0.6	0.535529	0.535522	1.096105	0.837624	0.231807
0.7	0.517454	0.517449	0.544199	0.394982	0.464200
0.8	0.490731	0.490726	0.543134	0.537530	0.483578
0.9	0.508520	0.508510	0.552991	0.474045	0.484907

Conclusion of numerical experiment

The numerical results shows that, in many cases, the weak convergence rate β estimated by the regression line are 0.5 even if the function f is smooth enough. Therefore, we guess that the weak convergence rate for the Euler-Maruyama approximation of SDEs with discontinuous coefficients may be theoretically 0.5.

3.4 SDEs with monotone Hölder continuous diffusion coefficinet

In this section, we will see that the arguments of section 3.3 are useful to improve the rate of convergence proved by Gyöngy and Rásonyi in [40] under the additional assumption that the diffusion coefficient σ is monotone Hölder continuous. Similar arguments are considered in recent paper [45].

To simplify the discussion, let us consider the SDE without drift coefficient:

$$X_t = x_0 + \int_0^t \sigma(X_s)\mathrm{d}W_s. \tag{3.72}$$

We assume the following assumption for the diffusion coefficient σ.

Assumption 3.4.1. *We assume that the diffusion coefficient σ satisfies the following conditions:*

(i) σ is bounded measurable and uniformly positive, i.e., there exist positive constants $\overline{\sigma}$ and $\underline{\sigma}$ such that $\underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}$.

(ii) σ is β-Hölder continuous function with $\beta \in (0,1)$, that is there exists a positive constant K such that

$$\sup_{x,y\in\mathbb{R},x\neq y} \frac{|\sigma(x)-\sigma(y)|}{|x-y|^{\beta}} \leq K.$$

(iii) there exists a bounded and strictly increasing function $g:\mathbb{R}\to\mathbb{R}$ such that for any $x,y\in\mathbb{R}$,

$$|\sigma(x)-\sigma(y)| \leq |g(x)-g(y)|.$$

Example 3.4.2. (i) Let $\beta \in (0,1)$ and define

$$\sigma(x) := \begin{cases} 1 & \text{if } x < -1, \\ \operatorname{sgn}(x)|x|^{\beta}+2 & \text{if } x \in [-1,1], \\ 3 & \text{if } 1 < x. \end{cases}$$

Then σ is β-Hölder continuous at the origin and satisfies Assumption 3.4.1.

(ii) Let $c:[0,1]\to[0,1]$ be the Cantor function and define

$$\sigma(x) := \begin{cases} 1 & \text{if } x < 0, \\ c(x)+1 & \text{if } x \in [0,1], \\ 2 & \text{if } 1 < x. \end{cases}$$

Then σ is $\log 2/\log 3$-Hölder continuous function and satisfies Assumption 3.4.1.

Remark 3.4.3. Under Assumption 3.4.1, since σ is of bounded variation, thus there exists a unique strong solution for the SDE (3.72), (see Le Gall [77] or Nakao [94]).

We obtain the following results on the rate of the Euler-Maruyama approximation in L^1-norm.

Theorem 3.4.4. *Under Assumption 3.4.1, there exists $C>0$ such that*

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t - X_t^{(n)}|] \leq \frac{C}{n^{\beta/2}}.$$

Remark 3.4.5. It has been shown in [64, 92] that if σ is bounded, uniformly elliptic and β-Hölder continuous with $\beta\in(0,1)$, then

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{(n)})]| \leq \frac{C}{n^{\beta/2}},$$

where $f \in C_b^2$ and the second derivative of f is β-Hölder continuous. Therefore, the strong rate of convergence obtained in Theorem 3.4.4 is the same as the weak convergence rate.

Recall that if σ are linear growth then for any $q > 0$, there exist $C \equiv C(q, K, T)$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^{(n)} - X_{\eta_n(t)}^{(n)}\right|^q\right] \leq \frac{C}{n^{q/2}},$$

(see Lemma 2.2.2). Hence if σ is β-Hölder continuous with $\beta \in (0,1)$, it holds that

$$\int_0^T \mathbb{E}\left[\left|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})\right|^2\right] \mathrm{d}s \leq \frac{CK^2T}{n^\beta}. \tag{3.73}$$

On the other hand, under Assumption 3.4.1, we can improve the standard convergence rate (3.73).

Lemma 3.4.6. *Suppose that Assumption 3.4.1 (i) and (ii) hold. Let f be a bounded, β-Hölder continuous function with $\beta \in (0,1)$ and let g be a bounded, monotone increasing function. Then for any $q \geq 0$, there exists a positive constant C such that*

$$\int_0^T \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \left|g(X_s^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right] \mathrm{d}s \leq C\left\{\frac{1}{n} + \frac{1}{n^{\frac{1+q\beta}{2}}}\right\}. \tag{3.74}$$

In particular, under Assumption 3.4.1, it holds that

$$\int_0^T \mathbb{E}\left[\left|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})\right|^2\right] \mathrm{d}s \leq \frac{C}{n^{\frac{1+\beta}{2}}}. \tag{3.75}$$

Proof. Since f and g are bounded, we have

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \left|g(X_s^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right] \mathrm{d}s \\
&\leq \frac{2^{q+1}\|f\|_\infty \|g\|_\infty T}{n} + \int_{T/n}^T \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \left|g(X_s^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right] \mathrm{d}s.
\end{aligned} \tag{3.76}$$

Hence, we estimate the second part of (3.76).

Since g is monotone increasing function, from Proposition 2.4.2 (ii), there exist a finite positive constant K and a sequence of functions $(g_N)_{N\in\mathbb{N}} \subset C^1(\mathbb{R})$ satisfying:

$$\begin{cases}
\mathcal{A}(i): & g_N \to g \text{ in } L^1_{\ell oc}(\mathbb{R}), \text{ as } N \to \infty, \\
\mathcal{A}(ii): & \sup_{N\in\mathbb{N}} |g_N(x)| \leq K, \\
\mathcal{A}(iii): & \sup\limits_{N\in\mathbb{N}, u>0, a\in\mathbb{R}} (1+\sqrt{u})^{-1} \displaystyle\int_{\mathbb{R}} |g_N'(x+a)| e^{-|x|^2/u} \mathrm{d}x \leq K.
\end{cases}$$

Using the above approximation, we first prove that the second part of the right hand side of (3.76) equals to

$$\lim_{N\to\infty} \int_{T/n}^T \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \cdot \left|g_N(X_s^{(n)}) - g_N(X_{\eta_n(s)}^{(n)})\right|\right] \mathrm{d}s. \tag{3.77}$$

From Corollary 2.2.6, for any $s \in [T/n, T]$, the density of $X_s^{(n)}$ satisfies the Gaussian upper bound, that is there exist $C \geq 1$ and $c > 0$ such that

$$p_s^{(n)}(x) \leq C p_c(s, x_0, x) := C\left(\frac{c}{2\pi s}\right)^{1/2} e^{-c\frac{|x-x_0|^2}{2s}}. \tag{3.78}$$

Therefore, by using Lemma 2.4.10, we have

$$\lim_{N\to\infty} \int_{T/n}^{T} \mathbb{E}\left[\left|g_N(X_s^{(n)}) - g(X_s^{(n)})\right|\right] \mathrm{d}s = 0 \tag{3.79}$$

and

$$\lim_{N\to\infty} \int_{T/n}^{T} \mathbb{E}\left[\left|g_N(X_{\eta_n(s)}^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right] \mathrm{d}s = 0. \tag{3.80}$$

Since σ is bounded, by using (3.79), (3.80) and $||a-b|-|a'-b'|| \leq |a-a'| + |b-b'|$, we have

$$\begin{aligned}
&\left|\int_{T/n}^{T} \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \cdot \left\{\left|g_N(X_s^{(n)}) - g_N(X_{\eta_n(s)}^{(n)})\right| - \left|g(X_s^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right\}\right] \mathrm{d}s\right| \\
&\leq \int_{T/n}^{T} \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \cdot \left\{\left|g_N(X_s^{(n)}) - g(X_s^{(n)})\right| + \left|g_N(X_{\eta_n(s)}^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right\}\right] \mathrm{d}s \\
&\leq 2^q \|f\|_\infty^q \int_{T/n}^{T} \left\{\mathbb{E}\left[\left|g_N(X_s^{(n)}) - g(X_s^{(n)})\right|\right] + \mathbb{E}\left[\left|g_N(X_{\eta_n(s)}^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right]\right\} \mathrm{d}s \to 0,
\end{aligned}$$

as $N \to \infty$. Hence we conclude that the second part of the right hand side of (3.76) equals to (3.77).

Now we consider (3.77). The following estimates are similar to Lemma 2.4.12. Since, σ is β-Hölder continuous, and $W_s - W_{\eta_n(s)}$ and $X_{\eta_n(s)}^{(n)}$ are independent, for any $s \in [T/n, T]$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \cdot \left|g_N(X_s^{(n)}) - g_N(X_{\eta_n(s)}^{(n)})\right|\right] \\
&\leq \|f\|_\beta^q \mathbb{E}\left[\left|X_s^{(n)} - X_{\eta_n(s)}^{(n)}\right|^{q\beta} \cdot \left|g_N(X_s^{(n)}) - g_N(X_{\eta_n(s)}^{(n)})\right|\right] \\
&= \|f\|_\beta^q \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y |\sigma(x)|^{q\beta} |y|^{q\beta} |g_N(x+\sigma(x)y) - g_N(x)| p_{\eta_n(s)}^{(n)}(x) \frac{\exp\left(-\frac{y^2}{2(s-\eta_n(s))}\right)}{\sqrt{2\pi(s-\eta_n(s))}}.
\end{aligned} \tag{3.81}$$

Using the Gaussian upper bounded (3.78) and applying the change of variables $z = \sigma(x)y$, (3.81) is bounded by

$$C \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}z |z|^{q\beta} |g_N(x+z) - g_N(x)| \frac{\exp\left(-c\frac{|x-x_0|^2}{2\eta_n(s)}\right)}{\sqrt{\eta_n(s)}} \frac{\exp\left(-\frac{z^2}{2|\sigma(x)|^2(s-\eta_n(s))}\right)}{\sqrt{s-\eta_n(s)}}. \tag{3.82}$$

Since σ is uniformly elliptic and using the mean value theorem for g_N, (3.82) is less that

$$C \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}z \int_0^1 \mathrm{d}\theta |z|^{1+q\beta} |g_N'(x+\theta z)| \frac{\exp\left(-c\frac{|x-x_0|^2}{2\eta_n(s)}\right)}{\sqrt{\eta_n(s)}} \frac{\exp\left(-\frac{z^2}{2\overline{\sigma}^2(s-\eta_n(s))}\right)}{\sqrt{s-\eta_n(s)}}$$

$$= \frac{C}{\sqrt{\eta_n(s)}} \int_{\mathbb{R}} \mathrm{d}z \int_0^1 \mathrm{d}\theta |z|^{1+q\beta} \int_{\mathbb{R}} \mathrm{d}w |g_N'(w + x_0 + \theta z)| \exp\left(-c\frac{w^2}{2\eta_n(s)}\right) \frac{\exp\left(-\frac{z^2}{2\overline{\sigma}^2(s-\eta_n(s))}\right)}{\sqrt{s-\eta_n(s)}}, \quad (3.83)$$

where we use the change of variable $w = x - x_0$. By using condition $\mathcal{A}$-(iii), (3.83) is bounded by

$$\frac{C}{\sqrt{\eta_n(s)}} \int_{\mathbb{R}} \mathrm{d}z |z|^{1+q\beta} \exp\left(-\frac{z^2}{4\overline{\sigma}^2(s-\eta_n(s))}\right) \frac{\exp\left(-\frac{z^2}{4\overline{\sigma}^2(s-\eta_n(s))}\right)}{\sqrt{s-\eta_n(s)}},$$

$$\leq \frac{C(s-\eta_n(s))^{\frac{1+q\beta}{2}}}{\sqrt{\eta_n(s)}} \int_{\mathbb{R}} \mathrm{d}z \frac{\exp\left(-\frac{z^2}{4\overline{\sigma}^2(s-\eta_n(s))}\right)}{\sqrt{s-\eta_n(s)}} \leq \frac{C(s-\eta_n(s))^{\frac{1+q\beta}{2}}}{\sqrt{\eta_n(s)}} \leq \frac{1}{\sqrt{\eta_n(s)}} \frac{C}{n^{\frac{1+q\beta}{2}}}, \quad (3.84)$$

where we use the fact that the function $|x|^{1+q\beta}\exp(-x^2)$ is bounded. Therefore, we have

$$\int_{T/n}^T \mathbb{E}\left[\left|f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)})\right|^q \left|g(X_s^{(n)}) - g(X_{\eta_n(s)}^{(n)})\right|\right] \mathrm{d}s \leq \frac{C}{n^{\frac{1+q\beta}{2}}} \int_{T/n}^T \frac{1}{\sqrt{\eta_n(s)}} \mathrm{d}s \leq \frac{C}{n^{\frac{1+q\beta}{2}}}. \quad (3.85)$$

This concludes the statement. □

Proof of Theorem 3.4.4. Let $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$. By using (3.4) and Itô's formula, we have

$$|X_t - X_t^{(n)}| \leq \varepsilon + N_t^{n,\delta,\varepsilon} + K_T^{n,\delta,\varepsilon}, \quad (3.86)$$

where

$$N_t^{n,\delta,\varepsilon} := \int_0^t \phi_{\delta,\varepsilon}'(X_s - X_s^{(n)})(\sigma(X_s) - \sigma(X_{\eta_n(s)}^{(n)}))\mathrm{d}W_s$$

and

$$K_t^{n,\delta,\varepsilon} := \frac{1}{2}\int_0^t \phi_{\delta,\varepsilon}''(X_s - X_s^{(n)})|\sigma(X_s) - \sigma(X_{\eta_n(s)}^{(n)})|^2 \mathrm{d}s.$$

Since σ and $\phi_{\delta,\varepsilon}'$ are bounded, $(N_t^{\delta,\varepsilon,n})_{0\leq t\leq T}$ is a martingale so $\mathbb{E}[N_t^{\delta,\varepsilon,n}] = 0$.

Now we estimate $|K_T^{n,\delta,\varepsilon}|$. By the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b > 0$, we have

$$K_T^{n,\delta,\varepsilon} \leq K_T^{1,n,\delta,\varepsilon} + K_T^{2,n,\delta,\varepsilon},$$

where

$$K_T^{1,n,\delta,\varepsilon} := \int_0^T \phi_{\delta,\varepsilon}''(X_s - X_s^{(n)})|\sigma(X_s) - \sigma(X_s^{(n)})|^2 \mathrm{d}s$$

and

$$K_T^{2,n,\delta,\varepsilon} := \int_0^T \phi_{\delta,\varepsilon}''(X_s - X_s^{(n)})|\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2 \mathrm{d}s.$$

We first estimate $K_T^{2,n,\delta,\varepsilon}$. From (3.6), we have

$$K_T^{2,n,\delta,\varepsilon} \leq \frac{\delta}{\varepsilon \log \delta} \int_0^T |\sigma(X_s^{(n)}) - \sigma(X_{\eta_n(s)}^{(n)})|^2 \mathrm{d}s.$$

Since σ satisfies Assumption 3.4.1, from Lemma 3.4.6, we have

$$\mathbb{E}[K_T^{2,n,\delta,\varepsilon}] \leq \frac{CT\delta}{\varepsilon \log \delta} \frac{1}{n^{\frac{1+\beta}{2}}}. \tag{3.87}$$

Now we consider $K_T^{1,n,\delta,\varepsilon}$. Recall that by Assumption 3.4.1 (iii), there exists a bounded and monotone increasing function $g : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| \leq |g(x) - g(y)|.$$

We consider approximation $g_\ell \in C^1(\mathbb{R})$ of g which is strictly increasing function and satisfies $\|g_\ell\|_\infty \leq \|g\|_\infty$ and $g_\ell \uparrow g$ on $\mathbb{R}$. Since σ is β-Hölder continuous, from (3.6) we have

$$\begin{aligned} K_T^{1,n,\delta,\varepsilon} &\leq \frac{2}{\log \delta} \int_0^T \mathbf{1}_{[\varepsilon/\delta, \varepsilon]}(|X_s - X_s^{(n)}|) \frac{|X_s - X_s^{(n)}|^\beta}{|X_s - X_s^{(n)}|} |g(X_s) - g(X_s^{(n)})| \mathrm{d}s \\ &\leq \frac{2\varepsilon^\beta}{\log \delta} \int_0^T \mathbf{1}_{[\varepsilon/\delta, \varepsilon]}(|X_s - X_s^{(n)}|) \frac{|g(X_s) - g(X_s^{(n)})|}{|X_s - X_s^{(n)}|} \mathrm{d}s. \end{aligned}$$

By using the mean value theorem, we have

$$\begin{aligned} K_t^{1,n,\delta,\varepsilon} &\leq \liminf_{\ell \to \infty} \frac{2\varepsilon^\beta}{\log \delta} \int_0^T \mathbf{1}_{[\varepsilon/\delta, \varepsilon]}(|X_s - X_s^{(n)}|) \frac{|g_\ell(X_s) - g_\ell(X_s^{(n)})|}{|X_s - X_s^{(n)}|} \mathrm{d}s \\ &\leq \liminf_{\ell \to \infty} \frac{2\varepsilon^\beta}{\log \delta} \int_0^T \mathrm{d}s \int_0^1 \mathrm{d}\theta g_\ell'(V_s^{(n)}(\theta)), \end{aligned}$$

where $V_t^{(n)}(\theta) = (V_t^{(n)}(\theta))_{0 \leq t \leq T}$ is defined by

$$V_t^{(n)}(\theta) := X_t + \theta(X_t^{(n)} - X_t) = x_0 + \int_0^t \left\{ (1-\theta)\sigma(X_s) + \theta\sigma(X_{\eta_n(s)}^{(n)}) \right\} \mathrm{d}W_s.$$

Since $\sigma \geq \underline{\sigma}$, the quadratic variation of $V^{(n)}(\theta)$ satisfies

$$\langle V^{(n)}(\theta) \rangle_t = \int_0^t \left\{ (1-\theta)\sigma(X_s) + \theta\sigma(X_{\eta_n(s)}^{(n)}) \right\}^2 \mathrm{d}s \geq \underline{\sigma}^2 t.$$

Therefore, from the occupation time formula, we have

$$\begin{aligned} \int_0^T \mathrm{d}s \int_0^1 \mathrm{d}\theta g_\ell'(V_s^{(n)}(\theta)) &\leq \underline{\sigma}^{-2} \int_0^1 \mathrm{d}\theta \int_0^T \mathrm{d}\langle V^{(n)}(\theta) \rangle_s g_\ell'(V_s^{(n)}(\theta)) \\ &= \underline{\sigma}^{-2} \int_{\mathbb{R}} \mathrm{d}a g_\ell'(a) \int_0^1 \mathrm{d}\theta L_T^a(V^{(n)}(\theta)). \end{aligned} \tag{3.88}$$

By using the symmetric Itô-Tanaka formula, we have

$$\begin{aligned}
L_T^a(V^{(n)}(\theta)) &= |V_T^{(n)}(\theta)-a| - |x_0-a| - \int_0^T \Big(\mathbf{1}(V_s^{(n)}(\theta)>a) - \mathbf{1}(V_s^{(n)}(\theta)<a)\Big)\,\mathrm{d}V_s^{(n)}(\theta) \\
&\leq |V_T^{(n)}(\theta)-x_0| + \left|\int_0^T \Big(\mathbf{1}(V_s^{(n)}(\theta)>a) - \mathbf{1}(V_s^{(n)}(\theta)<a)\Big)\,\mathrm{d}V_s^{(n)}(\theta)\right|.
\end{aligned}$$

Since σ is bounded, it follows from inequality $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b > 0$ and the Burkholder-Davis-Gundy inequality that,

$$\begin{aligned}
\sup_{\theta\in[0,1],a\in\mathbb{R}} \mathbb{E}[|L_T^a(V^{(n)}(\theta))|^2] &\leq 4c(2) \sup_{\theta\in[0,1],a\in\mathbb{R}} \int_0^T \mathbb{E}\left[\left|(1-\theta)\sigma(X_s)+\theta\sigma(X_s^{(n)})\right|^2\right]\mathrm{d}s \\
&\leq 4\overline{\sigma}^2 c(2)T. \qquad (3.89)
\end{aligned}$$

By Jensen's inequality and (3.89), we have

$$\sup_{\theta\in[0,1],a\in\mathbb{R}} \mathbb{E}[|L_T^a(V^{(n)}(\theta))|] \leq \sup_{\theta\in[0,1],a\in\mathbb{R}} \mathbb{E}[|L_T^a(V^{(n)}(\theta))|^2]^{1/2} \leq 2\overline{\sigma}\sqrt{c(2)T}. \qquad (3.90)$$

It follows from (3.88), (3.90) and $\|g'_\ell\|_{L^1(\mathbb{R})} \leq 2\|g_\ell\|_\infty \leq 2\|g\|_\infty$, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \mathrm{d}s \int_0^1 \mathrm{d}\theta g'_\ell(V_s^{(n)}(\theta))\right] &\leq \underline{\sigma}^{-2}\int_{\mathbb{R}} \mathrm{d}a g'_\ell(a)\int_0^1 \mathrm{d}\theta\mathbb{E}\left[L_T^a(V^{(n)}(\theta))\right] \\
&\leq 2\overline{\sigma}\sqrt{c(2)T}\underline{\sigma}^{-2}\|g'_\ell\|_{L^1(\mathbb{R})} \leq 4\|g\|_\infty\overline{\sigma}\sqrt{c(2)T}\underline{\sigma}^{-2}.
\end{aligned}$$

Hence, the expectation of $K_T^{1,n,\delta,\varepsilon}$ is bounded by

$$\mathbb{E}[|K_T^{1,n,\delta,\varepsilon}|] \leq \frac{4\|g\|_\infty\overline{\sigma}^3\sqrt{c(2)T}\underline{\sigma}^{-2}\varepsilon^\beta}{\log\delta}. \qquad (3.91)$$

Therefore, it holds from (3.86), (3.87) and (3.91) that

$$\sup_{0\leq t\leq T}\mathbb{E}[|X_t - X_t^{(n)}|] \leq \varepsilon + \frac{C}{\log\delta}\left(\frac{\delta}{\varepsilon}\frac{1}{n^{\frac{1+\beta}{2}}} + \varepsilon^\beta\right),$$

By choosing $\varepsilon = n^{-1/2}$ and $\delta = 2$, we have

$$\sup_{0\leq t\leq T}\mathbb{E}[|X_t - X_t^{(n)}|] \leq \frac{1}{n^{1/2}} + \frac{C}{n^{\beta/2}} \leq \frac{C}{n^{\beta/2}}.$$

This concludes the statement. □

Chapter 4

Approximation for non-smooth functionals of SDEs

4.1 Introduction

Let $(X_t)_{0\le t\le T}$ be the solution to

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x_0 \in \mathbb{R}^d,\ 0 \le t \le T,$$

where W is a d-dimensional Brownian motion. The diffusion $(X_t)_{0\le t\le T}$ is used to model many random dynamical phenomena in many fields of applications. In practice, one often encounters the problem of evaluating functionals of the type $\mathbb{E}[f(X)]$ for some given function $f : C[0,T] \to \mathbb{R}$. For example, in mathematical finance the function f is commonly referred as a *payoff* function. Since they are rarely analytically tractable, these expectations are usually approximated using numerical schemes. One of the most popular approximation methods is the Monte Carlo Euler-Maruyama method which consists of two steps:

1. The diffusion process $(X_t)_{0\le t\le T}$ is approximated using the Euler-Maruyama scheme $(X^h_t)_{0\le t\le T}$ with a small time step $h > 0$:

$$\mathrm{d}X^h_t = b(X^h_{\eta_h(t)})\mathrm{d}t + \sigma(X^h_{\eta_h(t)})\mathrm{d}W_t, \quad X^h_0 = x_0,\ \eta_h(t) = kh,$$

   for $t \in [kh, (k+1)h)$, $k \in \mathbb{N}$.

2. The expectation $\mathbb{E}[f(X)]$ is approximated using $\frac{1}{N}\sum_{i=1}^N f(X^{h,i})$ where $(X^{h,i})_{1\le i\le N}$ are N independent copies of X^h.

This approximation procedure is influenced by two sources of errors: a discretization error and a statistical error

$$Err(f,h) := Err(h) := \mathbb{E}[f(X)] - \mathbb{E}[f(X^h)] \text{ and } \mathbb{E}[f(X^h)] - \frac{1}{N}\sum_{i=1}^N f(X^{h,i}).$$

We say that the Euler-Maruyama approximation (X^h) is of *weak order* $\kappa > 0$ for a class $\mathcal{H}$ of functions f if there exists a constant $K(T)$ such that for any $f \in \mathcal{H}$,

$$|Err(f,h)| \leq K(T)h^{\kappa}.$$

The effect of the statistical error can be handled by the classical central limit theorem or large deviation theory. Roughly speaking, if $f(X_T^h)$ has a bounded variance, the L^2-norm of the statistical error is bounded by $N^{-1/2}Var(X_T^h)^{1/2}$. Hence, if the Euler-Maruyama approximation is of weak order κ, the optimal choice of the number of Monte Carlo iterations should be $N = O(h^{-2\kappa})$ in order to minimize the computational cost. Therefore, it is of both theoretical and practical importance to understand the weak order of the Euler-Maruyama approximation.

It has been shown that under sufficient regularity on the coefficients b and σ as well as f, the weak order of the Euler-Maruyama approximation is 1. This fact is proven by writing the discretization error $Err(f,h)$ as a sum of terms involving the solution of a parabolic partial differential equation (see [10, 35, 64, 93, 113]). It should be noted here that besides the Monte Carlo Euler-Maruyama method, there are many other related approximation schemes for $\mathbb{E}[f(X_T)]$ which have either higher weak order or lower computational cost. For example, one can use Romberg extrapolation technique to obtain very high weak order as long as $Err(h)$ can be expanded in terms of powers of h (see [113]). When f is a Lipschitz function and the strong rate of approximation is known, one can implement a Multi-level Monte Carlo simulation which can significantly reduce the computation cost of approximating $\mathbb{E}[f(X)]$ in many cases (see [32]). It is also worth looking at some algebraic schemes introduced in [74]. However, all the accelerated schemes mentioned above require sufficient regularity condition on the coefficients b, σ and the test function f.

The stochastic differential equations with non-smooth drift appear in many applications, especially when one wants to model sudden changes in the trend of a certain random dynamical phenomenon (see e.g., [64]). There are many papers studying the Euler-Maruyama approximations in this context. In [39] (see also [16]), it is shown that when the drift is only measurable, the diffusion coefficient is non-degenerate and Lipschitz continuous then the Euler-Maruyama approximations converges to the solution of stochastic differential equation. The weak order of the Euler-Maruyama scheme when both coefficients b and σ as well as payoff functions f are Hölder continuous has been studied in [64, 92]. In the papers [65] and [97], the authors studied the weak and strong convergent rates of the Euler-Maruyama scheme for specific classes of stochastic differential equations with discontinuous drift.

The aim of this chapter is to investigate the weak order of the Euler-Maruyama approximation for stochastic differential equations whose diffusion coefficient σ is constant, whereas the drift coefficient b may have a very low regularity, or could even be discontinuous. More precisely, we consider a class $\mathcal{A}$ of functions which contains not only smooth functions but also some discontinuous one such as indicator function. The drift b will then be assumed to be either in $\mathcal{A}$ or α-Hölder continuous. It should be noted that no smoothness assumption on the payoff function f is needed in our framework. As a by product of our method, we establish the weak order of the Euler-Maruyama approximation for some particular functionals f which include the path-wise maximum of the diffusion, integral of diffusion with respect to time as well as the approximation of a diffusion processes killed when it leaves an open set. We also apply our method to study the weak approximation of reflected stochastic differential equation whose drift is Hölder continuous.

4.2 Weak approximation of SDEs

4.2.1 Notations

For an invertible $d \times d$-matrix $A = (A_{i,j})_{1\le i,j\le d}$, we define

$$g_A(x,y) := \frac{\exp\left(-\frac{1}{2}\langle A^{-1}(y-x), y-x\rangle\right)}{(2\pi)^{d/2}\sqrt{\det A}}.$$

In particular we denote $g_c(x,y) = g_{cI}(x,y)$ for $c \in \mathbb{R}$ where the matrix I is the identity matrix.

A function $\zeta : \mathbb{R}^d \to \mathbb{R}$ is called *exponentially bounded* or *polynomially bounded* if there exist positive constants K, p such that $|\zeta(x)| \le Ke^{K|x|}$ or $|\zeta(x)| \le K(1+|x|^p)$, respectively.

Let $\mathcal{A}_{\exp}$ be a class of exponentially bounded functions $\zeta : \mathbb{R}^d \to \mathbb{R}$ such that there exists a sequence of functions $(\zeta_N) \subset C^1(\mathbb{R}^d)$ satisfying:

$$\begin{cases} \mathcal{A}_{\exp}(i): & \zeta_N \to \zeta \text{ in } L^1_{\ell oc}(\mathbb{R}^d), \\ \mathcal{A}_{\exp}(ii): & \sup_N |\zeta_N(x)| + |\zeta(x)| \le Ke^{K|x|}, \\ \mathcal{A}_{\exp}(iii): & \sup_{N,u>0;\ a\in\mathbb{R}^d} e^{-K|a|-Ku} \int_{\mathbb{R}^d} |\nabla\zeta_N(x+a)| \frac{e^{-|x|^2/u}}{u^{(d-1)/2}} \mathrm{d}x < K, \end{cases}$$

for some positive constant K. We call (ζ_N) an approximation sequence of ζ in $\mathcal{A}_{\exp}$.

The following propositions shows that this class is quite large.

Proposition 4.2.1. *i) If $\xi, \zeta \in \mathcal{A}_{\exp}$ then $\xi\zeta \in \mathcal{A}_{\exp}$ and $a_1\xi + a_2\zeta \in \mathcal{A}$ for any $a_1, a_2 \in \mathbb{R}$.*
ii) Suppose that A is a non-singular $d \times d$-matrix, $B \in \mathbb{R}^d$. Then $\zeta \in \mathcal{A}_{\exp}$ iff $\xi(x) := \zeta(Ax+B) \in \mathcal{A}_{\exp}$.

It is easy to verify that the class $\mathcal{A}_{\exp}$ contains all $C^1(\mathbb{R}^d)$ functions which has all first order derivatives polynomially bounded. Furthermore, the class $\mathcal{A}_{\exp}$ contains also some non-smooth functions of the type $\zeta(x) = (x_1 - a)^+$ or $\zeta(x) = I_{a<x<b}$ for some $a, b \in \mathbb{R}^d$. Moreover, we call a function $\zeta : \mathbb{R}^d \to \mathbb{R}$ *monotone in each variable separately* if for each $i = 1, \ldots, d$, the map $x_i \mapsto \zeta(x_1, \ldots, x_i, \ldots, x_n)$ is monotone for all $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d \in \mathbb{R}$.

Proposition 4.2.2. *The Class $\mathcal{A}_{\exp}$ contains all exponentially bounded functions which are monotone in each variable separately.*

The proofs of Propositions 4.2.1 and 4.2.2 are similar to the proof of Proposition 2.4.2 and therefore will be omitted.

We recall that a function $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ is called α-Hölder continuous for some $\alpha \in (0,1]$ if there exists a positive constant C such that $|\zeta(x) - \zeta(y)| \le C|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$. We denote by $\mathcal{B}(\alpha)$ the class of all measurable functions $b : \mathbb{R}^d \to \mathbb{R}^d$ such that $b = b^H + b^A$ where b^H is α-Hölder continuous for some $\alpha > 0$ and $b^A_j \in \mathcal{A}_{\exp}$ for $j = 1, \ldots, d$.

4.2.2 Error estimates for the Euler-Maruyama approximation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ be a filtered probability space and $(W_t)_{t\ge0}$ be a d-dimensional standard Brownian motion. We consider a d-dimensional stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \sigma W_t, \quad x_0 \in \mathbb{R}^d,\ t \in [0,T], \tag{4.1}$$

where σ is a $d \times d$ deterministic, uniformly elliptic matrix, that is, for the matrix $a := \sigma\sigma^*$, there exist $0 < \underline{a} < \overline{a} < \infty$ such that for any $\xi \in \mathbb{R}^d$,

$$\underline{a}|\xi|^2 \leq \langle a\xi, \xi \rangle \leq \overline{a}|\xi|^2,$$

and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a Borel measurable function. Let $X^h, h > 0$, denote the Euler-Maruyama approximation of X,

$$X_t^h = x_0 + \int_0^t b(X_{\eta_h(s)}^h)\mathrm{d}s + \sigma W_t, \quad t \in [0, T], \tag{4.2}$$

where $\eta_h(s) = kh$ if $kh \leq s < (k+1)h$ for some nonnegative integer k. In this chapter, we study the convergent rates of the error

$$Err(h) = \mathbb{E}[f(X)] - \mathbb{E}[f(X^h)]$$

as $h \to 0$ for some payoff function $f : C[0, T] \to \mathbb{R}$.

Remark 4.2.3. Note that the uniformly elliptic condition plays an important role in establishing the convergence of the Euler-Maruyama approximation for SDEs with non-Lipschitz coefficients. In fact, Hairer, Hutzenthaler and Jentzen [41, Theorem 5.1] constructed a class of 4-dimensional SDEs whose drift coefficient is a smooth, bounded and non-Lipschitz function, and diffusion coefficient is a deterministic non-uniformly elliptic matrix for which the Euler-Maruyama scheme does not converges with any polynomial rate, that is

$$\lim_{h \searrow 0} \frac{\mathbb{E}[|X_T - X_T^h|]}{h^\alpha} = \lim_{h \searrow 0} \frac{|\mathbb{E}[X_T] - \mathbb{E}[X_T^h]|}{h^\alpha} = \begin{cases} 0 & \text{if } \alpha = 0, \\ \infty & \text{if } \alpha > 0. \end{cases}$$

A Borel measurable function $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ is called *sub-linear growth* if ζ is bounded on compact sets and $\zeta(y) = o(|y|)$ as $y \to \infty$. ζ is called *linear growth* if $|\zeta(y)| < c_1|y| + c_2$ for some positive constants c_1, c_2. It has been shown recently in [48] that when b is of super-linear growth, i.e., there exist constants $C > 0$ and $\theta > 1$ such that $|\zeta(y)| \geq |y|^\theta$ for all $|y| > C$, then the Euler-Maruyama approximation (4.2) converges neither in the strong mean square sense nor in weak sense to the exact solution at a finite time point. It means that if $\mathbb{E}[|X_T|^p] < \infty$ for some $p \in [1, \infty)$ then

$$\lim_{h \to 0} \mathbb{E}\big[|X_T - X_T^h|^p\big] = \infty \quad \text{and} \quad \lim_{h \to 0} \big|\mathbb{E}\big[|X_T|^p - |X_T^h|^p\big]\big| = \infty.$$

Thus, in this chapter we will consider the case that b is of at most linear growth.

Remark 4.2.4. In the one-dimensional case, $d = 1$, it is well-known that if $\sigma \neq 0$ and b is of linear growth, then the strong existence and path-wise uniqueness hold for the equation (4.1) (see [18]).

In the multidimensional case, $d > 1$, it has been shown in [116] that if b is bounded then the equation (4.1) has a strong solution and the solution of (4.1) is strongly unique. Moreover, if σ is the identity matrix, then the equation (4.1) has a unique strong solution in the class of continuous processes such that $\mathbb{P}\big(\int_0^T |b(X_s)|^2\mathrm{d}s < \infty\big) = 1$ provided that $\int_{\mathbb{R}^d} |b(y)|^p\mathrm{d}y < \infty$ for some $p > d \vee 2$ (see [72]).

Throughout this chapter, we suppose that equation (4.1) has a weak solution which is unique in the sense of probability law (see Chapter 5 [60])).

Change of Measures

From now on, we will use the following notations

$$Z_t = e^{Y_t}, \quad Y_t = \int_0^t (\sigma^{-1}b)_j(x_0+\sigma W_s)\mathrm{d}W_s^j - \frac{1}{2}\int_0^t |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s,$$
$$Z_t^h = e^{Y_t^h}, \quad Y_t^h = \int_0^t (\sigma^{-1}b)_j(x_0+\sigma W_{\eta_h(s)})\mathrm{d}W_s^j - \frac{1}{2}\int_0^t |\sigma^{-1}b(x_0+\sigma W_{\eta_h(s)})|^2\mathrm{d}s,$$

where we use Einstein's summation convention on repeated indices. We also use the following auxiliary stopping times

$$\tau_D = \inf\{t>0 : X_t \notin D\} \;\text{ and }\; \tau_D^h = \inf\{kh>0 : X_{kh}^h \notin D, k=0,1,\ldots\}$$
$$\tau_D^W = \inf\{t\geq 0 : x_0+\sigma W_t \notin D\} \;\text{ and }\; \tau_D^{W,h} = \inf\{kh\geq 0 : x_0+\sigma W_{kh} \notin D, k=0,1,\ldots\}.$$

Lemma 4.2.5. *Suppose that b is a function with at most linear growth, then we have the following representations*

$$\mathbb{E}[f(X)] - \mathbb{E}[f(X^h)] = \mathbb{E}[f(x_0+\sigma W)(Z_T - Z_T^h)], \tag{4.3}$$

and

$$\begin{aligned}&\mathbb{E}[g(X_T)\mathbf{1}_{(\tau_D>T)}] - \mathbb{E}[g(X_T^h)\mathbf{1}_{(\tau_D^h>T)}]\\&\qquad = \mathbb{E}[g(x_0+\sigma W_T)(Z_T\mathbf{1}_{(\tau_D^W>T)} - Z_T^h\mathbf{1}_{(\tau_D^{W,h}>T)})],\end{aligned} \tag{4.4}$$

for all measurable functions $f : C[0,T]\to\mathbb{R}$ and $g:\mathbb{R}^d\to\mathbb{R}$ provided that all the above expectations are integrable.

Proof. Let σ^{-1} be the inverse matrix of σ. Since b is of linear growth, so is $\sigma^{-1}b$. Thus, there exist constants $c_1, c_2 > 0$ such that $|b(x)| < c_1|x| + c_2$ for any $x\in\mathbb{R}^d$. For any $0\leq t\leq t_0\leq T$,

$$|X_t| \leq |x_0| + |\sigma W_t| + \int_0^t |b(X_s)|\mathrm{d}s \leq |x_0| + |\sigma|\sup_{0\leq s\leq t_0}|W_s| + c_2t_0 + c_1\int_0^t |X_s|\mathrm{d}s.$$

Applying Gronwall's inequality for $t\in[0,t_0]$, one obtains

$$\begin{aligned}|X_{t_0}| &\leq (|x_0| + |\sigma|\sup_{0\leq s\leq t_0}|W_s| + c_2t_0)e^{c_1t_0}\\&\leq (|x_0| + c_2T)e^{c_1T} + |\sigma|e^{c_1T}\sup_{0\leq s\leq t_0}|W_s|.\end{aligned} \tag{4.5}$$

On the other hand, for each integer $k\geq 1$, one has

$$\begin{aligned}|X_{kh}^h| &\leq |X_{(k-1)h}^h| + h|b(X_{(k-1)h}^h)| + 2|\sigma|\sup_{0\leq t\leq kh}|W_t|\\&\leq (1+hc_1)|X_{(k-1)h}^h| + hc_2 + 2|\sigma|\sup_{0\leq t\leq kh}|W_t|.\end{aligned}$$

Hence, a simple iteration yields that

$$|X_{kh}^h| \le (1+hc_1)^k |x_0| + (hc_2 + 2|\sigma| \sup_{0\le t\le kh} |W_t|)\frac{(1+hc_1)^{k-1}-1}{hc_1}.$$

Thus, for any $t \in (0,T]$,

$$|X_{\eta_h(t)}^h| \le (1+hc_1)^{T/h}|x_0| + \frac{c_2(1+hc_1)^{T/h}}{c_1} + 2|\sigma|\frac{(1+hc_1)^{T/h}}{hc_1}\sup_{0\le s\le \eta_h(t)}|W_s|.$$

Moreover,

$$|X_t^h - X_{\eta_h(t)}^h| \le c_1 h|X_{\eta_h(t)}^h| + c_2 h + 2|\sigma|\sup_{0\le s\le t}|W_t|.$$

Therefore, for any $t\in(0,T]$, we have

$$|X_t^h| \le (1+c_1h)^{1+T/h}\frac{c_1|x_0|+c_2}{c_1} + c_2 h + \frac{2|\sigma|(1+hc_1)^{1+T/h}+2hc_1}{hc_1}\sup_{0\le s\le t}|W_s|. \tag{4.6}$$

We define new measures $\mathbb{Q}$ and $\mathbb{Q}^h$ as

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} &= \exp\Big(-\int_0^T (\sigma^{-1}b)_j(X_s)\mathrm{d}W_s^j - \frac{1}{2}\int_0^T |\sigma^{-1}b(X_s)|^2\mathrm{d}s\Big),\\
\frac{\mathrm{d}\mathbb{Q}^h}{\mathrm{d}\mathbb{P}} &= \exp\Big(-\int_0^T (\sigma^{-1}b)_j(X_{\eta_h(s)}^h)\mathrm{d}W_s^j - \frac{1}{2}\int_0^T |\sigma^{-1}b(X_{\eta_h(s)}^h)|^2\mathrm{d}s\Big).
\end{aligned}$$

It follows from Corollary 3.5.16 [60] together with estimates (4.5) and (4.6) that $\mathbb{Q}$ and $\mathbb{Q}^h$ are probability measures. Furthermore, it follows from Girsanov theorem that processes $B = \{(B_t^1,\dots,B_t^d),\ 0\le t\le T\}$ and $B^h = \{(B_t^{h,1},\dots,B_t^{h,d}),\ 0\le t\le T\}$ defined by

$$B_t^j = W_t^j + \int_0^t (\sigma^{-1}b)_j(X_s)\mathrm{d}s,\ B_t^{h,j} = W_t^j + \int_0^t (\sigma^{-1}b)_j(X_{\eta_h(s)})\mathrm{d}s,\ 1\le j\le d, 0\le t\le T,$$

are d-dimensional Brownian motions with respect to $\mathbb{Q}$ and $\mathbb{Q}^h$, respectively. Note that $X_s = x_0 + \sigma B_s$ and $X_s^h = x_0 + \sigma B_s^h$. Therefore,

$$\begin{aligned}
\mathbb{E}[f(X)] &= \mathbb{E}_{\mathbb{Q}}\Big[f(X)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big]\\
&= \mathbb{E}_{\mathbb{Q}}\Big[f(x_0+\sigma B)\exp\Big(\int_0^T (\sigma^{-1}b)_j(x_0+\sigma B_s)\mathrm{d}B_s^j - \frac{1}{2}\int_0^T |\sigma^{-1}b(x_0+\sigma B_s)|^2\mathrm{d}s\Big)\Big]\\
&= \mathbb{E}[f(x_0+\sigma W)Z_T].
\end{aligned}$$

Repeating the previous argument leads to $\mathbb{E}[f(X^h)] = \mathbb{E}[f(x_0+\sigma W)Z_T^h]$, which implies (4.3). The proof of (4.4) is similar and will be omitted. □

From now on, we will use the representation formulas in Lemma 4.2.5 to analyze the weak rate of convergence. We need the following estimates on the moments of Z and Z^h.

Lemma 4.2.6. *Suppose that b is of sub-linear growth. Then for any $p > 0$,*

$$\mathbb{E}[|Z_T|^p + |Z_T^h|^p] \le C < \infty,$$

for some constant C which is not depend on h.

Proof. It suffices to proof the statement for $p \ge 1$. Using Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}[e^{pY_T}] &= \mathbb{E}\Big[\exp\Big(p\int_0^T (\sigma^{-1}b)_j(x_0+\sigma W_s)\mathrm{d}W_s^j - \frac{p}{2}\int_0^T |\sigma^{-1}b(x+\sigma W_s)|^2\mathrm{d}s\Big)\Big] \\
&= \mathbb{E}\Big[\exp\Big(p\int_0^T (\sigma^{-1}b)_j(x_0+\sigma W_s)\mathrm{d}W_s^j - p^2\int_{t_{n-1}}^{t_n} |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s+ \\
&\quad + (p^2-\frac{p}{2})\int_0^T |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s\Big)\Big] \\
&\le \Big\{\mathbb{E}\Big[\exp\Big(2p\int_0^T (\sigma^{-1}b)_j(x_0+\sigma W_s)\mathrm{d}W_s^j - 2p^2\int_0^T |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s\Big)\Big]\Big\}^{1/2} \\
&\quad \times \Big\{\mathbb{E}\Big[\exp\Big((2p^2-p)\int_0^T |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s\Big)\Big]\Big\}^{1/2}.
\end{aligned}$$

Since b is of linear growth, so is $\sigma^{-1}b$ and it follows from Corollary 3.5.16 [60] that

$$\mathbb{E}\Big[\exp\Big(2p\int_0^T (\sigma^{-1}b)_j(x_0+\sigma W_s)\mathrm{d}W_s^j - 2p^2\int_0^T |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s\Big)\Big] = 1. \tag{4.7}$$

On the other hand, since b is bounded on compact sets and $b(y) = o(|y|)$ as $y \to \infty$, for any $\delta > 0$ sufficiently small, there exists a constant $M > 0$ such that $|\sigma^{-1}b(x_0+\sigma y)|^2 \le \delta|y|^2 + M$ for any $y \in \mathbb{R}^d$. Thus,

$$\begin{aligned}
\int_0^T |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s \le \int_0^T (\delta|W_s|^2 + M)\mathrm{d}s &\le TM + T\delta\sup_{s\le T}|W_s|^2 \\
&\le TM + T\delta\sum_{j=1}^d ((\sup_{s\le T} W_s^j)^2 + (\inf_{s\le T} W_s^j)^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big((2p^2-p)\int_0^T |\sigma^{-1}b(x_0+\sigma W_s)|^2\mathrm{d}s\Big)\Big] \\
&\le e^{(2p^2-p)MT}\mathbb{E}\Big[\exp\Big(T\delta(2p^2-p)\sum_{j=1}^d ((\sup_{s\le T} W_s^j)^2 + (\inf_{s\le T} W_s^j)^2)\Big)\Big] \\
&\le e^{(2p^2-p)MT}\Big(\mathbb{E}\Big[\exp\Big(2T\delta(2p^2-p)|W_T^1|^2\Big)\Big]\Big)^{d/2},
\end{aligned}$$

where the last inequality follows from Hölder's inequality and the fact that

$$\sup_{s\le T} W_s^j \overset{law}{=} -\inf_{s\le T} W_s^j \overset{law}{=} |W_T^1|.$$

Because $\mathbb{E}\Big[\exp\Big(2T\delta(2p^2-p)W_T^2\Big)\Big] < \infty$ if one chooses $\delta < (4T^2(2p^2-p))^{-1}$, we obtain $\mathbb{E}[|Z_T|^p] < \infty$. Furthermore, since equation (4.7) still holds if one replaces $b(x_0+\sigma W_s)$ with $b(x_0+\sigma W_{\eta_h(s)})$, a similar argument yields $\mathbb{E}[|Z_T^h|^p] < \infty$. □

Remark 4.2.7. In general, the conclusion of Lemma 4.2.6 is no longer correct if we only suppose that b is of linear growth or even Lipschitz.

Indeed, consider the particular case that $d=1, \sigma=1$ and $b(x)=x$, which is a Lipschitz function. It follows from Hölder's inequality that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\frac{p}{2}\int_0^T W_s^2 \mathrm{d}s\Big)\Big]\mathbb{E}\Big[\exp\Big(p\int_0^T W_s \mathrm{d}W_s - \frac{p}{2}\int_0^T W_s^2\mathrm{d}s\Big)\Big]\\
&= e^{-pT/2}\mathbb{E}\Big[\exp\Big(\frac{p}{2}\int_0^T W_s^2 \mathrm{d}s\Big)\Big]\mathbb{E}\Big[\exp\Big(\frac{p}{2}W_T^2 - \frac{p}{2}\int_0^T W_s^2\mathrm{d}s\Big)\Big]\\
&\geq e^{-pT/2}\Big(\mathbb{E}\big[e^{pW_T^2/4}\big]\Big)^2.
\end{aligned}$$

Furthermore, for any $p, T > 0$ such that $pT \geq 2$ and $pT^2 < 1/2$, we have $\mathbb{E}\big[e^{pW_T^2/4}\big] = \infty$, whereas

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(\frac{p}{2}\int_0^T W_s^2\mathrm{d}s\Big)\Big] &\leq \mathbb{E}\Big[\exp\Big(\frac{pT}{2}\sup_{s\leq T}|W_s|^2\Big)\Big]\\
&\leq \mathbb{E}\Big[\exp\Big(\frac{pT}{2}(\sup_{s\leq T} W_s)^2 + \frac{pT}{2}(\inf_{s\leq T} W_s)^2)\Big)\Big]\\
&\leq \Big(\mathbb{E}\big[e^{pT|W_T|^2}\big]\Big)^2 < \infty.
\end{aligned}$$

Therefore,

$$\mathbb{E}\Big[\exp\Big(p\int_0^T W_s\mathrm{d}W_s - \frac{p}{2}\int_0^T W_s^2\mathrm{d}s\Big)\Big] = \infty, \quad \text{if} \quad pT\geq 2,\ pT^2 < \frac{1}{2}.$$

Some auxiliary estimates

From now on, we will repeatedly use without mentioning the following elementary estimate

$$\sup_{x\in\mathbb{R}} |x|^p e^{k|x|-x^2} < \infty, \quad \text{for any } p\geq 0,\ k\in\mathbb{R}. \tag{4.8}$$

Throughout this section, a symbol C stands for a positive generic constant independent of the discretization parameter h, which nonetheless may depend on time T, coefficients b, σ and payoff function f.

The following result plays a crucial role in our argument.

Lemma 4.2.8. *For any $\zeta \in \mathcal{A}_{\exp}$, any $p\geq 1$, $t > s > 0$,*

$$\mathbb{E}[|\zeta(W_t)-\zeta(W_s)|^p] \leq C_p\frac{\sqrt{t-s}}{\sqrt{s}}, \tag{4.9}$$

for some constant C_p not depending on neither t nor s. On the other hand, if ζ is α-Hölder continuous then

$$\mathbb{E}[|\zeta(W_t)-\zeta(W_s)|^p] \leq C_p(t-s)^{p/2}. \tag{4.10}$$

Proof. If $\zeta \in \mathcal{A}_{\exp}$, let (ζ_N) be an approximate sequence of ζ in $\mathcal{A}_{\exp}$. Since $\zeta_N \to \zeta$ in $L^1_{\ell oc}(\mathbb{R}^d)$ and ζ and ζ_N are uniformly exponential bounded, we have

$$\mathbb{E}[|\zeta(W_t) - \zeta(W_s)|^p)] = \lim_{N\to\infty} \mathbb{E}[|\zeta_N(W_t) - \zeta_N(W_s)|^p]. \tag{4.11}$$

Next, we will show that

$$\sup_{N\in\mathbb{N}} \mathbb{E}[|\zeta_N(W_t) - \zeta_N(W_s)|^p] \leq C\frac{\sqrt{t-s}}{\sqrt{s}}. \tag{4.12}$$

Indeed, we write

$$\begin{aligned}
&\mathbb{E}[|\zeta_N(W_t) - \zeta_N(W_s)|^p] \\
&= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y |\zeta_N(x+y) - \zeta_N(x)|^p \frac{e^{-|x|^2/2s}}{(2\pi s)^{d/2}} \frac{e^{-|y|^2/2(t-s)}}{(2\pi(t-s))^{d/2}} \\
&\leq C \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y |\zeta_N(x+y) - \zeta_N(x)| (e^{K(|x+y|} + e^{K|y|})^{p-1} \frac{e^{-|x|^2/2s}}{(2\pi s)^{d/2}} \frac{e^{-|y|^2/2(t-s)}}{(2\pi(t-s))^{d/2}} \\
&\leq C \sum_{i=1}^d \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \int_0^1 \mathrm{d}\theta \Big| y_i \frac{\partial \zeta_N(x+\theta y)}{\partial x_i} \Big| e^{K(p-1)(|x|+2|y|)} \frac{e^{-|x|^2/2s}}{(2\pi s)^{d/2}} \frac{e^{-|y|^2/2(t-s)}}{(2\pi(t-s))^{d/2}} \\
&\leq C\sqrt{t-s} \sum_{i=1}^d \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \int_0^1 \mathrm{d}\theta \Big| \frac{\partial \zeta_N(x+\theta y)}{\partial x_i} \Big| \frac{e^{-|x|^2/4s}}{(2\pi s)^{d/2}} \frac{e^{-|y|^2/4(t-s)}}{(2\pi(t-s))^{d/2}}.
\end{aligned}$$

It follows from $\mathcal{A}_{\exp}(iii)$ that

$$\sup_N \sum_{i=1}^d \int_{\mathbb{R}^d} \Big| \frac{\partial \zeta_N(x+\theta y)}{\partial x_i} \Big| \frac{e^{-|x|^2/4s}}{(2\pi s)^{(d-1)/2}} \mathrm{d}x \leq Ce^{K|\theta y|},$$

thus

$$\begin{aligned}
\mathbb{E}[|\zeta_N(W_t) - \zeta_N(W_s)|^p] &\leq C\frac{\sqrt{t-s}}{\sqrt{s}} \int_{\mathbb{R}^d} \mathrm{d}y \int_0^1 \mathrm{d}\theta C e^{K|\theta y|} \frac{e^{-|y|^2/4(t-s)}}{(2\pi(t-s))^{d/2}} \\
&\leq C\frac{\sqrt{t-s}}{\sqrt{s}}.
\end{aligned}$$

From (4.11) and (4.12) we get (4.9). The proof of (4.10) is straightforward. ☐

Lemma 4.2.9. *Suppose $\zeta_A \in \mathcal{A}_{\exp}$ and $\zeta_H : \mathbb{R}^d \to \mathbb{R}$ is α-Hölder continuous with $\alpha \in (0,1]$. Let M be a non-negative constant. Then there exists $C > 0$ such that for any $0 < t_1 < t_2 < t_3 < t_4 \leq T$,*

$$\mathbb{E}\left[|\zeta_A(W_{t_2}) - \zeta_A(W_{t_1})||\zeta_A(W_{t_4}) - \zeta_A(W_{t_3})| \sum_{i=1}^4 e^{M|W_{t_i}|} \right] \leq \frac{C\sqrt{t_4-t_3}\sqrt{t_2-t_1}}{\sqrt{t_3-t_2}\sqrt{t_1}}, \tag{4.13}$$

$$\mathbb{E}\left[|\zeta_A(W_{t_2}) - \zeta_A(W_{t_1})||\zeta_H(W_{t_4}) - \zeta_H(W_{t_3})| \sum_{i=1}^4 e^{M|W_{t_i}|} \right] \leq \frac{C(t_4-t_3)^{\alpha/2}\sqrt{t_2-t_1}}{\sqrt{t_1}}, \tag{4.14}$$

$$\mathbb{E}\left[|\zeta_H(W_{t_2})-\zeta_H(W_{t_1})||\zeta_A(W_{t_4})-\zeta_A(W_{t_3})|\sum_{i=1}^4 e^{M|W_{t_i}|}\right] \le \frac{C\sqrt{t_4-t_3}(t_2-t_1)^{\alpha/2}}{\sqrt{t_3-t_2}}, \tag{4.15}$$

$$\mathbb{E}\left[|\zeta_H(W_{t_2})-\zeta_H(W_{t_1})||\zeta_H(W_{t_4})-\zeta_H(W_{t_3})|\sum_{i=1}^4 e^{M|W_{t_i}|}\right] \le C(t_4-t_3)^{\alpha/2}(t_2-t_1)^{\alpha/2}. \tag{4.16}$$

Proof. Let $(\zeta_{A,N})$ be an approximate sequence of ζ_A in $\mathcal{A}_{\exp}$. Since $\zeta_{A,N} \to \zeta_A$ in $L^1_{\ell oc}(\mathbb{R}^d)$ and ζ_A and $\zeta_{A,N}$ are uniformly exponential bounded, we have

$$\begin{aligned}
&\mathbb{E}\left[|\zeta_A(W_{t_2})-\zeta_A(W_{t_1})||\zeta_A(W_{t_4})-\zeta_A(W_{t_3})|\sum_{i=1}^4 e^{M|W_{t_i}|}\right]\\
&=\lim_{N\to\infty}\mathbb{E}\left[|\zeta_{A,N}(W_{t_2})-\zeta_{A,N}(W_{t_1})||\zeta_{A,N}(W_{t_4})-\zeta_{A,N}(W_{t_4})|\sum_{i=1}^4 e^{M|W_{t_i}|}\right].
\end{aligned} \tag{4.17}$$

Next, we will show that

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left[|\zeta_{A,N}(W_{t_2})-\zeta_{A,N}(W_{t_1})||\zeta_{A,N}(W_{t_4})-\zeta_{A,N}(W_{t_4})|\sum_{i=1}^4 e^{M|W_{t_i}|}\right] \le \frac{C\sqrt{t_4-t_3}\sqrt{t_2-t_1}}{\sqrt{t_3-t_2}\sqrt{t_1}}. \tag{4.18}$$

We observe that

$$\begin{aligned}
&\mathbb{E}\left[|\zeta_{A,N}(W_{t_2})-\zeta_{A,N}(W_{t_1})||\zeta_{A,N}(W_{t_4})-\zeta_{A,N}(W_{t_4})|\sum_{i=1}^4 e^{M|W_{t_i}|}\right]\\
&=\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d}\mathrm{d}z\int_{\mathbb{R}^d}\mathrm{d}w|\zeta_{A,N}(z+w)-\zeta_{A,N}(w)||\zeta_{A,N}(x+y+z+w)-\zeta_{A,N}(y+z+w)|\\
&\quad\times\{e^{M|w|}+e^{M|z+w|}+e^{M|y+z+w|}+e^{M|x+y+z+w|}\}g_{t_4-t_3}(x)g_{t_3-t_2}(y)g_{t_2-t_1}(z)g_{t_1}(w)\\
&\le C\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d}\mathrm{d}z\int_{\mathbb{R}^d}\mathrm{d}w|\zeta_{A,N}(z+w)-\zeta_{A,N}(w)||\zeta_{A,N}(x+y+z+w)-\zeta_{A,N}(y+z+w)|\\
&\quad\times g_{c(t_4-t_3)}(x)g_{c(t_3-t_2)}(y)g_{c(t_2-t_1)}(z)g_{ct_1}(w).
\end{aligned} \tag{4.19}$$

Using the mean-value theorem, (4.19) is bounded by

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d}\mathrm{d}z\int_{\mathbb{R}^d}\mathrm{d}w\int_0^1\mathrm{d}\theta\int_0^1\mathrm{d}\delta\left|z_i\frac{\partial\zeta_{A,N}(w+\theta z)}{\partial w_i}\right|\left|x_i\frac{\partial\zeta_{A,N}(y+z+w+\delta x)}{\partial y_i}\right|\\
&\quad\times g_{c(t_4-t_3)}(x)g_{c(t_3-t_2)}(y)g_{c(t_2-t_1)}(z)g_{ct_1}(w)\\
&\le C\sqrt{t_4-t_3}\sqrt{t_2-t_1}\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d}\mathrm{d}z\int_{\mathbb{R}^d}\mathrm{d}w\int_0^1\mathrm{d}\theta\int_0^1\mathrm{d}\delta\\
&\quad\times\left|\frac{\partial\zeta_{A,N}(w+\theta z)}{\partial w_i}\right|\left|\frac{\partial\zeta_{A,N}(y+z+w+\delta x)}{\partial y_i}\right|g_{c(t_4-t_3)}(x)g_{c(t_3-t_2)}(y)g_{c(t_2-t_1)}(z)g_{ct_1}(w)
\end{aligned} \tag{4.20}$$

It follows from $\mathcal{A}_{\exp}(iii)$ that

$$\int_{\mathbb{R}^d}\left|\frac{\partial\zeta_{A,N}(w+\theta z)}{\partial w_i}\right|\frac{e^{-\frac{|w|^2}{2ct_1}}}{(2c\pi t_1)^{(d-1)/2}}\mathrm{d}w \le Ce^{K|\theta z|},$$

$$\int_{\mathbb{R}^d} \left| \frac{\partial \zeta_{A,N}(y+z+w+\delta x)}{\partial y_i} \right| \frac{e^{-\frac{|y|^2}{2c(t_3-t_2)}}}{(2c\pi(t_3-t_2))^{(d-1)/2}} \mathrm{d}y \leq C e^{K|z+w+\delta x|},$$

thus (4.20) is bounded by

$$\begin{aligned}
&\frac{C\sqrt{t_4-t_3}\sqrt{t_2-t_1}}{\sqrt{t_3-t_2}} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} \mathrm{d}w \int_0^1 \mathrm{d}\theta \left| \frac{\partial \zeta_{A,N}(w+\theta z)}{\partial w_i} \right| g_{c(t_4-t_3)}(x) g_{c(t_2-t_1)}(z) g_{ct_1}(w) \\
&\leq \frac{C\sqrt{t_4-t_3}\sqrt{t_2-t_1}}{\sqrt{t_3-t_2}\sqrt{t_1}} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}z g_{c(t_4-t_3)}(x) g_{c(t_2-t_1)}(z) \\
&= \frac{C\sqrt{t_4-t_3}\sqrt{t_2-t_1}}{\sqrt{t_3-t_2}\sqrt{t_1}}.
\end{aligned}$$

From (4.17) and (4.18) we get (4.13).

The proof of (4.14), (4.15) and (4.16) follows from similar arguments. □

Lemma 4.2.10. *Suppose that $b \in \mathcal{B}(\alpha)$.*

(i) If b is bounded, then for any $p \geq 2$, there exists a constant $C > 0$ such that

$$\mathbb{E}[|Y_T - Y_T^h|^p] \leq Ch^{1/2} + Ch^{p\alpha/2}.$$

(ii) If b is of linear growth, then there exists a constant $C > 0$ such that

$$\mathbb{E}[|Y_T - Y_T^h|^2] \leq Ch^{1/2} + Ch^{\alpha}$$

Proof. Using Minkowski's inequality, we obtain $\mathbb{E}[|Y_T - Y_T^h|^p] \leq C\{S_1(p) + S_2(p)\}$ where

$$\begin{aligned}
S_1(p) &= \mathbb{E}\Big[\Big| \int_0^T \big\{(\sigma^{-1}b)_j(x_0+\sigma W_s) - (\sigma^{-1}b)_j(x_0+\sigma W_{\eta_h(s)})\big\} \mathrm{d}W_s^j \Big|^p\Big], \\
S_2(p) &= \mathbb{E}\Big[\Big| \int_0^T \big\{|\sigma^{-1}b(x_0+\sigma W_s)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(s)})|^2\big\} \mathrm{d}s \Big|^p\Big].
\end{aligned}$$

It follows from Burkholder-Davis-Gundy's inequality that,

$$S_1(p) \leq C \sum_{j=1}^d \int_0^T \mathbb{E}\Big[|(\sigma^{-1}b)_j(x_0+\sigma W_s) - (\sigma^{-1}b)_j(x_0+\sigma W_{\eta_h(s)})|^p\Big] \mathrm{d}s.$$

Since b is of linear growth,

$$\sum_{j=1}^d \int_0^h \mathbb{E}\Big[|(\sigma^{-1}b)_j(x_0+\sigma W_s) - (\sigma^{-1}b)_j(x_0+\sigma W_{\eta_h(s)})|^p\Big] \mathrm{d}s \leq Ch.$$

Furthermore, it follows from Proposition 4.2.1 ii) and Lemma 4.2.8 that

$$\sum_{j=1}^d \int_h^T \mathbb{E}\Big[|(\sigma^{-1}b)_j(x_0+\sigma W_s) - (\sigma^{-1}b)_j(x_0+\sigma W_{\eta_h(s)})|^p\Big] \mathrm{d}s$$

$$\leq C\int_h^T \left\{ \frac{\sqrt{s-\eta_h(s)}}{\sqrt{\eta_h(s)}} + |s-\eta_n(s)|^{p\alpha} \right\} \mathrm{d}s \leq C(h^{1/2}+h^{p\alpha/2}).$$

Therefore, $S_1(p) \leq C(h^{1/2}+h^{p\alpha})$.

Proof of (i). We assume that b is bounded. Using Hölder's inequality, we obtain

$$\begin{aligned} S_2(p) &\leq \mathbb{E}\Big[\int_0^T \Big| |\sigma^{-1}b(x_0+\sigma W_s)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(s)})|^2 \Big|^p \mathrm{d}s \Big] \\ &\leq C\sum_{j=1}^d \int_0^T \mathbb{E}\Big[\Big| |(\sigma^{-1}b)_j(x_0+\sigma W_s)|^2 - |(\sigma^{-1}b)_j(x_0+\sigma W_{\eta_h(s)})|^2 \Big|^p \Big] \mathrm{d}s. \end{aligned}$$

Since b is bounded, it holds that for any $x,y\in\mathbb{R}$ and $j=1,\ldots,d$,

$$\Big| |(\sigma^{-1}b)_j(x)|^2 - |(\sigma^{-1}b)_j(y)|^2 \Big|^p \leq C\left\{ |(\sigma^{-1}b_A)_j(x) - (\sigma_A^{-1}b)_j(y)|^p + |(\sigma^{-1}b_H)_j(x) - (\sigma_H^{-1}b)_j(y)|^p \right\}.$$

Thus, by dividing the integral into two parts: from 0 to h and from h to T, and applying a similar argument as above, we obtain

$$S_2(p) \leq Ch + C\int_h^T \frac{\sqrt{s-\eta_h(s)}}{\sqrt{\eta_h(s)}} \mathrm{d}s + Ch^{p\alpha/2} \leq C(h^{1/2}+h^{p\alpha/2}).$$

Thus, $\mathbb{E}[|Y_T - Y_T^h|^p] \leq Ch^{1/2} + Ch^{p\alpha/2}$.

Proof of (ii). We assume that b is of linear growth. We observe that

$$\begin{aligned} S_2(2) = \int_0^T \mathrm{d}u \int_0^u \mathrm{d}s \mathbb{E}\Big[&\left\{ |\sigma^{-1}b(x_0+\sigma W_s)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(s)})|^2 \right\} \\ &\times \left\{ |\sigma^{-1}b(x_0+\sigma W_u)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(u)})|^2 \right\} \Big]. \end{aligned}$$

Let n be a natural number such that $(n-1)h < T \leq nh$. Define $t_i^h = ih$ for $i=0,\ldots,n-1$ and $t_n^h = T$. Since b is of linear growth, we have

$$\begin{aligned} &S_2(2) \\ &\leq Ch + \sum_{i=1}^{n-1} \int_{t_i^h}^{t_{i+1}^h} \mathrm{d}u \left\{ \int_0^{t_1^h} \mathrm{d}s + \int_{t_1^h}^{t_i^h} \mathrm{d}s + \int_{t_i^h}^u \mathrm{d}s \right\} \\ &\quad \times \mathbb{E}\Big[\left\{ |\sigma^{-1}b(x_0+\sigma W_s)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(s)})|^2 \right\} \left\{ |\sigma^{-1}b(x_0+\sigma W_u)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(u)})|^2 \right\} \Big] \\ &\leq Ch + \sum_{i=1}^{n-1} \int_{t_i^h}^{t_{i+1}^h} \mathrm{d}u \int_{t_1^h}^{t_i^h} \mathrm{d}s \\ &\quad \times \mathbb{E}\Big[\left\{ |\sigma^{-1}b(x_0+\sigma W_s)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(s)})|^2 \right\} \left\{ |\sigma^{-1}b(x_0+\sigma W_u)|^2 - |\sigma^{-1}b(x_0+\sigma W_{\eta_h(u)})|^2 \right\} \Big]. \end{aligned}$$

Since that for any $x,y\in\mathbb{R}^d$,

$$|\sigma^{-1}b(x) - \sigma^{-1}b(y)|^2$$

$$= \sum_{j=1}^{d} \Big\{ |(\sigma^{-1}b_A)_j(x)|^2 - |(\sigma^{-1}b_A)_j(y)|^2$$
$$+ \{(\sigma^{-1}b_H)_j(x) + \{(\sigma^{-1}b_H)_j(y)\}\{(\sigma^{-1}b_H)_j(x) - \{(\sigma^{-1}b_H)_j(y)\}$$
$$+ 2(\sigma^{-1}b_H)_j(x)\{(\sigma^{-1}b_A)_j(x) - (\sigma^{-1}b_A)_j(y)\} + 2(\sigma^{-1}b_A)_j(y)\{(\sigma^{-1}b_H)_j(x) - (\sigma^{-1}b_H)_j(y)\} \Big\}.$$

by using Lemma 4.2.9 with $t_1 = \eta_h(s)$, $t_2 = s$, $t_3 = \eta_h(u)$ and $t_4 = u$, we obtain

$$S_2(2) \leq Ch + C \sum_{i=1}^{n-1} \int_{t_i^h}^{t_{i+1}^h} \mathrm{d}u \int_{t_1^h}^{t_i^h} \mathrm{d}s \left\{ \frac{h}{\sqrt{\eta_h(u) - s}\sqrt{\eta_h(s)}} + \frac{h^{\frac{1}{2}+\frac{\alpha}{2}}}{\sqrt{\eta_h(s)}} + \frac{h^{\frac{1}{2}+\frac{\alpha}{2}}}{\sqrt{\eta_h(u) - s}} + h^{\alpha} \right\}$$

Therefore from Lemma 4.4.1, we have

$$S_2(2) \leq C(h + h^{\frac{1}{2}+\frac{\alpha}{2}} + h^{\alpha}).$$

Thus, $\mathbb{E}[|Y_T - Y_T^h|^2] \leq Ch^{1/2} + Ch^{\alpha}$. □

Weak rate of convergence

The following results requires no assumption on the smoothness of f.

Theorem 4.2.11. *Suppose that $b \in \mathcal{B}(\alpha)$ and b is of linear growth. Moreover, assume that $f : C[0,T] \to \mathbb{R}$ is bounded. Then*

$$\lim_{h\to 0} \mathbb{E}[f(X^h)] = \mathbb{E}[f(X)].$$

Proof. It follows from Lemma 4.2.10 that Y_T^h converges in probability to Y_T as $h \to 0$. Thus Z_T^h also converges in probability to Z_T as $h \to 0$. Moreover $\mathbb{E}[Z_T^h] = \mathbb{E}[Z_T] = 1$ for all $h > 0$. Therefore, it follows from Proposition 4.12 [57] that

$$\lim_{h\to 0} \mathbb{E}[|Z_T^h - Z_T|] = 0. \tag{4.21}$$

On the other hand, since f is bounded, it follows from (4.3) that

$$|\mathbb{E}[f(X) - f(X^h)]| \leq C\mathbb{E}[|Z_T^h - Z_T|].$$

This estimate together with (4.21) implies the desired result. □

If b is of sub-linear growth, we can obtain the rate of weak convergence as follows.

Theorem 4.2.12. *Suppose that $b \in \mathcal{B}(\alpha)$ and b is of sub-linear growth. Moreover, assume that $f : C[0,T] \to \mathbb{R}$ satisfies $\mathbb{E}[|f(x_0 + \sigma W)|^r] < \infty$ for some $r > 2$. Then there exists a constant C which does not depend of h such that*

$$|\mathbb{E}[f(X)] - \mathbb{E}[f(X^h)]| \leq Ch^{\frac{\alpha}{2} \wedge \frac{1}{4}}.$$

Proof. It is clear that $|e^x - e^y| \leq (e^x + e^y)|x - y|$. This estimate and Hölder's inequality imply that $|\mathbb{E}[f(X) - f(X^h)]|$ is bounded by

$$\mathbb{E}\big[\big|f(x_0 + \sigma W)(Z_T + Z_T^h)(Y_T - Y_T^h)\big|\big]$$

$$\leq \big\|f(x_0+\sigma W)(Z_T+Z_T^h)\big\|_2\|Y_T-Y_T^h\|_2$$
$$\leq \Big(\mathbb{E}\big[|f(x_0+\sigma W)|^r\big]\Big)^{2/r}\Big(\mathbb{E}\big[|Z_T+Z_T^h|^{2r/(r-2)}\big]\Big)^{(r-2)/r}\|Y_T-Y_T^h\|_2.$$

Thanks to the integrability condition of f and Lemma 4.2.6,

$$\Big(\mathbb{E}\big[|f(x_0+\sigma W)|^r\big]\Big)^{2/r}\Big(\mathbb{E}\big[|Z_T+Z_T^h|^{2r/(r-2)}\big]\Big)^{(r-2)/r}\leq C<\infty.$$

This together with Lemma 4.2.10 implies the desired result. □

Remark 4.2.13. In the paper [84], the author considered the weak rate of convergence of the Euler-Maruyama scheme for equation (4.1) in the case of a one-dimensional diffusion. It was claimed that if b was Lipschitz continuous, the weak rate of approximation is of order 1. However, we would like to point out that the given proof contains several gaps (see for instance Lemma 2 of [84] and Remark 4.2.7 below) which leave us unsure about the claim.

Remark 4.2.14. It has been shown in [64, 92] that for a stochastic differential equation with α-Hölder continuous drift and diffusion coefficients with $\alpha\in(0,1)$, one has

$$|\mathbb{E}[f(X_T)]-\mathbb{E}[f(X_T^h)]|\leq Ch^{\alpha/2},$$

where $f\in C_b^2$ and the second derivative of f is α-Hölder continuous. On the other hand, in [40], Gyöngy and Rásonyi have obtained the strong rate of convergence for a one-dimensional stochastic differential equation whose drift is the sum of a Lipschitz continuous and a monotone decreasing Hölder continuous function, and its diffusion coefficient is Hölder continuous. Recall that in Chapter 2, we improve the results in [40]. More precisely, we assume that the drift coefficient b is a bounded and one-sided Lipschitz function, i.e., there exists a positive constant L such that for any $x,y\in\mathbb{R}^d$, $\langle x-y,b(x)-b(y)\rangle_{\mathbb{R}^d}\leq L|x-y|^2$, $b_j\in\mathcal{A}_{\exp}$ for any $j=1,\ldots,d$ and the diffusion coefficient σ is bounded, uniformly elliptic and $1/2+\alpha$-Hölder continuous with $\alpha\in[0,1/2]$. Then for $h=T/n$, it holds that

$$\mathbb{E}[|X_T-X_T^h|]\leq\begin{cases}C(\log 1/h)^{-1} & \text{if } \alpha=0 \text{ and } d=1,\\ Ch^{\alpha} & \text{if } \alpha\in(0,1/2] \text{ and } d=1,\\ Ch^{1/2} & \text{if } \alpha=1/2 \text{ and } d\geq 2.\end{cases}$$

Therefore, if the payoff function f is Lipschitz continuous, it is straightforward to verify that

$$|\mathbb{E}[f(X_T)-f(X_T^h)]|\leq\begin{cases}C(\log 1/h)^{-1} & \text{if } \alpha=0 \text{ and } d=1,\\ Ch^{\alpha} & \text{if } \alpha\in(0,1/2] \text{ and } d=1,\\ Ch^{1/2} & \text{if } \alpha=1/2 \text{ and } d\geq 2.\end{cases}$$

The following result concerns with the approximation of maximum of SDEs.

Corollary 4.2.15. *Assume the hypotheses of Theorem 4.2.12. Moreover, suppose that $g:\mathbb{R}\to\mathbb{R}$ is β-Hölder continuous with $\beta\in(0,1]$. Then there exists a constant C which does not depend of h such that*

$$\left|\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X_s|\right)\right]-\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X^h_{\eta_h(s)}|\right)\right]\right|\leq C\left\{h^{\frac{\alpha}{2}\wedge\frac14}+(h\log(1/h))^{\beta/2}\right\}.$$

Proof. Since g is β-Hölder continuous, it holds that $\mathbb{E}\left[|g\left(\max_{0\leq s\leq T}|x_0+\sigma W_s|\right)|^r\right]<\infty$ for any $r>2$. Thanks to Theorem 4.2.12, it remains to estimate

$$\left|\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X_s^h|\right)\right]-\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X_{\eta_h(s)}^h|\right)\right]\right|.$$

Since g is β-Hölder continuous and b is sub-linear growth, we have

$$\begin{aligned}
&\left|\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X_s^h|\right)\right]-\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X_{\eta_h(s)}^h|\right)\right]\right|\\
&\leq C\mathbb{E}\left[\left|\max_{0\leq s\leq T}|X_s^h|-\max_{0\leq s\leq T}|X_{\eta_h(s)}^h|\right|^\beta\right]\leq C\mathbb{E}\left[\max_{0\leq s\leq T}|X_s^h-X_{\eta_h(s)}^h|\right]^\beta\\
&\leq C\mathbb{E}\left[\max_{0\leq s\leq T}\{|b(X_{\eta_h(s)}^h)|(s-\eta_h(s))+|\sigma(W_s-W_{\eta_h(s)})|\}\right]^\beta\\
&\leq Ch^\beta+C\mathbb{E}\left[\max_{0\leq s\leq T}|W_s-W_{\eta_h(s)}|\}\right]^\beta. \qquad (4.22)
\end{aligned}$$

By modulus continuity of Brownian motion (e.g. Lemma 4.4 in [102]), we have

$$\mathbb{E}\left[\max_{0\leq s\leq t\leq T,|t-s|\leq h}|W_t-W_s|^2\right]\leq Ch\log\left(1/h\right),$$

Thus from (4.22), we obtain

$$\left|\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X_s^h|\right)\right]-\mathbb{E}\left[g\left(\max_{0\leq s\leq T}|X_{\eta_h(s)}^h|\right)\right]\right|\leq C\left\{h^\beta+(h\log(1/h))^{\beta/2}\right\},$$

which implies the proof of the statement. □

For an integral type functional, we obtain the following corollary.

Corollary 4.2.16. *Let $h=T/n$ for some $n\in\mathbb{N}$. If the drift coefficient $b\in\mathcal{B}(\alpha)$ is bounded, then for any Lipschitz continuous function f and $g\in\mathcal{B}(\beta)$ with $\beta\in(0,1]$, there exists a constant C which does not depend of h such that*

$$\left|\mathbb{E}\left[f\left(\int_0^T g(X_s)\mathrm{d}s\right)\right]-\mathbb{E}\left[f\left(\int_0^T g(X_{\eta_h(s)}^h)\mathrm{d}s\right)\right]\right|\leq Ch^{\frac{\alpha}{2}\wedge\frac{\beta}{2}\wedge\frac{1}{4}}.$$

Proof. We first note that if b is bounded, then it holds from Theorem 2.1 in [80] (see also Corollary 3.2 in [97]) that there exists a density function p_t^h of X_t^h for $t\in(0,T]$ and it satisfies the following Gaussian upper bound, i.e.,

$$p_t^h(x)\leq C\frac{e^{-\frac{|x-x_0|^2}{2ct}}}{t^{d/2}}.$$

for some positive constants C and c.

Now we prove that $\mathbb{E}\left[\left|f\left(\int_0^T g(x_0+W_s)\mathrm{d}s\right)\right|^r\right]$ is finite for any $r>2$. Since $|g(x)|\leq Ke^{K|x|}$, it follows from Jensen's inequality that for any $r>2$,

$$\mathbb{E}\left[\left|f\left(\int_0^T g(x_0+W_s)\mathrm{d}s\right)\right|^r\right]\leq C+C\int_0^T\mathbb{E}[|g(x+W_s)|^r]\mathrm{d}s\leq C+C\int_0^T\prod_{i=1}^d\mathbb{E}[e^{rKW_s^i}]\mathrm{d}s.$$

Since $x^2/(4s)+K^2r^2s\geq Krx$, we have

$$\mathbb{E}\left[\left|f\left(\int_0^T g(x_0+W_s)\mathrm{d}s\right)\right|^r\right]\leq C+C\int_0^T e^{dKrs}\mathrm{d}s<\infty.$$

Thanks to Theorem 4.2.12, it remains to prove that

$$\left|\mathbb{E}\left[f\left(\int_0^T g(X_s^h)\mathrm{d}s\right)\right]-\mathbb{E}\left[f\left(\int_0^T g(X_{\eta_h(s)}^h)\mathrm{d}s\right)\right]\right|\leq Ch^{\beta/2}.$$

Since f is a Lipschitz continuous function, we have

$$\left|\mathbb{E}\left[f\left(\int_0^T g(X_s^h)\mathrm{d}s\right)\right]-\mathbb{E}\left[f\left(\int_0^T g(X_{\eta_h(s)}^h)\mathrm{d}s\right)\right]\right|\leq C\int_0^T\mathbb{E}\left[\left|g(X_s^h)-g(X_{\eta_h(s)}^h)\right|\right]\mathrm{d}s$$

If $s\in(0,h]$, then by using the Gaussian upper bound for $p_s^h(x)$, we have

$$\begin{aligned}\int_0^h\mathbb{E}\left[\left|g(X_s^h)-g(X_{\eta_h(s)}^h)\right|\right]\mathrm{d}s&\leq\int_0^h\mathbb{E}[|g(X_s^h)|]\mathrm{d}s+|g(x_0)|h\\&\leq C\int_0^h\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}x|g(x)|\frac{e^{-\frac{|x-x_0|^2}{2cs}}}{s^{d/2}}+|g(x_0)|h\\&\leq Ch.\end{aligned}$$

On the other hand, for $s\in[h,T]$, using the Gaussian upper bound for $p_{\eta_h(s)}^h$ and following the proof of Lemma 4.2.8 (see also Lemma 3.5 of [97]), we have

$$\mathbb{E}[|g(X_s^h)-g(X_{\eta_h(s)}^h)|]\leq\frac{C\sqrt{s-\eta_h(s)}}{\sqrt{\eta_h(s)}}+Ch^{\beta/2}.$$

Therefore, we conclude the proof of the statement. □

In the following we consider a special case of the functional f. More precise, we are interested in the law at time T of the diffusion X killed when it leaves an open set. Let D be an open subset of $\mathbb{R}^d$ and recall that $\tau_D=\inf\{t>0:X_t\notin D\}$. Quantities of the type $\mathbb{E}[g(X_T)\mathbf{1}_{(\tau_D>T)}]$ appear in many domains, e.g. in financial mathematics when one computes the price of a barrier option on a d-dimensional asset price random variable X_t with characteristics f,T and D (see [34, 36] and the references therein for more detail). We approximate τ_D by $\tau_D^h=\inf\{kh>0:X_{kh}^h\notin D,k=0,1,\ldots\}$.

Corollary 4.2.17. *Assume the hypotheses of Theorem 4.2.12. Furthermore, we assume* *(i) D is of class C^∞ and ∂D is a compact set (see [31] or [34]);*

(ii) $g:\mathbb{R}^d\to\mathbb{R}$ is a measurable function, satisfying $\mathrm{d}(\mathrm{Supp}(g),\partial D)\geq 2\varepsilon$ for some $\varepsilon>0$ and $\|g\|_\infty=\sup_{x\in\mathbb{R}^d}|g(x)|<\infty$.

Then for any $p>1$, there exist constants C and C_p independent of h such that

$$\Big|\mathbb{E}[g(X_T)\mathbf{1}_{(\tau_D>T)}]-\mathbb{E}[g(X_T^h)\mathbf{1}_{(\tau_D^h>T)}]\Big|\leq Ch^{\frac{\alpha}{2}\wedge\frac{1}{4}}+\frac{C_p\|g^p\|_\infty}{1\wedge\varepsilon^{4/p}}h^{\frac{1}{2p}}. \tag{4.23}$$

Proof. It suffices to proof the statement for the case that g is positive. It follows from (4.4) that $\mathbb{E}[g(X_T)\mathbf{1}_{(\tau_D>T)}]-\mathbb{E}[g(X_T^h)\mathbf{1}_{(\tau_D^h>T)}]=E_1+E_2$ where

$$E_1=\mathbb{E}[g(x_0+\sigma W_T)(Z_T-Z_T^h)\mathbf{1}_{(\tau_D^{W,h}>T)}],$$
$$E_2=\mathbb{E}[g(x_0+\sigma W_T)Z_T(\mathbf{1}_{(\tau_D^W>T)}-\mathbf{1}_{(\tau_D^{W,h}>T)})].$$

It follows from the proof of Theorem 4.2.12 that

$$|E_1|\leq\mathbb{E}[|g(x_0+\sigma W_T)(Z_T-Z_T^h)|]\leq Ch^{\frac{\alpha}{2}\wedge\frac{1}{4}}. \tag{4.24}$$

Applying Hölder's inequality, we have

$$|E_2|\leq\|Z_T\|_q\|g(x_0+\sigma W_T)(\mathbf{1}_{(\tau_D^W>T)}-\mathbf{1}_{(\tau_D^{W,h}>T)})\|_p,$$

where q is the conjugate of p. Thanks to Lemma 4.2.6 and the fact $\tau_D^{W,h}\geq\tau_D^W$, we have

$$|E_2|\leq C_p\Big(\mathbb{E}\Big[g^p(x_0+\sigma W_T)\mathbf{1}_{(\tau_D^{W,h}\geq T)}\Big]-\mathbb{E}\Big[g^p(x_0+\sigma W_T)\mathbf{1}_{(\tau_D^W\geq T)}\Big]\Big)^{1/p}.$$

It follows from Theorem 2.4 in [34] that there exists a constant $K(T)$ such that

$$|E_2|\leq C_p\Big(\frac{K(T)\|g^p\|_\infty}{1\wedge\varepsilon^4}\Big)^{1/p}h^{\frac{1}{2p}}.$$

Combining this estimate with (4.24) completes the proof. □

Finally, we consider the approximation for the density of SDE (4.1). Let $p_t(x_0,\cdot)$ and $p_t^h(x_0,\cdot)$ be the density functions of X_t and X_t^h respectively. Then we have the following rate of convergence.

Theorem 4.2.18. *Suppose that $b\in\mathcal{B}(\alpha)$ and bounded. Then for any $p>d$ and $r>1$, there exists constants $C_{p,r}$ and c_p such that for any $y\in\mathbb{R}^d$ and $h\in(0,T/2)$,*

$$|p_T(x_0,y)-p_T^h(x_0,y)|\leq C_{p,r}g_{c_pT}(x_0,y)\left\{\frac{h}{T^{1/2}}+h^{\alpha/2}+h^{1/(2pr)}\right\}.$$

Remark 4.2.19. (i) Note that if $d=1$, we can choose $p\in(1,2)$ and $r=2/p$, and then

$$|p_T(x_0,y)-p_T^h(x_0,y)|\leq C_{p,r}g_{c_pT}(x_0,y)\left\{\frac{h}{T^{1/2}}+h^{\alpha/2}+h^{1/4}\right\}.$$

(ii) Konakov and Menozzi [70] obtain a better rate of convergence under further assumption that the drift coefficient is piecewise smooth (see Theorem 2 in [70]). However, in our setting, the drift coefficient may have infinite number of discontinuous points.

The proof of Theorem 4.2.18 is the based on the perturbation or Levi's parametrix method (see [28]) for the density functions $p_t(x_0,y)$ and $p_t^h(x_0,y)$. It is known that when the drift coefficient b is bounded and Hölder continuous, it holds that for any $t \in (0,T]$

$$\begin{aligned} p_t(x_0,y) &= g_{ta}(x_0,y) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}z \left\langle \nabla_x g_{(t-s)a}(z,y), b(z) \right\rangle p_t(x_0,z) \\ &= g_{ta}(x_0,y) + \int_0^t \mathbb{E}\left[\left\langle \nabla_x g_{(t-s)a}(X_s,y), b(X_s)\right\rangle\right] \mathrm{d}s, \end{aligned} \tag{4.25}$$

where $a = \sigma\sigma^*$ and $g_{ta}(x_0,y)$ is called the parametrix (see (1.9) and Chapter 1 in [28]).

We first consider a similar representation (4.25) for the density functions $p_t(x_0,y)$ and $p_t^h(x_0,y)$ under the assumption that the drift coefficient is bounded measurable. Recently, Makhlouf [85, Theorem 3.1] prove that the representation (4.25) also holds for a Brownian motion with random drift $b = (b_t)_{0\le t\le T}$ under the suitable growth condition. For the convenience of the reader, we will give a proof below.

Proposition 4.2.20 ([85]). *Let $W = (W_t)_{0\le t\le T}$ be a d-dimensional $(\mathcal{F}_t)_{0\le t\le T}$-Brownian motion. Suppose that the stochastic process $b = (b_t)_{0\le t\le T}$ is adapted to $(\mathcal{F}_t)_{0\le t\le T}$ and there exists a constant $K > 0$ such that $\sup_{0\le t\le T} |b_t| \le K$ almost surely. Then, for any $t \in (0,T]$ and $x \in \mathbb{R}^d$, the stochastic process $Y_t := x + \int_0^t b_s \mathrm{d}s + \sigma W_t$ admits a density $\gamma_t(x,\cdot)$ with respect to Lebesgue measure and for any $y \in \mathbb{R}^d$,*

$$\gamma_t(x,y) = g_{ta}(x,y) + \int_0^t \mathbb{E}\left[\left\langle \nabla_x g_{(t-s)a}(Y_s,y), b_s\right\rangle\right] \mathrm{d}s. \tag{4.26}$$

Proof. It suffices to prove that for any $f : \mathbb{R}^d \to \mathbb{R}$ infinitely differentiable functions with compact support contained in $\mathbb{R}$,

$$\mathbb{E}[f(Y_t)] = \int_{\mathbb{R}^d} f(y) g_{ta}(x,y)\mathrm{d}y + \int_{\mathbb{R}^d} \mathrm{d}y f(y) \int_0^t \mathrm{d}s \mathbb{E}\left[\langle \nabla_x g_{(t-s)a}(Y_s,y), b_s\rangle\right].$$

It is well-known that the function $u(s,x) := \mathbb{E}\left[f\left(x+\sigma W_{t-s}\right)\right]$ is a solution to the following partial differential equation:

$$\begin{aligned} \partial_s u(s,x) + \frac{1}{2}\sum_{i,j=1}^d a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} u(s,x) = 0,\ (s,x) \in [0,t) \times \mathbb{R}^d, \\ u(t,x) = f(x),\ x \in \mathbb{R}^d. \end{aligned} \tag{4.27}$$

Hence we have

$$\mathbb{E}\left[f\left(x+\sigma W_t\right)\right] = u(0,x), \tag{4.28}$$

$$\mathbb{E}[f(Y_t)] = \mathbb{E}\left[u(t,Y_t)\right]. \tag{4.29}$$

By using Itô's formula and (4.27), it holds that for any $\varepsilon \in (0,t)$,

$$u(t-\varepsilon, Y_{t-\varepsilon}) = u(0,x) + \int_0^{t-\varepsilon} \langle \nabla_x u(s,Y_s), b_s\rangle \mathrm{d}s + \sum_{i,j=1}^d \int_0^{t-\varepsilon} \sigma_{i,j} \frac{\partial}{\partial x_i} u(s,Y_s) \mathrm{d}W_s^j. \tag{4.30}$$

Since for any $i = 1, \dots, d$, $s \in [0, t)$ and $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial x_i} u(s,x) = \frac{\partial}{\partial x_i} \mathbb{E}\left[f\left(x + \sigma W_{t-s}\right)\right] = \int_{\mathbb{R}^d} f(y) \frac{\partial}{\partial x_i} g_{(t-s)a}(x,y) \mathrm{d}y \tag{4.31}$$

and

$$\left| \frac{\partial}{\partial x_i} u(s,x) \right| \le \frac{C \|f\|_\infty}{(t-s)^{1/2}},$$

for some constant $C > 0$, the stochastic integral in (4.30) is martingale. By taking the expectation and Fubini's theorem, we have from (4.28)

$$\mathbb{E}[u(t-\varepsilon, Y_{t-\varepsilon})] = \mathbb{E}\left[f\left(x + \sigma W_t\right)\right] + \int_0^{t-\varepsilon} \mathbb{E}\left[\langle \nabla_x u(s, Y_s), b_s \rangle\right] \mathrm{d}s.$$

Taking $\varepsilon \to 0$ and using the dominated convergence theorem, we have from (4.29) and (4.31),

$$\begin{aligned}
\mathbb{E}[f(Y_t)] &= \lim_{\varepsilon \to 0+} \mathbb{E}[u(t-\varepsilon, Y_{t-\varepsilon})] = \mathbb{E}\left[f\left(x + \sigma W_t\right)\right] + \int_0^t \mathbb{E}\left[\langle \nabla_x u(s, Y_s), b_s \rangle\right] \mathrm{d}s \\
&= \int_{\mathbb{R}^d} f(y) g_{ta}(x,y) \mathrm{d}y + \int_{\mathbb{R}^d} \mathrm{d}y f(y) \int_0^t \mathrm{d}s \mathbb{E}\left[\langle \nabla_x g_{(t-s)a}(Y_s, y), b_s \rangle\right].
\end{aligned}$$

This concludes the proof. $\square$

Proof of Theorem 4.2.18. Using Proposition 4.2.20, we have

$$\begin{aligned}
p_T(x_0, y) &= g_{Ta}(x_0, y) + \int_0^T \mathbb{E}\left[\left\langle \nabla_x g_{(T-s)a}(X_s, y), b(X_s) \right\rangle\right] \mathrm{d}s, \\
p_T^h(x_0, y) &= g_{Ta}(x_0, y) + \int_0^T \mathbb{E}\left[\left\langle \nabla_x g_{(T-s)a}(X_s^h, y), b(X_{\eta_h(s)}^h) \right\rangle\right] \mathrm{d}s.
\end{aligned}$$

Moreover, from Lemma 4.2.5, we have

$$p_T(x_0, y) - p_T^h(x_0, y) = \int_0^T \mathbb{E}\left[\left\langle \nabla_x g_{(T-s)a}(x_0 + \sigma W_s, y), Z_T b(x_0 + \sigma W_s) - Z_T^h b(x_0 + \sigma W_{\eta_h(s)}) \right\rangle\right] \mathrm{d}s.$$

By using Jensen's inequality and Schwarz's inequality, we have there exists $c > 0$ such that

$$\begin{aligned}
\left| p_T(x_0, y) - p_T^h(x_0, y) \right| &\le \int_0^T \mathbb{E}\left[\left| \nabla_x g_{(T-s)a}(x_0 + W_s, y) \right| \left| Z_T b(x_0 + \sigma W_s) - Z_T^h b(x_0 + \sigma W_{\eta_h(s)}) \right|\right] \mathrm{d}s \\
&\le C \int_0^T \frac{1}{\sqrt{T-s}} \mathbb{E}\left[g_{c(T-s)}(x_0 + W_s, y) \left| Z_T b(x_0 + \sigma W_s) - Z_T^h b(x_0 + \sigma W_{\eta_h(s)}) \right|\right] \mathrm{d}s \\
&\le C \int_0^T \frac{1}{\sqrt{T-s}} \{A_s + B_s\} \mathrm{d}s,
\end{aligned}$$

where

$$A_s := \mathbb{E}\left[g_{c(T-s)}(x_0 + \sigma W_s, y) \left| Z_T - Z_T^h \right|\right]$$

$$B_s := \mathbb{E}\left[g_{c(T-s)}(x_0+\sigma W_s, y)\,|Z_T|\,\left|b(x_0+\sigma W_s) - b(x_0+\sigma W_{\eta_h(s)})\right|\right].$$

Note that for any $q>1$, there exist C_q and c_q such that

$$\mathbb{E}\left[|g_{c(T-s)}(x_0+\sigma W_s, y)|^q\right]^{1/q} \leq C_q\left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}} g_{c_qT}(x_0,y). \tag{4.32}$$

We first consider the upper bounded for A_s. By using $|e^x-e^y| \leq (e^x+e^y)|x-y|$ and Hölder's inequality, for any $p,q>1$ with $1/p+1/q=1$ and $r>1$, we have

$$\begin{aligned}
A_s &\leq \mathbb{E}\left[|g_{c(T-s)}(x_0+\sigma W_s, y)|^q\right]^{1/q}\mathbb{E}\left[|Z_T+Z_T^h|^p\left|Y_T-Y_T^h\right|^p\right]^{1/p} \\
&\leq C_q\left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}} g_{c_qT}(x_0,y)\mathbb{E}\left[|Z_T+Z_T^h|^{pr/(r-1)}\right]^{(r-1)/(pr)}\mathbb{E}\left[\left|Y_T-Y_T^h\right|^{pr}\right]^{1/(pr)}
\end{aligned}$$

Therefore, it holds that

$$\int_0^T \frac{1}{\sqrt{T-s}}A_s\mathrm{d}s \leq C_{p,r}C_q g_{c_qT}(x_0,y)\int_0^T\frac{1}{\sqrt{T-s}}\left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}}\mathrm{d}s\mathbb{E}\left[\left|Y_T-Y_T^h\right|^{pr}\right]^{1/(pr)}. \tag{4.33}$$

By choosing $p>d$ that is $q=p/(p-1)<d/(d-1)$, from Lemma 4.2.10, we have

$$\begin{aligned}
\int_0^T \frac{1}{\sqrt{T-s}}A_s\mathrm{d}s &\leq C_{p,r}C_q g_{c_qT}(x_0,y)T^{1/2}\mathbb{E}\left[\left|Y_T-Y_T^h\right|^{pr}\right]^{1/(pr)} \\
&\leq C_{p,q,r}g_{c_qT}(x_0,y)T^{1/2}\{h^{\alpha/2}+h^{1/(2pr)}\}.
\end{aligned} \tag{4.34}$$

Now we consider the upper bounded for B_s. By using Hölder's inequality, for any $p>d$ and $q=p/(p-1)$ and $r>1$, we have

$$\begin{aligned}
B_s &\leq \mathbb{E}\left[|g_{c(T-s)}(x_0+\sigma W_s, y)|^q\right]^{1/q}\mathbb{E}\left[|Z_T|^{pr/(r-1)}\right]^{(r-1)/(pr)}\mathbb{E}\left[\left|b(x_0+\sigma W_s)-b(x_0+\sigma W_{\eta_h(s)})\right|^{pr}\right]^{1/(pr)} \\
&\leq C_{p,q,r}\left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}} g_{c_qT}(x_0,y)\mathbb{E}\left[\left|b(x_0+\sigma W_s)-b(x_0+\sigma W_{\eta_h(s)})\right|^{pr}\right]^{1/(pr)}.
\end{aligned}$$

By Lemma 4.2.8 for any $s \geq h$,

$$\mathbb{E}\left[\left|b(x_0+\sigma W_s)-b(x_0+\sigma W_{\eta_h(s)})\right|^{pr}\right]^{1/(pr)} \leq C\{h^{\alpha/2}+\frac{h^{1/(2pr)}}{\eta_h(s)^{1/(2pr)}}\}.$$

Since b is bounded, we have

$$\begin{aligned}
\int_0^T \frac{1}{\sqrt{T-s}}B_s\mathrm{d}s \leq& C_{p,q,r}g_{c_qT}(x_0,y)\int_0^h\frac{1}{\sqrt{T-s}}\left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}}\mathrm{d}s \\
&+ C_{p,q,r}g_{c_qT}(x_0,y)h^{\alpha/2}\int_h^T\frac{1}{\sqrt{T-s}}\left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}}\mathrm{d}s
\end{aligned}$$

$$+ C_{p,q,r} g_{c_q T}(x_0, y) h^{1/(2pr)} \int_h^T \frac{1}{\sqrt{T-s}} \left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}} \frac{1}{\eta_h(s)^{1/(2pr)}} \mathrm{d}s. \tag{4.35}$$

By the assumption $h \in (0, T/2)$, it holds that

$$\int_0^h \frac{1}{\sqrt{T-s}} \left(\frac{T}{T-s}\right)^{\frac{d(q-1)}{2q}} \mathrm{d}s \leq 2^{\frac{1}{2}+\frac{d(q-1)}{2q}} \frac{h}{T^{1/2}}.$$

Since $q < d/(d-1)$, we have

$$\int_h^T \frac{1}{\sqrt{T-s}} \left(\frac{1}{T-s}\right)^{\frac{d(q-1)}{2q}} \mathrm{d}s = \frac{2q}{q-dq+d}(T-h)^{\frac{q-dq+d}{2q}}$$

and

$$\int_h^T \frac{1}{\sqrt{T-s}} \left(\frac{1}{T-s}\right)^{\frac{d(q-1)}{2q}} \frac{1}{\eta_h(s)^{1/(2pr)}} \mathrm{d}s \leq (T-h)^{1-1/(2pr)} B\left(\frac{1}{2} + \frac{d(q-1)}{2q}, \frac{1}{2pr}\right),$$

where $B(x, y)$ is the beta function. Therefore, we obtain

$$\int_0^T \frac{1}{\sqrt{T-s}} B_s \mathrm{d}s \leq C_{p,q,r} g_{c_p T}(x_0, y) \left\{\frac{h}{T^{1/2}} + h^{\alpha/2} + h^{1/(2pr)}\right\}, \tag{4.36}$$

which concludes the proof. □

4.3 Weak approximation of reflected SDEs

We first recall the Skorohod problem.

Lemma 4.3.1 ([60], Lemma III.6.14). *Let $z \geq 0$ be a given number and $y : [0, \infty) \to \mathbb{R}$ be a continuous function with $y_0 = 0$. Then there exists unique continuous function $\ell = (\ell_t)_{t \geq 0}$ satisfying the following conditions:*

(i) $x_t := z + y_t + \ell_t \geq 0, 0 \leq t < \infty$;

(ii) ℓ *is a non-decreasing function with* $\ell_0 = 0$ *and* $\ell_t = \int_0^t \mathbf{1}(x_s = 0) d\ell_s$.

Moreover, $\ell = (\ell_t)_{t \geq 0}$ *is given by*

$$\ell_t = \max\{0, \max_{0 \leq s \leq t}(-z - y_s)\} = \max_{0 \leq s \leq t} \max(0, \ell_s - x_s).$$

Let us consider the following one-dimensional reflected stochastic differential equation valued in $[0, \infty)$ such that

$$X_t = x_0 + \int_0^t b(X_s) \mathrm{d}s + \sigma W_t + L_t^0(X), x_0 \in [0, \infty), t \in [0, T], \tag{4.37}$$

$$L_t^0(X) = \int_0^t \mathbf{1}_{(X_s=0)} \mathrm{d}L_s^0(X),$$

where $(L_t^0(X))_{0\le t\le T}$ is a non-decreasing continuous process stating at the origin and it is called local time of X at the origin. In this chapter, we assume that the SDE (4.37) has a weak solution and the uniqueness in the sense of probability law holds (see [104, 115]). Using Lemma 4.3.1, we have

$$L_t^0(X) = \sup_{0\le s\le t} \max\left(0, L_s^0(X) - X_s\right).$$

Now we define the Euler-Maruyama scheme $X^h = (X_t^h)_{0\le t\le T}$ for the reflected stochastic differential equation (4.37). Let $X_0^h := x_0$ and define

$$X_t^h = x_0 + \int_0^t b(X_{\eta_h(s)}^h)\mathrm{d}s + \sigma W_t + L_t^0(X^h).$$

The existence of the pair $(X_t^h, L_t^0(X^h))_{0\le t\le T}$ is deduced from Lemma 4.3.1. Moreover

$$L_t^0(X^h) = \int_0^t \mathbf{1}_{(X_s^h=0)} \mathrm{d}L_s^0(X^h).$$

By the definition of the Euler-Maruyama scheme, we have the following representation. For each $k = 0, 1, \ldots,$

$$X_{(k+1)h}^h = X_{kh}^h + b(X_{kh}^h)h + \sigma(W_{(k+1)h} - W_{kh}) + \max(0, A_k - X_{kh}^h),$$

where

$$A_k := \sup_{kh\le s<(k+1)h} \left(-b(X_{kh}^h)(s-kh) - \sigma(W_s - W_{kh})\right).$$

Though A_k is defined by the supremum of a stochastic process, it can be simulated by using the following lemma.

Lemma 4.3.2 ([82], Theorem 1)**.** *Let $t \in [0,T]$ and $a, c \in \mathbb{R}$. Define $S_t := \sup_{0\le s\le t}(aW_s + cs)$. Let U_t be a centered Gaussian random variable with variance t and let V_t be an exponential random variable with parameter $1/(2t)$ independent from U_t. Define*

$$Y_t := \frac{1}{2}(aU_t + ct + (a^2V_t + (aU_t + ct)^2)^{1/2}).$$

Then the processes $(W_t, S_t)_{t\in[0,T]}$ and $(U_t, Y_t)_{t\in[0,T]}$ have the same law.

Under the Lipschitz condition for the coefficients of the reflected SDE (4.37), Lépingle [83] shows that

$$\mathbb{E}[\sup_{0\le t\le T} |X_t - X_t^h|^2]^{1/2} \le Ch^{1/2},$$

for some constant C.

Change of measures

In the same way as in subsection 4.2.2, we have the following Lemma.

Lemma 4.3.3. *If b is a measurable function with sub-linear growth then*

$$\mathbb{E}[f(X)] - \mathbb{E}[f(X^h)] = \mathbb{E}[f(U)(\hat{Z}_T - \hat{Z}_T^h)]$$

for all measurable functions $f : C[0,T] \to \mathbb{R}$ provided that the above expectations are integrable. Here the process $U = (U_t)_{0 \le t \le T}$ is the unique solution of the equation $U_t = x_0 + \sigma W_t + L_t^0(U)$ and

$$\hat{Z}_t := e^{\hat{Y}_t}, \quad \hat{Y}_t := \int_0^t b(U_s)\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2(U_s)\mathrm{d}s$$
$$\hat{Z}_t^h := e^{\hat{Y}_t^h}, \quad \hat{Y}_t^h := \int_0^t b(U_{\eta_h(s)})\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2(U_{\eta_h(s)})\mathrm{d}s,$$

Proof. We define new measures $\hat{\mathbb{Q}}$ and $\hat{\mathbb{Q}}^h$ as

$$\frac{\mathrm{d}\hat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \exp\Big(- \int_0^T \sigma^{-1} b(X_s)\mathrm{d}W_s - \frac{1}{2}\int_0^T |\sigma^{-1}b(X_s)|^2 \mathrm{d}s\Big),$$
$$\frac{\mathrm{d}\hat{\mathbb{Q}}^h}{\mathrm{d}\mathbb{P}} = \exp\Big(- \int_0^T \sigma^{-1} b(X^h_{\eta_h(s)})\mathrm{d}W_s - \frac{1}{2}\int_0^T |\sigma^{-1}b(X^h_{\eta_h(s)})|^2 \mathrm{d}s\Big).$$

Since b is of sub-linear growth and the fact that $0 \le L_t^0(X) \le |\sigma| \sup_{0 \le s \le t} |W_s|$, by following the proof of Lemma 4.2.5 we can show that $\hat{\mathbb{Q}}$ and $\hat{\mathbb{Q}}^h$ are probability measures. Furthermore, it follows from the Girsanov theorem that the processes $\hat{B} = (\hat{B}_t)_{0 \le t \le T}$ and $\hat{B}^h = (\hat{B}_t^h)_{0 \le t \le T}$ defined by

$$\hat{B}_t = W_t + \int_0^t \sigma^{-1}b(X_s)\mathrm{d}s, \ \hat{B}_t^h = W_t + \int_0^t \sigma^{-1}b(X^h_{\eta_h(s)})\mathrm{d}s, 0 \le t \le T,$$

are Brownian motions with respect to $\hat{\mathbb{Q}}$ and $\hat{\mathbb{Q}}^h$ respectively. Note that $X_s = x_0 + \sigma \hat{B}_s + L_s^0(X)$ and $X_s^h = x_0 + \sigma \hat{B}_s^h + L_s^0(X^h)$. Therefore,

$$\begin{aligned}
&\mathbb{E}[f(X)] = \mathbb{E}_{\hat{\mathbb{Q}}}\Big[f(X)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\hat{\mathbb{Q}}}\Big] \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\Big[f(X)\exp\Big(\int_0^T \sigma^{-1}b(X_s)\mathrm{d}\hat{B}_s - \frac{1}{2}\int_0^T |\sigma^{-1}b(X_s)|^2\mathrm{d}s\Big)\Big] \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\Big[f(x_0 + \sigma\hat{B} + L^0(X))\exp\Big(\int_0^T \sigma^{-1}b(x_0 + \sigma\hat{B}_s + L_s^0(X))\mathrm{d}\hat{B}_s - \frac{1}{2}\int_0^T |\sigma^{-1}b(x_0 + \sigma\hat{B}_s + L_s^0(X))|^2\mathrm{d}s\Big)\Big].
\end{aligned}$$

Since $(X, \hat{B})|_{\hat{\mathbb{Q}}} \overset{d}{=} (U, W)|_{\mathbb{P}}$, the above term equals to

$$\begin{aligned}
&\mathbb{E}\Big[f(x_0 + \sigma W + L^0(U))\exp\Big(\int_0^T \sigma^{-1}b(x_0 + \sigma W_s + L_s^0(U))\mathrm{d}W_s - \frac{1}{2}\int_0^T |\sigma^{-1}b(x_0 + \sigma W_s + L_s^0(U))|^2\mathrm{d}s\Big)\Big] \\
&= \mathbb{E}\Big[f(U)\exp\Big(\int_0^T \sigma^{-1}b(U_s)\mathrm{d}W_s - \frac{1}{2}\int_0^T |\sigma^{-1}b(U_s)|^2\mathrm{d}s\Big)\Big]
\end{aligned}$$

$= \mathbb{E}[f(U)\hat{Z}_T]$.

Repeating the previous argument leads to $\mathbb{E}[f(X^h)] = \mathbb{E}[f(U)\hat{Z}^h_T]$, which concludes the statement. □

In the same way as Lemma 4.2.6, we have the following estimate of the moments of $\hat{Z}$ and $\hat{Z}^h$.

Lemma 4.3.4. *Suppose that b is of sub-linear growth. Then for any $p > 0$,*

$$\mathbb{E}[|\hat{Z}_T|^p + |\hat{Z}^h_T|^p] \leq C < \infty,$$

for some constant C which is not depend on h.

Finally, we introduce the following auxiliary estimate.

Lemma 4.3.5. *Let U as in Lemma 4.3.3. Suppose that ζ is α-Hölder continuous with $\alpha \in (0,1]$, then for any $t > s > 0$,*

$$\mathbb{E}[|\zeta(U_t) - \zeta(U_s)|^p] \leq C_p(t-s)^{p\alpha/2}.$$

Proof. By Hölder continuity of ζ, we have

$$\mathbb{E}[|\zeta(U_t) - \zeta(U_s)|^p)] \leq \mathbb{E}[|U_t - U_s|^{\alpha p}] \leq C\mathbb{E}[|W_t - W_s|^{p\alpha}] + C\mathbb{E}[|L^0_t(U) - L^0_s(U)|^{p\alpha}].$$

Hence it is sufficient to prove that

$$\mathbb{E}[|L^0_t(U) - L^0_s(U)|^{p\alpha}] \leq C_p(t-s)^{p\alpha/2}.$$

Using Lemma 4.3.1, we have

$$L^0_s(U) \leq L^0_t(U) \leq L^0_s(U) + \sup_{s \leq u \leq t} \max\left(0, -\sigma(W_u - W_s)\right).$$

Therefore, $|L^0_t(U) - L^0_s(U)| \leq |\sigma| \sup_{s\leq u\leq t} |W_u - W_s|$. Hence applying Burkholder-Davis-Gundy's inequality, we have

$$\mathbb{E}[|L^0_t(U) - L^0_s(U)|^{p\alpha}] \leq C\mathbb{E}[\sup_{s\leq u\leq t} |W_u - W_s|^{p\alpha}] \leq C(t-s)^{p\alpha/2}.$$

This concludes the proof. □

We obtain the following result on the weak convergence for the Euler-Maruyama scheme for a reflected SDE with non-Lipschitz coefficient.

Theorem 4.3.6. *Suppose that the drift coefficient b is of sub-linear growth and α-Hölder continuous with $\alpha \in (0,1]$. Moreover, assume that $f : C[0,T] \to \mathbb{R}$ is bounded. Then there exists a constant C not depend of h such that*

$$|\mathbb{E}[f(X)] - \mathbb{E}[f(X^h)]| \leq Ch^{\alpha/2}.$$

Proof. Using Lemmas 4.3.3 and the elementary estimate $|e^x - e^y| \leq (e^x + e^y)|x - y|$, we have

$$|\mathbb{E}[f(X)] - \mathbb{E}[f(X^h)]| \leq \mathbb{E}\Big[\big|f(U)(\hat{Z}_T + \hat{Z}_T^h)(\hat{Y}_T - \hat{Y}_T^h)\big|\Big].$$

Thanks to Lemma 4.3.4 and the Hölder's inequality, for some $r > 2$, we have

$$|\mathbb{E}[f(X)] - \mathbb{E}[f(X^h)]| \leq C\mathbb{E}[|f(U)|^r]^{2/r}||\hat{Y}_T - \hat{Y}_T^h||_2.$$

By a similar argument as the proof of Lemma 4.2.10, we can show that

$$||\hat{Y}_T - \hat{Y}_T^h||_2 \leq Ch^{\alpha/2},$$

which concludes the proof. □

Remark 4.3.7. The conclusion of Theorem 4.3.6 still holds if we relax the condition f bounded to $\mathbb{E}[|f(U)|^r] < \infty$ for some $r > 2$.

4.4 Appendix

Lemma 4.4.1. *Let n be a natural number such that $(n-1)h < T \leq nh$. Define $t_i^h = ih$ for $i = 0, \dots, n-1$ and $t_n^h = T$. Then it holds that*

$$\sum_{i=1}^{n-1} \int_{t_i^h}^{t_{i+1}^h} \mathrm{d}u \int_{t_1^h}^{t_i^h} \mathrm{d}s \left\{ \frac{1}{\sqrt{\eta_h(u) - s}\sqrt{\eta_h(s)}} + \frac{1}{\sqrt{\eta_h(s)}} + \frac{1}{\sqrt{\eta_h(u) - s}} \right\} \leq 4\sqrt{2} + 2\sqrt{T} + \frac{4T^{3/2}}{3}.$$

Proof. We first note that if $s \geq t_1^h$ then $\eta_h(s) \geq s/2$. The first integral is estimated as follows

$$\begin{aligned}
\sum_{i=1}^{n-1} \int_{t_i^h}^{t_{i+1}^h} \mathrm{d}u \int_{t_1^h}^{t_i^h} \mathrm{d}s \frac{1}{\sqrt{\eta_h(u) - s}\sqrt{\eta_h(s)}} &\leq \sqrt{2}h \sum_{i=2}^{n-1} \left\{ \int_{t_1^h}^{t_i^h/2} \mathrm{d}s + \int_{t_i^h/2}^{t_i^h} \mathrm{d}s \right\} \frac{1}{\sqrt{t_i^h - s}\sqrt{s}} \\
&\leq \sqrt{2}h \sum_{i=2}^{n-1} \sqrt{\frac{2}{t_i}} \left\{ \int_{t_1^h}^{t_i^h/2} \frac{1}{\sqrt{s}} \mathrm{d}s + \int_{t_i^h/2}^{t_i^h} \frac{1}{\sqrt{t_i^h - s}} \mathrm{d}s \right\} \\
&\leq 4\sqrt{2}.
\end{aligned}$$

The second integral is estimated as follows

$$\sum_{i=1}^{n-1} \int_{t_i^h}^{t_{i+1}^h} \mathrm{d}u \int_{t_1^h}^{t_i^h} \mathrm{d}s \frac{1}{\sqrt{\eta_h(s)}} \leq \sqrt{2}h \sum_{i=1}^{n-1} \int_{t_1^h}^{t_i^h} \frac{\mathrm{d}s}{\sqrt{s}} \leq 2\sqrt{T}.$$

The third integral is estimated as follows

$$\begin{aligned}
\sum_{i=1}^{n-1} \int_{t_i^h}^{t_{i+1}^h} \mathrm{d}u \int_{t_1^h}^{t_i^h} \mathrm{d}s \frac{1}{\sqrt{\eta_h(u) - s}} &\leq h \sum_{i=2}^{n-1} \int_{t_1^h}^{t_i^h} \frac{\mathrm{d}s}{\sqrt{t_i^h - s}} = 2h \sum_{i=2}^{n-1} \sqrt{t_i^h - h} = 2 \sum_{i=2}^{n-1} \int_{t_{i-1}^h}^{t_i^h} \sqrt{t_{i-1}^h} \mathrm{d}s \\
&\leq 2 \int_0^T \sqrt{s} \mathrm{d}s = \frac{4T^{3/2}}{3}.
\end{aligned}$$

This concludes the proof of the statement. □

Chapter 5

Stability problem for SDEs with discontinuous drift

5.1 Introduction

Let $X = (X_t)_{0 \leq t \leq T}$ be a solution of the one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s,\ x_0 \in \mathbb{R},\ t \in [0, T], \tag{5.1}$$

where $W := (W_t)_{0 \leq t \leq T}$ is a standard one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions. The drift coefficient b and the diffusion coefficient σ are Borel-measurable functions from $\mathbb{R}$ into $\mathbb{R}$. The diffusion process X is used in many fields of application, for example, mathematical finance, optimal control and filtering.

Let $X^{(n)}$ be a solution of the SDE (5.1) with drift coefficient b_n and diffusion coefficient σ_n. We consider the stability problem for $(X, X^{(n)})$ when the pair of coefficients (b_n, σ_n) converges to (b, σ). Stroock and Varadhan introduced the stability problem in the weak sense in order to consider the martingale problem with continuous and locally bounded coefficients (see chapter 11 of [111]). In [61], Kawabata and Yamada consider the strong convergence of the stability problem under the condition that the drift coefficients b and b_n are Lipschitz continuous functions, the diffusion coefficients σ and σ_n are Hölder continuous and (b_n, σ_n) locally uniformly converges to (b, σ) (see [61], example 1). Kaneko and Nakao [59] prove that if the coefficients b_n and σ_n are uniformly bounded, σ_n is uniformly elliptic and (b_n, σ_n) tends to (b, σ) in L^1-sense, then $(X^{(n)})_{n \in \mathbb{N}}$ converges to X in L^2-sense. Moreover they also prove that the solution of the SDE (5.1) can be constructed as the limit of the Euler-Maruyama approximation under the condition that the coefficients b and σ are continuous and of linear growth (see [59], Theorem D). Recently, under the Nakao-Le Gall condition, Hashimoto and Tsuchiya [44] prove that $(X^{(n)})_{n \in \mathbb{N}}$ converges to X in L^p sense for any $p \geq 1$ and give the rate of convergence under the condition that $b_n \to b$ and $\sigma_n \to \sigma$ in L^1 and L^2 sense, respectively. Their proof is based on the Yamada-Watanabe approximation technique which was introduced in [120] and some estimates for the local time.

On a related study, the convergence for the Euler-Maruyama approximation with non-Lipschitz coefficients has been studied recently. Yan [121] has proven that if the sets of discontinuous points of b and σ are countable, then the Euler-Maruyama approximation converges weakly to the unique weak solution of the corresponding SDE. Kohatsu-Higa, Lejay and Yasuda [65] have studied the weak approximation error for the one-dimensional SDE with the drift $\mathbf{1}_{(-\infty,0]}(x) - \mathbf{1}_{(0,+\infty)}(x)$ and constant diffusion. Gyöngy and Rásonyi [40] give the order of the strong rate of convergence for a class of one-dimensional SDEs whose drift is the sum of a Lipschitz continuous function and a monotone decreasing Hölder continuous function and its diffusion coefficient is a Hölder continuous function. The Yamada-Watanabe approximation technique is a key idea to obtain their results. In [97], Ngo and Taguchi extend the results in [40] for SDEs with discontinuous drift. They prove that if the drift coefficient b is bounded and one-sided Lipschitz function, and the diffusion coefficient is bounded, uniformly elliptic and η-Hölder continuous, then there exists a positive constant C such that

$$\sup_{0\le t\le T} \mathbb{E}[|X_t - \overline{X}_t^{(n)}|] \le \begin{cases} \dfrac{C}{n^{\eta-1/2}}, & \text{if } \eta \in (1/2,1], \\ \dfrac{C}{\log n}, & \text{if } \eta = 1/2, \end{cases}$$

where $\overline{X}^{(n)}$ is the Euler-Maruyama approximation for SDE (5.1). This fact implies that the strong rate of convergence for the stability problem may also depend on the Hölder exponent of the diffusion coefficient.

The goal of this chapter is to estimate the difference between two SDEs using the norm of the difference of coefficients. More precisely, let us consider another SDE given by

$$\hat{X}_t = x_0 + \int_0^t \hat{b}(\hat{X}_s)\mathrm{d}s + \int_0^t \hat{\sigma}(\hat{X}_s)\mathrm{d}W_s. \tag{5.2}$$

We will prove the following inequality:

$$\sup_{0\le t\le T} \mathbb{E}[|X_t - \hat{X}_t|] \le \begin{cases} C(||b-\hat{b}||_1 \vee ||\sigma-\hat{\sigma}||_2^2)^{(2\eta-1)/(2\eta)}, & \text{if } \eta\in(1/2,1], \\ \dfrac{C}{\log(1/(||b-\hat{b}||_1 \vee ||\sigma-\hat{\sigma}||_2^2))}, & \text{if } \eta = 1/2, \end{cases} \tag{5.3}$$

where η is the Hölder exponent of the diffusion coefficients, C is a positive constant and $||\cdot||_p$ is a L^p-norm which will be defined by (5.4). We will also estimate $\mathbb{E}[\sup_{0\le t\le T}|X_t - \hat{X}_t|^p]$ for any $p \ge 1$. It is worth noting that in the papers [59] and [61], the authors only prove the strong convergence for the stability problem. On the other hand, applying our main results, we are able to establish the strong rate of convergence for the stability problem (see section 5.3). In order to obtain (5.3), we use the Yamada-Watanabe approximation technique and a Gaussian upper bound for the density of SDE (5.2) (see [7], [75], [85] and [109]).

Finally, we note that SDEs with discontinuous drift coefficient have many applications in mathematical finance [1] and [51], optimal control problems [14] and other domains (see also [16] and [72]).

5.2 L^p-difference between two solutions of SDEs

5.2.1 Notations and Assumptions

We will assume that the drift coefficient b belongs to the class of one-sided Lipschitz functions which is defined as follows.

Definition 5.2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is called a* one-sided Lipschitz function *if there exists a positive constant L such that for any $x, y \in \mathbb{R}$,*

$$(x-y)(f(x)-f(y)) \leq L|x-y|^2.$$

Let $\mathcal{L}$ be the class of all one-sided Lipschitz functions.

Remark 5.2.2. By the definition of the class $\mathcal{L}$, if $f, g \in \mathcal{L}$ and $\alpha \geq 0$, then $f+g$, $\alpha f \in \mathcal{L}$. The one-sided Lipschitz property is closely related to the monotonicity condition. Actually, any monotone decreasing function is one-sided Lipschitz. Moreover, any Lipschitz continuous function is also a one-sided Lipschitz.

Now we give assumptions for the coefficients $b, \hat{b}, \sigma$ and $\hat{\sigma}$.

Assumption 5.2.3. *We assume that the coefficients $b, \hat{b}, \sigma$ and $\hat{\sigma}$ satisfy the following conditions:*

A-(i) : $b \in \mathcal{L}$.

A-(ii) : *b and $\hat{b}$ are measurable and there exists $K > 0$ such that*

$$\sup_{x \in \mathbb{R}} \left(|b(x)| \vee |\hat{b}(x)| \right) \leq K.$$

A-(iii) : *σ and $\hat{\sigma}$ are $\eta := (1/2+\alpha)$-Hölder continuous with some $\alpha \in [0, 1/2]$, i.e., there exists $K > 0$ such that*

$$\sup_{x,y \in \mathbb{R}, x \neq y} \left(\frac{|\sigma(x)-\sigma(y)|}{|x-y|^\eta} \vee \frac{|\hat{\sigma}(x)-\hat{\sigma}(y)|}{|x-y|^\eta} \right) \leq K.$$

A-(iv) : *$a = \sigma^2$ and $\hat{a} = \hat{\sigma}^2$ are bounded and uniformly elliptic, i.e., there exists $\lambda \geq 1$ such that for any $x \in \mathbb{R}$,*

$$\lambda^{-1} \leq a(x) \leq \lambda \text{ and } \lambda^{-1} \leq \hat{a}(x) \leq \lambda.$$

Remark 5.2.4. Assume that A-(ii), A-(iii) and A-(iv) hold. Then the SDE (5.1) and the SDE (5.2) have unique strong solution (see [122]). Note that the one-sided Lipschitz property is used only in (5.7) for b, so we don't need to assume $\hat{b} \in \mathcal{L}$.

5.2.2 Gaussian upper bound for the density of SDEs

A Gaussian upper bounded for the density of X_t is well-known under suitable conditions for the coefficients. If coefficients b and σ are Hölder continuous and σ is bounded and uniformly elliptic, then a Gaussian type estimate holds for the fundamental solution of parabolic type partial differential equations (see [28], Theorem 11, chapter 1). Under A-(ii), (iii) and (iv), the density function $p_t(x_0, \cdot)$ of X_t exists for any $t \in (0, T]$ and there exist positive constants $\overline{C}$ and c_* such that for any $y \in \mathbb{R}$ and $t \in (0, T]$,

$$p_t(x_0, y) \leq \overline{C} p_{c_*}(t, x_0, y),$$

where $p_c(t, x, y) := \frac{e^{-\frac{(y-x)^2}{2ct}}}{\sqrt{2\pi ct}}$ (see [7], [75], [85], [109] and Remark 4.1 [80]).

Using a Gaussian upper bound for the density of X_t, we can prove the following estimate.

Lemma 5.2.5. *Let $p \geq 1$. Assume that A-(ii), A-(iii) and A-(iv) hold. Then we have*

$$\int_0^T \mathbb{E}[|b(\hat{X}_s) - \hat{b}(\hat{X}_s)|^p]\mathrm{d}s \leq C_T ||b - \hat{b}||_p^p$$

and

$$\int_0^T \mathbb{E}[|\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^{2p}]\mathrm{d}s \leq C_T ||\sigma - \hat{\sigma}||_{2p}^{2p},$$

where $C_T := \overline{C}\sqrt{\frac{2T}{\pi c_*}}$ and for any bounded measurable function f, $||\cdot||_p$ is defined by

$$||f||_p := \left(\int_{\mathbb{R}} |f(x)|^p e^{-\frac{|x-x_0|^2}{2c_* T}} \mathrm{d}x \right)^{1/p}. \tag{5.4}$$

Proof. We only prove the first estimate. The second one can be obtained by using a similar argument. From a Gaussian upper bound for the density of $\hat{X}_t$, for any $x \in \mathbb{R}$ and $s \in (0, T]$, we have

$$\hat{p}_s(x_0, x) \leq \overline{C} p_{c_*}(s, x_0, x) \leq \frac{\overline{C}}{\sqrt{2\pi c_* s}} e^{-\frac{|x-x_0|^2}{2c_* T}},$$

where $\hat{p}_s(x_0, \cdot)$ is a density function of $\hat{X}_s$. Hence we obtain

$$\begin{aligned}
\int_0^T \mathbb{E}[|b(\hat{X}_s) - \hat{b}(\hat{X}_s)|^p]\mathrm{d}s &= \int_0^T \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}x |b(x) - \hat{b}(x)|^p \hat{p}_s(x_0, x) \\
&\leq \int_0^T \mathrm{d}s \frac{\overline{C}}{\sqrt{2\pi c_* s}} \int_{\mathbb{R}} \mathrm{d}x |b(x) - \hat{b}(x)|^p e^{-\frac{|x-x_0|^2}{2c_* T}} \qquad (5.5) \\
&= C_T ||b - \hat{b}||_p^p.
\end{aligned}$$

This concludes the proof. ☐

Remark 5.2.6. Our proof of Lemma 5.2.5 is based on the fact that we are in the one-dimensional setting. In multi-dimensional case, the integrand of (5.5) is not integrable with respect to s in general. This is the main reason for restricting our discussion to the one-dimensional SDE case.

5.2.3 Rate of convergence for the $L^p(\Omega)$-difference between two SDEs

For any $p \geq 1$, we define

$$\varepsilon_p := ||b - \hat{b}||_p^p \vee ||\sigma - \hat{\sigma}||_{2p}^{2p}.$$

Then we have the following estimate for the difference between two SDEs.

Theorem 5.2.7. *Suppose that Assumption 5.2.3 holds. We assume that $\varepsilon_1 < 1$ if $\alpha \in (0, 1/2]$ and $1/\log(1/\varepsilon_1) < 1$ if $\alpha = 0$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, \alpha, \lambda$ and x_0 such that*

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[|X_\tau - \hat{X}_\tau|] \leq \begin{cases} C\varepsilon_1^{2\alpha/(2\alpha+1)} & \text{if } \alpha \in (0, 1/2], \\ \dfrac{C}{\log(1/\varepsilon_1)} & \text{if } \alpha = 0, \end{cases}$$

where $\mathcal{T}$ is the set of all stopping times $\tau \leq T$.

Proof. We set $Y_t := X_t - \hat{X}_t$ for $t \in [0,T]$. We again use the Yamada and Watanabe approximation technique in one-dimensional setting (see subsection 3.2.2). Let $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$. From Itô's formula, (3.4) and (3.5), we have

$$\begin{aligned}
|Y_t| \leq & \varepsilon + \phi_{\delta,\varepsilon}(Y_t) \\
= & \varepsilon + \int_0^t \phi'_{\delta,\varepsilon}(Y_s)(b(X_s) - \hat{b}(\hat{X}_s))\mathrm{d}s + \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(Y_s)|\sigma(X_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s + M_t^{\delta,\varepsilon} \\
= & \varepsilon + \int_0^t \phi'_{\delta,\varepsilon}(Y_s)(b(X_s) - b(\hat{X}_s))\mathrm{d}s + \int_0^t \phi'_{\delta,\varepsilon}(Y_s)(b(\hat{X}_s) - \hat{b}(\hat{X}_s))\mathrm{d}s \\
& + \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(Y_s)|\sigma(X_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s + M_t^{\delta,\varepsilon} \\
\leq & \varepsilon + \int_0^t \phi'_{\delta,\varepsilon}(Y_s)(b(X_s) - b(\hat{X}_s))\mathrm{d}s + \int_0^T |b(\hat{X}_s) - \hat{b}(\hat{X}_s)|\mathrm{d}s \\
& + \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(Y_s)|\sigma(X_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s + M_t^{\delta,\varepsilon},
\end{aligned} \tag{5.6}$$

where

$$M_t^{\delta,\varepsilon} := \int_0^t \phi'_{\delta,\varepsilon}(Y_s)(\sigma(X_s) - \hat{\sigma}(\hat{X}_s))\mathrm{d}W_s.$$

Note that since σ, $\hat{\sigma}$ and $\phi'_{\delta,\varepsilon}$ are bounded, $(M_t^{\delta,\varepsilon})_{0\leq t\leq T}$ is a martingale so $\mathbb{E}[M_t^{\delta,\varepsilon}] = 0$. Since $b \in \mathcal{L}$, it follows from $\phi'_{\delta,\varepsilon}(x)/x > 0$ for $x \neq 0$ (see (2.8)) and (3.5) that for any $x, y \in \mathbb{R}$ with $x \neq y$,

$$\phi'_{\delta,\varepsilon}(x-y)(b(x) - b(y)) = \frac{\phi'_{\delta,\varepsilon}(x-y)}{x-y}(x-y)(b(x)-b(y)) \leq L\frac{\phi'_{\delta,\varepsilon}(x-y)}{x-y}|x-y|^2 \leq L|x-y|. \tag{5.7}$$

Therefore we get

$$\int_0^t \phi'_{\delta,\varepsilon}(Y_s)(b(X_s) - b(\hat{X}_s))\mathrm{d}s \leq L\int_0^t |Y_s|\mathrm{d}s. \tag{5.8}$$

Using Lemma 5.2.5 with $p = 1$, we have

$$\int_0^T \mathbb{E}[|b(\hat{X}_s) - \hat{b}(\hat{X}_s)|]\mathrm{d}s \leq C_T||b - \hat{b}||_1. \tag{5.9}$$

From (3.6) and $(x+y)^2 \leq 2x^2 + 2y^2$ for any $x, y \geq 0$, we have

$$\begin{aligned}
& \frac{1}{2}\int_0^t \phi''_{\delta,\varepsilon}(Y_s)|\sigma(X_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s \leq \int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s|)}{|Y_s|\log\delta}|\sigma(X_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s \\
& \leq 2\int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s|)}{|Y_s|\log\delta}|\sigma(X_s) - \sigma(\hat{X}_s)|^2\mathrm{d}s + 2\int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s|)}{|Y_s|\log\delta}|\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s \\
& \leq 2\int_0^t \frac{\mathbf{1}_{[\varepsilon/\delta,\varepsilon]}(|Y_s|)}{|Y_s|\log\delta}|\sigma(X_s) - \sigma(\hat{X}_s)|^2\mathrm{d}s + \frac{2\delta}{\varepsilon\log\delta}\int_0^T |\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s.
\end{aligned} \tag{5.10}$$

Again using Lemma 5.2.5 with $p = 1$, we have

$$\frac{2\delta}{\varepsilon \log \delta} \int_0^T \mathbb{E}[|\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^2] \mathrm{d}s \leq \frac{2C_T \delta}{\varepsilon \log \delta} ||\sigma - \hat{\sigma}||_2^2. \tag{5.11}$$

Since σ is $(1/2 + \alpha)$-Hölder continuous, we have

$$2 \int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta, \varepsilon]}(|Y_s|)}{|Y_s| \log \delta} |\sigma(X_s) - \sigma(\hat{X}_s)|^2 \mathrm{d}s \leq 2K^2 \int_0^T \frac{\mathbf{1}_{[\varepsilon/\delta, \varepsilon]}(|Y_s|)}{|Y_s| \log \delta} |Y_s|^{1+2\alpha} \mathrm{d}s \leq \frac{2TK^2 \varepsilon^{2\alpha}}{\log \delta}. \tag{5.12}$$

Let τ be a stopping time with $\tau \leq T$ and $Z_t := |Y_{t \wedge \tau}|$. From (5.6), (5.8), (5.9), (5.11) and (5.12), we obtain

$$\begin{aligned}
\mathbb{E}[Z_t] &\leq \varepsilon + L \int_0^t \mathbb{E}[Z_s] \mathrm{d}s + C_T ||b - \hat{b}||_1 + \frac{2C_T \delta}{\varepsilon \log \delta} ||\sigma - \hat{\sigma}||_2^2 + \frac{2TK^2 \varepsilon^{2\alpha}}{\log \delta} \\
&\leq \varepsilon + L \int_0^t \mathbb{E}[Z_s] \mathrm{d}s + C_T \varepsilon_1 + \frac{2C_T \delta}{\varepsilon \log \delta} \varepsilon_1 + \frac{2TK^2 \varepsilon^{2\alpha}}{\log \delta}.
\end{aligned}$$

If $\alpha \in (0, 1/2]$, then since $\varepsilon_1 < 1$, by choosing $\delta = 2$ and $\varepsilon = \varepsilon_1^{1/(2\alpha+1)}$, we have

$$\begin{aligned}
\mathbb{E}[Z_t] &\leq L \int_0^t \mathbb{E}[Z_s] \mathrm{d}s + \varepsilon_1^{1/(2\alpha+1)} + C_T \varepsilon_1 + \frac{4C_T \varepsilon_1^{1-1/(2\alpha+1)}}{\log 2} + \frac{2TK^2 \varepsilon_1^{2\alpha/(2\alpha+1)}}{\log 2} \\
&\leq L \int_0^t \mathbb{E}[Z_s] \mathrm{d}s + C_1(\alpha, T) \varepsilon_1^{2\alpha/(2\alpha+1)},
\end{aligned}$$

where

$$C_1(\alpha, T) := 1 + C_T + \frac{4C_T}{\log 2} + \frac{2TK^2}{\log 2}.$$

By Gronwall's inequality, we get

$$\mathbb{E}[Z_t] \leq C_1(\alpha, T) e^{LT} \varepsilon_1^{2\alpha/(2\alpha+1)}.$$

Therefore by the dominated convergence theorem, we conclude the statement by taking $t \to T$.

If $\alpha = 0$, then since $1/\log(1/\varepsilon_1) < 1$, by choosing $\delta = \varepsilon_1^{-1/2}$ and $\varepsilon = 1/\log(1/\varepsilon_1)$, we have

$$\mathbb{E}[Z_t] \leq L \int_0^t \mathbb{E}[Z_s] \mathrm{d}s + \frac{1}{\log(1/\varepsilon_1)} + C_T \varepsilon_1 + 4C_T \varepsilon_1^{1/2} + \frac{4TK^2}{\log(1/\varepsilon_1)} \leq L \int_0^t \mathbb{E}[Z_s] \mathrm{d}s + \frac{C_1(0, T)}{\log(1/\varepsilon_1)},$$

where

$$C_1(0, T) := 1 + 5C_T + 4TK^2.$$

By Gronwall's inequality, we obtain

$$\mathbb{E}[Z_t] \leq \frac{C_1(0, T) e^{LT}}{\log(1/\varepsilon_1)}.$$

Therefore by the dominated convergence theorem, we conclude the statement by taking $t \to T$. $\square$

Theorem 5.2.8. *Suppose that Assumption 5.2.3 holds. We assume that $\varepsilon_1 < 1$ if $\alpha \in (0,1/2]$ and $1/\log(1/\varepsilon_1) < 1$ if $\alpha = 0$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, \alpha, \lambda$ and x_0 such that*

$$\mathbb{E}[\sup_{0\le t\le T} |X_t - \hat{X}_t|] \le \begin{cases} C\varepsilon_1^{4\alpha^2/(2\alpha+1)} & \text{if } \alpha \in (0,1/2], \\ \dfrac{C}{\sqrt{\log(1/\varepsilon_1)}} & \text{if } \alpha = 0. \end{cases}$$

Before proving Theorem 5.2.8, we estimate the expectation of $\sup_{0\le s\le t}|M_s^{\delta,\varepsilon}|$ for any $t \in [0,T]$, $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$.

Lemma 5.2.9. *Suppose that the assumption of Theorem 5.2.8 hold. Then for any $t \in [0,T]$, $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$, we have*

$$\mathbb{E}[\sup_{0\le s\le t} |M_s^{\delta,\varepsilon}|] \le \begin{cases} \dfrac{1}{2}\mathbb{E}[V_t] + C_3(\alpha,T)\varepsilon_1^{4\alpha^2/(2\alpha+1)} & \text{if } \alpha \in (0,1/2], \\ \dfrac{C_3(0,T)}{\sqrt{\log(1/\varepsilon_1)}} & \text{if } \alpha = 0, \end{cases}$$

where

$$C_3(\alpha,T) := \begin{cases} \hat{C}_1^2 K^2 T C_1(\alpha,T)^{2\alpha} e^{2\alpha LT} + \sqrt{2}\hat{C}_1 C_T^{1/2}, & \text{if } \alpha \in (0,1/2], \\ \sqrt{2}\hat{C}_1 K T^{1/2} C_1(0,T)^{1/2} e^{LT/2} + \sqrt{2}\hat{C}_1 C_T^{1/2}, & \text{if } \alpha = 0, \end{cases}$$

and $\hat{C}_p$ is the constant of Burkholder-Davis-Gundy's inequality with $p > 0$.

Proof. Recall that for each $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$,

$$M_t^{\delta,\varepsilon} = \int_0^t \phi'_{\delta,\varepsilon}(Y_s)(\sigma(X_s) - \hat{\sigma}(\hat{X}_s))\mathrm{d}W_s.$$

Hence the quadratic variation of $M_t^{\delta,\varepsilon}$ is given by

$$\langle M^{\delta,\varepsilon}\rangle_t = \int_0^t |\phi'_{\delta,\varepsilon}(Y_s)|^2 |\sigma(X_s) - \hat{\sigma}(\hat{X}_s)|^2 \mathrm{d}s.$$

Thus from Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned} \mathbb{E}[\sup_{0\le s\le t} |M_s^{\delta,\varepsilon}|] &\le \hat{C}_1 \mathbb{E}[\langle M^{\delta,\varepsilon}\rangle_t^{1/2}] \le \hat{C}_1 \mathbb{E}\left[\left(\int_0^t |\sigma(X_s) - \hat{\sigma}(\hat{X}_s)|^2 \mathrm{d}s\right)^{1/2}\right] \\ &\le \sqrt{2}\hat{C}_1 \mathbb{E}\left[\left(\int_0^t |\sigma(X_s) - \sigma(\hat{X}_s)|^2 \mathrm{d}s\right)^{1/2}\right] + \sqrt{2}\hat{C}_1 \mathbb{E}\left[\left(\int_0^T |\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^2 \mathrm{d}s\right)^{1/2}\right]. \end{aligned}$$

From Jensen's inequality and Lemma 5.2.5, we have

$$\mathbb{E}\left[\left(\int_0^T |\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^2 \mathrm{d}s\right)^{1/2}\right] \le \left(\int_0^T \mathbb{E}\left[|\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^2\right] \mathrm{d}s\right)^{1/2} \le C_T^{1/2} ||\sigma - \hat{\sigma}||_2.$$

Since σ is $(1/2+\alpha)$-Hölder continuous, we obtain

$$\mathbb{E}[\sup_{0\le s\le t}|M_s^{\delta,\varepsilon}|] \le \sqrt{2}\hat{C}_1 K\mathbb{E}\left[\left(\int_0^t |Y_s|^{1+2\alpha}\mathrm{d}s\right)^{1/2}\right] + \sqrt{2}\hat{C}_1 C_T^{1/2}||\sigma-\hat{\sigma}||_2. \tag{5.13}$$

If $\alpha \in (0,1/2]$, then we get

$$\sqrt{2}\hat{C}_1 K\mathbb{E}\left[\left(\int_0^t |Y_s|^{1+2\alpha}\mathrm{d}s\right)^{1/2}\right] \le \sqrt{2}\hat{C}_1 K\mathbb{E}\left[V_t^{1/2}\left(\int_0^t |Y_s|^{2\alpha}\mathrm{d}s\right)^{1/2}\right].$$

Using Young's inequality $xy \le \frac{x^2}{2\sqrt{2}\hat{C}_1 K} + \frac{\sqrt{2}\hat{C}_1 K y^2}{2}$ for any $x,y\ge 0$ and Jensen's inequality, we obtain

$$\begin{aligned}\sqrt{2}\hat{C}_1 K\mathbb{E}\left[\left(\int_0^t |Y_s|^{1+2\alpha}\mathrm{d}s\right)^{1/2}\right] &\le \frac{1}{2}\mathbb{E}[V_t] + \frac{2\hat{C}_1^2K^2}{2}\int_0^T \mathbb{E}[|Y_s|^{2\alpha}]\mathrm{d}s \\ &\le \frac{1}{2}\mathbb{E}[V_t] + \hat{C}_1^2K^2T^{1-2\alpha}\left(\int_0^T \mathbb{E}[|Y_s|]\mathrm{d}s\right)^{2\alpha}.\end{aligned}$$

From Theorem 5.2.7 with $\tau=s$, we have

$$\sqrt{2}\hat{C}_1 K\mathbb{E}\left[\left(\int_0^t |Y_s|^{1+2\alpha}\mathrm{d}s\right)^{1/2}\right] \le \frac{1}{2}\mathbb{E}[V_t] + \hat{C}_1^2K^2TC_1(\alpha,T)^{2\alpha}e^{2\alpha LT}\varepsilon_1^{4\alpha^2/(2\alpha+1)}. \tag{5.14}$$

Since $4\alpha^2/(2\alpha+1)\le\alpha\le 1/2$, from (5.13) and (5.14), we get

$$\mathbb{E}[\sup_{0\le s\le t}|M_s^{\delta,\varepsilon}|] \le \frac{1}{2}\mathbb{E}[V_t] + C_3(\alpha,T)\varepsilon_1^{4\alpha^2/(2\alpha+1)}$$

which concludes the statement for $\alpha\in(0,1/2]$.

If $\alpha=0$, then from Jensen's inequality and Theorem 5.2.7 with $\tau=s$, we get

$$\sqrt{2}\hat{C}_1 K\mathbb{E}\left[\left(\int_0^t |Y_s|\mathrm{d}s\right)^{1/2}\right] \le \sqrt{2}\hat{C}_1 K\left(\int_0^T \mathbb{E}[|Y_s|]\mathrm{d}s\right)^{1/2} \le \frac{\sqrt{2}\hat{C}_1KT^{1/2}C_1(0,T)^{1/2}e^{LT/2}}{\sqrt{\log(1/\varepsilon_1)}}.$$

Therefore we have

$$\mathbb{E}[\sup_{0\le s\le T}|M_s^{\delta,\varepsilon}|] \le \frac{\sqrt{2}\hat{C}_1KT^{1/2}C_1(0,T)^{1/2}e^{LT/2}}{\sqrt{\log(1/\varepsilon_1)}} + \sqrt{2}\hat{C}_1C_T^{1/2}||\sigma-\hat{\sigma}||_2 \le \frac{C_3(0,T)}{\sqrt{\log(1/\varepsilon_1)}}.$$

This concludes the statement for $\alpha=0$. □

Using the above estimate, we can prove Theorem 5.2.8.

Proof of Theorem 5.2.8. Let $V_t := \sup_{0\le s\le t}|Y_s|$. From (5.6), (5.8), (5.10) and (5.12), we have

$$\begin{aligned} V_t \le& \varepsilon + L\int_0^t V_s \mathrm{d}s + \int_0^T |b(\hat{X}_s) - \hat{b}(\hat{X}_s)|\mathrm{d}s \\ &+ \frac{2\delta}{\varepsilon\log\delta}\int_0^T |\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^2\mathrm{d}s + \frac{2TK^2\varepsilon^{2\alpha}}{\log\delta} + \sup_{0\le s\le t}|M_s^{\delta,\varepsilon}|. \end{aligned} \tag{5.15}$$

If $\alpha\in(0,1/2]$, then from (5.15), Lemma 5.2.5 and Lemma 5.2.9, we have

$$\begin{aligned} \mathbb{E}[V_t] \le& \varepsilon + L\int_0^t \mathbb{E}[V_s]\mathrm{d}s + C_T||b-\hat{b}||_1 + \frac{2C_T\delta}{\varepsilon\log\delta}||\sigma-\hat{\sigma}||_2^2 + \frac{2TK^2\varepsilon^{2\alpha}}{\log\delta} \\ &+ \frac{1}{2}\mathbb{E}[V_t] + C_3(\alpha,T)\varepsilon_1^{4\alpha^2/(2\alpha+1)} \\ \le& \varepsilon + L\int_0^t \mathbb{E}[V_s]\mathrm{d}s + C_T\varepsilon_1 + \frac{2C_T\delta}{\varepsilon\log\delta}\varepsilon_1 + \frac{2TK^2\varepsilon^{2\alpha}}{\log\delta} + \frac{1}{2}\mathbb{E}[V_t] + C_3(\alpha,T)\varepsilon_1^{4\alpha^2/(2\alpha+1)}. \end{aligned}$$

Hence we get

$$\mathbb{E}[V_t] \le 2\varepsilon + 2L\int_0^t \mathbb{E}[V_s]\mathrm{d}s + 2C_T\varepsilon_1 + \frac{4C_T\delta}{\varepsilon\log\delta}\varepsilon_1 + \frac{4TK^2\varepsilon^{2\alpha}}{\log\delta} + 2C_3(\alpha,T)\varepsilon_1^{4\alpha^2/(2\alpha+1)}.$$

Note that $0<4\alpha^2/(2\alpha+1)\le\alpha\le 1/2$. Taking $\delta=2$ and $\varepsilon=\varepsilon_1^{1/2}$, we have

$$\begin{aligned} \mathbb{E}[V_t] &\le 2L\int_0^t \mathbb{E}[V_s]\mathrm{d}s + 2\left(1 + C_T + \frac{4C_T}{\log 2}\right)\varepsilon_1^{1/2} + \frac{4TK^2}{\log 2}\varepsilon_1^{\alpha} + 2C_3(\alpha,T)\varepsilon_1^{4\alpha^2/(2\alpha+1)} \\ &\le 2L\int_0^t \mathbb{E}[V_s]\mathrm{d}s + C_4(\alpha,T)\varepsilon_1^{4\alpha^2/(2\alpha+1)}, \end{aligned}$$

where

$$C_4(\alpha,T) := 2\left(1 + C_T + \frac{4C_T + 2TK^2}{\log 2} + C_3(\alpha,T)\right).$$

By Gronwall's inequality, we obtain

$$\mathbb{E}[V_t] \le C_4(\alpha,T)e^{2LT}\varepsilon_1^{4\alpha^2/(2\alpha+1)}.$$

If $\alpha=0$, then from (5.15), Lemma 5.2.5 and Lemma 5.2.9, we have

$$\mathbb{E}[V_t] \le \varepsilon + L\int_0^t \mathbb{E}[V_s]\mathrm{d}s + C_T\varepsilon_1 + \frac{2C_T\delta}{\varepsilon\log\delta}\varepsilon_1 + \frac{2TK^2}{\log\delta} + \frac{C_3(0,T)}{\sqrt{\log(1/\varepsilon_1)}}.$$

Taking $\delta=\varepsilon_1^{-1/2}$ and $\varepsilon = 1/\log(1/\varepsilon_1)$, we get

$$\mathbb{E}[V_t] \le L\int_0^t \mathbb{E}[V_s]\mathrm{d}s + \frac{C_4(0,T)}{\sqrt{\log(1/\varepsilon_1)}},$$

where

$$C_4(0,T) := 1 + 5C_T + 4TK^2 + C_3(0,T).$$

By Gronwall's inequality, we obtain

$$\mathbb{E}[V_t] \leq \frac{C_4(0,T)e^{LT}}{\sqrt{\log(1/\varepsilon_1)}}.$$

Hence we conclude the proof of Theorem 5.2.8. □

Theorem 5.2.10. *Suppose that Assumption 5.2.3 holds and $p \geq 2$. We assume that $\varepsilon_p < 1$ if $\alpha \in (0,1/2]$ and $1/\log(1/\varepsilon_p) < 1$ if $\alpha = 0$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, p, \alpha, \lambda$ and x_0 such that*

$$\mathbb{E}[\sup_{0\leq t\leq T} |X_t - \hat{X}_t|^p] \leq \begin{cases} C\varepsilon_p^{1/2} & \text{if } \alpha = 1/2, \\ C\varepsilon_1^{2\alpha/(2\alpha+1)} & \text{if } \alpha \in (0,1/2), \\ \dfrac{C}{\log(1/\varepsilon_1)} & \text{if } \alpha = 0. \end{cases}$$

Using Jensen's inequality, we can extend Theorem 5.2.10 as follows.

Corollary 5.2.11. *Suppose that Assumption 5.2.3 holds and $p \in (1,2)$. We assume that $\varepsilon_{2p} < 1$ if $\alpha \in (0,1/2]$ and $1/\log(1/\varepsilon_{2p}) < 1$ if $\alpha = 0$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, p, \alpha, \lambda$ and x_0 such that*

$$\mathbb{E}[\sup_{0\leq t\leq T} |X_t - \hat{X}_t|^p] \leq \begin{cases} C\varepsilon_{2p}^{1/2} & \text{if } \alpha = 1/2, \\ C\varepsilon_1^{\alpha/(2\alpha+1)} & \text{if } \alpha \in (0,1/2), \\ \dfrac{C}{\sqrt{\log(1/\varepsilon_1)}} & \text{if } \alpha = 0. \end{cases}$$

In order to prove Theorem 5.2.10, we first estimate the expectation of $\sup_{0\leq s\leq t} |M_s^{\delta,\varepsilon}|^p$ for any $p \geq 2$, $t \in [0,T]$, $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$.

Lemma 5.2.12. *Let $p \geq 2$. Assume that A-(ii), A-(iii) and A-(iv) hold. Then for any $t \in [0,T]$, $\delta \in (1,\infty)$ and $\varepsilon \in (0,1)$, we have*

$$\mathbb{E}[\sup_{0\leq s\leq t} |M_s^{\delta,\varepsilon}|^p] \leq C_5(p,T)\mathbb{E}\left[\left(\int_0^t |Y_s|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right] + C_6(p,T)||\sigma - \hat{\sigma}||_{2p}^p,$$

where $C_5(p,T) := 2^{p/2}C_pK^p$ and $C_6(p,T) := 2^{p/2}T^{\frac{p-1}{2}}C_pC_T^{1/2}$. In particular, if $\alpha = 1/2$, we have

$$\mathbb{E}[\sup_{0\leq s\leq t} |M_s^{\delta,\varepsilon}|^p] \leq \frac{1}{2\cdot 5^{p-1}}\mathbb{E}[V_t^p] + \frac{5^{p-1}C_5(p,T)^2T^{p-1}}{2}\int_0^t \mathbb{E}[V_s^p]\mathrm{d}s + C_6(p,T)||\sigma - \hat{\sigma}||_{2p}^p.$$

Proof of Lemma 5.2.12. From Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}\mathbb{E}[\sup_{0\le s\le t}|M_s^{\delta,\varepsilon}|^p] &\le C_p\mathbb{E}[\langle M^{\delta,\varepsilon}\rangle_t^{p/2}] \le C_p\mathbb{E}\left[\left(\int_0^t|\sigma(X_s)-\hat\sigma(\hat X_s)|^2\mathrm{d}s\right)^{p/2}\right]\\ &\le 2^{p/2}C_p\left(\mathbb{E}\left[\left(\int_0^t|\sigma(X_s)-\sigma(\hat X_s)|^2\mathrm{d}s\right)^{p/2}\right]+\mathbb{E}\left[\left(\int_0^T|\sigma(\hat X_s)-\hat\sigma(\hat X_s)|^2\mathrm{d}s\right)^{p/2}\right]\right).\end{aligned}$$

From Jensen's inequality and Lemma 5.2.5, we have

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T|\sigma(\hat X_s)-\hat\sigma(\hat X_s)|^2\mathrm{d}s\right)^{p/2}\right] &\le T^{\frac{p-1}{2}}\left(\int_0^T\mathbb{E}[|\sigma(\hat X_s)-\hat\sigma(\hat X_s)|^{2p}]\mathrm{d}s\right)^{1/2}\\ &\le T^{\frac{p-1}{2}}C_T^{1/2}||\sigma-\hat\sigma||_{2p}^p.\end{aligned}$$

Since σ is $(1/2+\alpha)$-Hölder continuous, we get

$$\mathbb{E}[\sup_{0\le s\le t}|M_s^{\delta,\varepsilon}|^p] \le C_5(p,T)\mathbb{E}\left[\left(\int_0^t|Y_s|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right]+C_6(p,T)||\sigma-\hat\sigma||_{2p}^p.$$

This concludes the first statement.

In particular, if $\alpha=1/2$, then we get from definition of V_t,

$$C_5(p,T)\mathbb{E}\left[\left(\int_0^t|Y_s|^2\mathrm{d}s\right)^{p/2}\right]\le C_5(p,T)\mathbb{E}\left[(V_t)^{p/2}\left(\int_0^t|Y_s|\mathrm{d}s\right)^{p/2}\right].$$

Using Young's inequality $xy\le\frac{x^2}{2\cdot5^{p-1}C_5(p,T)}+\frac{5^{p-1}C_5(p,T)y^2}{2}$ for any $x,y\ge0$ and Jensen's inequality, we obtain

$$\begin{aligned}C_5(p,T)\mathbb{E}\left[\left(\int_0^t|Y_s|^2\mathrm{d}s\right)^{p/2}\right] &\le \frac{1}{2\cdot5^{p-1}}\mathbb{E}[V_t^p]+\frac{5^{p-1}C_5(p,T)^2}{2}\mathbb{E}\left[\left(\int_0^t|Y_s|\mathrm{d}s\right)^{p}\right]\\ &\le \frac{1}{2\cdot5^{p-1}}\mathbb{E}[V_t^p]+\frac{5^{p-1}C_5(p,T)^2T^{p-1}}{2}\int_0^t\mathbb{E}[V_s^p]\mathrm{d}s,\end{aligned}$$

which concludes the second statement. □

To prove Theorem 5.2.10, we recall the following Gronwall type inequality.

Lemma 5.2.13 ([40], Lemma 3.2. (ii))**.** *Let $(A_t)_{0\le t\le T}$ be a nonnegative continuous stochastic process and set $B_t:=\sup_{0\le s\le t}A_s$. Assume that for some $r>0$, $q\ge1$, $\rho\in[1,q]$ and $C_1,\xi\ge0$,*

$$\mathbb{E}[B_t^r]\le C_1\mathbb{E}\left[\left(\int_0^tB_s\mathrm{d}s\right)^r\right]+C_1\mathbb{E}\left[\left(\int_0^tA_s^\rho\mathrm{d}s\right)^{r/q}\right]+\xi<\infty$$

for all $t \in [0,T]$*. If* $r \geq q$ *or* $q+1-\rho < r < q$ *hold, then there exists constant* C_2 *depending on* r, q, ρ, T *and* C_1 *such that*

$$\mathbb{E}[B_T^r] \leq C_2\xi + C_2 \int_0^T \mathbb{E}[A_s]\mathrm{d}s.$$

Now using Lemma 5.2.12 and Lemma 5.2.13, we can prove Theorem 5.2.10.

Proof of Theorem 5.2.10. From (5.15) and the inequality $\left(\sum_{i=1}^m a_i\right)^p \leq m^{p-1}\sum_{i=1}^m a_i^p$ for any $p \geq 2$ $a_i > 0$ and $m \in \mathbb{N}$, and Jensen's inequality, we have

$$\begin{aligned} V_t^p \leq & 5^{p-1}\Bigg(\varepsilon^p + \left(L\int_0^t V_s \mathrm{d}s\right)^p + T^{p-1}\int_0^T |b(\hat{X}_s) - \hat{b}(\hat{X}_s)|^p \mathrm{d}s \\ & + \frac{2T^{p-1}\delta^p}{\varepsilon^p(\log\delta)^p}\int_0^T |\sigma(\hat{X}_s) - \hat{\sigma}(\hat{X}_s)|^{2p}\mathrm{d}s + \frac{(2TK^2)^p\varepsilon^{2p\alpha}}{(\log\delta)^p} + \sup_{0\leq s\leq t}|M_s^{\delta,\varepsilon}|^p\Bigg). \end{aligned}$$

From Lemma 5.2.5 with $p \geq 2$, we have

$$\begin{aligned} \mathbb{E}[V_t^p] \leq & 5^{p-1}\varepsilon^p + 5^{p-1}L^p\mathbb{E}\left[\left(\int_0^t V_s \mathrm{d}s\right)^p\right] + (5T)^{p-1}C_T||b - \hat{b}||_p^p \\ & + \frac{2(5T)^{p-1}C_T\delta^p}{\varepsilon^p(\log\delta)^p}||\sigma - \hat{\sigma}||_{2p}^{2p} + \frac{5^{p-1}(2TK^2)^p\varepsilon^{2p\alpha}}{(\log\delta)^p} + 5^{p-1}\mathbb{E}[\sup_{0\leq s\leq t}|M_s^{\delta,\varepsilon}|^p]. \end{aligned}$$

If $\alpha = 1/2$, using Lemma 5.2.12, we have

$$\begin{aligned} \mathbb{E}[V_t^p] \leq & 5^{p-1}\varepsilon^p + (5T)^{p-1}\left(L^p + \frac{C_5(p,T)^2}{2}\right)\int_0^t \mathbb{E}[V_s^p]\mathrm{d}s + (5T)^{p-1}C_T||b - \hat{b}||_p^p \\ & + \frac{2(5T)^{p-1}C_T\delta^p}{\varepsilon^p(\log\delta)^p}||\sigma - \hat{\sigma}||_{2p}^{2p} + \frac{5^{p-1}(2TK^2)^p\varepsilon^p}{(\log\delta)^p} + \frac{1}{2}\mathbb{E}[V_T^p] + 5^{p-1}C_6(p,T)||\sigma - \hat{\sigma}||_{2p}^p. \end{aligned}$$

Hence we get

$$\begin{aligned} \mathbb{E}[V_t^p] \leq & 2\cdot 5^{p-1}\varepsilon^p + (5T)^{p-1}\left(2L^p + C_5(p,T)^2\right)\int_0^t \mathbb{E}[V_s^p]\mathrm{d}s + 2(5T)^{p-1}C_T||b - \hat{b}||_p^p \\ & + \frac{4(5T)^{p-1}C_T\delta^p}{\varepsilon^p(\log\delta)^p}||\sigma - \hat{\sigma}||_{2p}^{2p} + \frac{2\cdot 5^{p-1}(2TK^2)^p\varepsilon^p}{(\log\delta)^p} + 2\cdot 5^{p-1}C_6(p,T)||\sigma - \hat{\sigma}||_{2p}^p \\ \leq & 2\cdot 5^{p-1}\varepsilon^p + (5T)^{p-1}\left(2L^p + C_5(p,T)^2\right)\int_0^t \mathbb{E}[V_s^p]\mathrm{d}s + 2(5T)^{p-1}C_T\varepsilon_p + \frac{4(5T)^{p-1}C_T\delta^p}{\varepsilon^p(\log\delta)^p}\varepsilon_p \\ & + \frac{2\cdot 5^{p-1}(2TK^2)^p\varepsilon^p}{(\log\delta)^p} + 2\cdot 5^{p-1}C_6(p,T)\varepsilon_p^{1/2}. \end{aligned}$$

Taking $\delta = 2$ and $\varepsilon = \varepsilon_p^{1/(2p)}$, we have

$$\mathbb{E}[V_t^p] \leq (5T)^{p-1}\left(2L^p + C_5(p,T)^2\right)\int_0^t \mathbb{E}[V_s^p]\mathrm{d}s + C_7(1/2,p,T)\varepsilon_p^{1/2},$$

where

$$C_7(1/2,p,T) := 2\cdot 5^{p-1} + 2(5T)^{p-1}C_T + \frac{4\cdot 2^p(5T)^{p-1} + 2\cdot 5^{p-1}(2TK^2)^p}{(\log 2)^p} + 2\cdot 5^{p-1}C_6(p,T).$$

By Gronwall's inequality, we obtain

$$\mathbb{E}[V_t^p] \le C_7(1/2,p,T)\exp(5^{p-1}T^p\left(2L^p + C_5(p,T)^2\right))\varepsilon_p^{1/2}.$$

If $\alpha \in [0,1/2)$, using Lemma 5.2.12, we have

$$\begin{aligned}
\mathbb{E}[V_t^p] \le & 5^{p-1}\varepsilon^p + 5^{p-1}L^p\mathbb{E}\left[\left(\int_0^t V_s \mathrm{d}s\right)^p\right] + (5T)^{p-1}C_T||b-\hat{b}||_p^p + \frac{2(5T)^{p-1}C_T\delta^p}{\varepsilon^p(\log\delta)^p}||\sigma-\hat{\sigma}||_{2p}^{2p} \\
& + \frac{5^{p-1}(2TK^2)^p\varepsilon^{2p\alpha}}{(\log\delta)^p} + 5^{p-1}C_5(p,T)\mathbb{E}\left[\left(\int_0^t |Y_s|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right] + 5^{p-1}C_6(p,T)||\sigma-\hat{\sigma}||_{2p}^p \\
\le & 5^{p-1}L^p\mathbb{E}\left[\left(\int_0^t V_s \mathrm{d}s\right)^p\right] + 5^{p-1}C_5(p,T)\mathbb{E}\left[\left(\int_0^t |Y_s|^{1+2\alpha}\mathrm{d}s\right)^{p/2}\right] \\
& + 5^{p-1}\varepsilon^p + ((5T)^{p-1}C_T + 5^{p-1}C_6(p,T))\varepsilon_p^{1/2} + \frac{2(5T)^{p-1}C_T\delta^p}{\varepsilon^p(\log\delta)^p}\varepsilon_p + \frac{5^{p-1}(2TK^2)^p\varepsilon^{2p\alpha}}{(\log\delta)^p}.
\end{aligned}$$

Now we apply Theorem 5.2.7 with $\tau = s$ and Lemma 5.2.13 with $r = p$, $q = 2$, $\rho = 1+2\alpha$ and

$$\xi = 5^{p-1}\varepsilon^p + ((5T)^{p-1}C_T + 5^{p-1}C_6(p,T))\varepsilon_p^{1/2} + \frac{2(5T)^{p-1}C_T\delta^p}{\varepsilon^p(\log\delta)^p}\varepsilon_p + \frac{5^{p-1}(2TK^2)^p\varepsilon^{2p\alpha}}{(\log\delta)^p}.$$

Then there exists $C_7(\alpha,p,T)$ which depends on p,α,T,L and $C_5(p,T)$ such that

$$\begin{aligned}
\mathbb{E}[V_T^p] \le & C_7(\alpha,p,T)\left(\varepsilon^p + \varepsilon_p^{1/2} + \frac{\delta^p\varepsilon_p}{\varepsilon^p(\log\delta)^p} + \frac{\varepsilon^{2p\alpha}}{(\log\delta)^p}\right) + C_7(\alpha,p,T)\int_0^T \mathbb{E}[|Y_s|]\mathrm{d}s \\
\le & C_7(\alpha,p,T)\left(\varepsilon^p + \varepsilon_p^{1/2} + \frac{\delta^p\varepsilon_p}{\varepsilon^p(\log\delta)^p} + \frac{\varepsilon^{2p\alpha}}{(\log\delta)^p}\right) \\
& + \begin{cases} C_7(\alpha,p,T)C_1(\alpha,T)e^{LT}T\varepsilon_1^{2\alpha/(2\alpha+1)} & \text{if } \alpha\in(0,1/2), \\ \dfrac{C_7(0,p,T)C_1(0,T)e^{LT}T}{\log(1/\varepsilon_1)} & \text{if } \alpha = 0. \end{cases}
\end{aligned}$$

Taking $\delta = 2$ and $\varepsilon = \varepsilon_p^{1/(2p)}$ if $\alpha\in(0,1/2)$ and $\delta = \varepsilon_p^{-1/(2p)}$ and $\varepsilon = 1/\log(1/\varepsilon_p)$ if $\alpha = 0$, we get

$$\mathbb{E}[V_T^p] \le \begin{cases} C_8(\alpha,p,T)\varepsilon_1^{2\alpha/(2\alpha+1)} & \text{if } \alpha\in(0,1/2), \\ \dfrac{C_8(\alpha,p,T)}{\log(1/\varepsilon_1)} & \text{if } \alpha = 0, \end{cases}$$

where

$$C_8(\alpha,p,T) := \begin{cases} C_7(\alpha,p,T)\left(2 + \dfrac{2^p+1}{(\log 2)^p} + C_1(\alpha,T)e^{LT}T\right) & \text{if } \alpha\in(0,1/2), \\ C_7(\alpha,p,T)\left(2 + 2(2p)^p + C_1(\alpha,T)e^{LT}T\right) & \text{if } \alpha = 0, \end{cases}$$

Hence we conclude the proof of Theorem 5.2.10. $\square$

Next, we will find a bound for $\mathbb{E}[|g(X_T) - g(\hat{X}_T)|^r]$ where g is a function of bounded variation and $r \geq 1$.

Definition 5.2.14. *For a function $f : \mathbb{R} \to \mathbb{R}$, we define*

$$T_f(x) := \sup \sum_{j=1}^{N} |f(x_j) - f(x_{j-1})|.$$

Here the supremum is taken over all positive integers N and all partitions $-\infty < x_0 < x_1 < \cdots < x_N = x < \infty$. We call f a function of bounded variation, if

$$V(f) := \lim_{x \to \infty} T_f(x) < \infty.$$

Denote by BV the class of all functions of bounded variation.

Corollary 5.2.15. *Suppose that Assumption 5.2.3 holds. Furthermore assume that $\varepsilon_1 < 1$ if $\alpha \in (0, 1/2]$ and $1/\log(1/\varepsilon_1) < 1$ if $\alpha = 0$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, \alpha, \lambda$ and x_0 such that for any $g \in BV$ and $r \geq 1$,*

$$\mathbb{E}[|g(X_T) - g(\hat{X}_T)|^r] \leq \begin{cases} 3^{r+1} V(g)^r C \varepsilon_1^{\alpha/(2\alpha+1)} & \textit{if } \alpha \in (0, 1/2], \\ \dfrac{3^{r+1} V(g)^r C}{\sqrt{\log(1/\varepsilon_1)}} & \textit{if } \alpha = 0. \end{cases}$$

Remark 5.2.16. In the proof of all results, we calculate the constant C explicitly. In Theorem 5.2.7, 5.2.8, 5.2.10 and Corollary 5.2.11, the constant C does not blow up when $T \to 0$. On the other hand, in Corollary 5.2.15, the constant C may tend to infinity as $T \to 0$ because we use a Gaussian upper bound for the density of X_T in (5.16).

To prove Corollary 5.2.15, we recall the upper bound for $\mathbb{E}[|g(X) - g(\hat{X})|^r]$ where g is a function of bounded variation, $r \geq 1$, X and $\hat{X}$ are random variables.

Lemma 5.2.17 ([8], Theorem 4.3). *Let X and $\hat{X}$ be random variables. Assume that X has a bounded density p_X. If $g \in BV$ and $r \geq 1$, then for every $p \geq 1$, we have*

$$\mathbb{E}[|g(X) - g(\hat{X})|^r] \leq 3^{r+1} V(g)^r \left(\sup_{x \in \mathbb{R}} p_X(x) \right)^{\frac{p}{p+1}} \mathbb{E}[|X - \hat{X}|^p]^{1/(p+1)}.$$

Using the above Lemma, we can prove Corollary 5.2.15.

Proof of Corollary 5.2.15. From the Gaussian upper bound for the density $p_T(x_0, \cdot)$ of X_T, we have for any $y \in \mathbb{R}$,

$$p_T(x_0, y) \leq \overline{C} p_{c_*}(T, x_0, y) \leq \frac{\overline{C}}{\sqrt{2\pi c_* T}}. \tag{5.16}$$

This means that the density $p_T(x_0, \cdot)$ of X_T is bounded. Hence from Lemma 5.2.17 with $p = 1$ and Theorem 5.2.7 with $\tau = T$, for any $g \in BV$ and $r \geq 1$, we have

$$\mathbb{E}[|g(X_T) - g(\hat{X}_T)|^r] \leq \frac{3^{r+1} V(g)^r \overline{C}^{1/2}}{(2\pi c_* T)^{1/4}} \mathbb{E}[|X_T - \hat{X}_T|]^{1/2}$$

$$\leq \begin{cases} 3^{r+1}V(g)^r C_2(\alpha,T)\varepsilon_1^{\alpha/(2\alpha+1)} & \text{if } \alpha \in (0,1/2], \\ \dfrac{3^{r+1}V(g)^r C_2(0,T)}{\sqrt{\log(1/\varepsilon_1)}} & \text{if } \alpha = 0, \end{cases}$$

where

$$C_2(\alpha,T) := \frac{\overline{C}^{1/2} C_1(\alpha,T)^{1/2} e^{LT/2}}{(2\pi c_* T)^{1/4}}, \text{ for } \alpha \in [0,1/2].$$

This concludes the proof of statement. □

5.3 Application to the stability problem

In this section, we apply our main results to the stability problem. For any $n \in \mathbb{N}$, we consider the one-dimensional stochastic differential equation

$$X_t^{(n)} = x_0 + \int_0^t b_n(X_s^{(n)})\mathrm{d}s + \int_0^t \sigma_n(X_t^{(n)})\mathrm{d}W_s.$$

Assumption 5.3.1. *We assume that the coefficients b, σ and the sequence of coefficients $(b_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ satisfy the following conditions:*

A'-(i) : $b \in \mathcal{L}$.

A'-(ii) : b and b_n are bounded measurable i.e., there exists $K > 0$ such that

$$\sup_{n\in\mathbb{N}, x\in\mathbb{R}} (|b_n(x)| \vee |b(x)|) \leq K.$$

A'-(iii) : σ and σ_n are $\eta = 1/2 + \alpha$-Hölder continuous with $\alpha \in [0,1/2]$, i.e., there exists $K > 0$ such that

$$\sup_{n\in\mathbb{N}, x,y\in\mathbb{R}, x\neq y} \left(\frac{|\sigma(x)-\sigma(y)|}{|x-y|^\eta} \vee \frac{|\sigma_n(x)-\sigma_n(y)|}{|x-y|^\eta} \right) \leq K.$$

A'-(iv) : $a = \sigma$ and $a_n := \sigma_n^2$ are bounded and uniformly elliptic, i.e., there exists $\lambda \geq 1$ such that for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\lambda^{-1} \leq a(x) \leq \lambda \text{ and } \lambda^{-1} \leq a_n(x) \leq \lambda.$$

A'-(p) : For given $p > 0$,

$$\varepsilon_{p,n} := ||b - b_n||_p^p \vee ||\sigma - \sigma_n||_{2p}^{2p} \to 0$$

as $n \to \infty$.

For $p \geq 1$ and $\alpha \in [0, 1/2]$, we define $N_{\alpha,p}$ by

$$N_{\alpha,p} := \begin{cases} \min\{n \in \mathbb{N} : \varepsilon_{p,m} < 1, \forall m \geq n\}, & \text{if } \alpha \in (0, 1/2], \\ \min\{n \in \mathbb{N} : \varepsilon_{p,m} < 1/e, \forall m \geq n\}, & \text{if } \alpha = 0. \end{cases}$$

Then using Theorem 5.2.7, 5.2.8, 5.2.10 and Corollary 5.2.11, 5.2.15, we have the following corollaries.

Corollary 5.3.2. *Suppose that Assumption 5.3.1 holds with $p = 1$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, \alpha, \lambda$ and x_0 such that for any $n \geq N_{\alpha,1}$,*

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[|X_\tau - X_\tau^{(n)}|] \leq \begin{cases} C\varepsilon_{1,n}^{2\alpha/(2\alpha+1)} & \text{if } \alpha \in (0, 1/2], \\ \dfrac{C}{\log(1/\varepsilon_{1,n})} & \text{if } \alpha = 0 \end{cases}$$

and

$$\mathbb{E}[\sup_{0 \leq t \leq T} |X_t - X_t^{(n)}|] \leq \begin{cases} C\varepsilon_{1,n}^{4\alpha^2/(2\alpha+1)} & \text{if } \alpha \in (0, 1/2], \\ \dfrac{C}{\sqrt{\log(1/\varepsilon_{1,n})}} & \text{if } \alpha = 0 \end{cases}$$

and for any $g \in BV$ and $r \geq 1$, we have

$$\mathbb{E}[|g(X_T) - g(X_T^{(n)})|^r] \leq \begin{cases} 3^{r+1}V(g)^r C\varepsilon_{1,n}^{\alpha/(2\alpha+1)} & \text{if } \alpha \in (0, 1/2], \\ \dfrac{3^{r+1}V(g)^r C}{\sqrt{\log(1/\varepsilon_{1,n})}} & \text{if } \alpha = 0. \end{cases}$$

Corollary 5.3.3. *Suppose that Assumption 5.3.1 holds with $p \geq 2$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, p, \alpha, \lambda$ and x_0 such that for any $n \geq N_{\alpha,p}$,*

$$\mathbb{E}[\sup_{0 \leq t \leq T} |X_t - X_t^{(n)}|^p] \leq \begin{cases} C\varepsilon_{p,n}^{1/2} & \text{if } \alpha = 1/2, \\ C\varepsilon_{1,n}^{2\alpha/(2\alpha+1)} & \text{if } \alpha \in (0, 1/2), \\ \dfrac{C}{\log(1/\varepsilon_{1,n})} & \text{if } \alpha = 0. \end{cases}$$

Corollary 5.3.4. *Suppose that Assumption 5.3.1 holds with $2p$ for $p \in (1, 2)$. Then there exists a positive constant C which depends on $\overline{C}, c_*, K, L, T, p, \alpha, \lambda$ and x_0 such that for any $n \geq N_{\alpha,2p}$,*

$$\mathbb{E}[\sup_{0 \leq t \leq T} |X_t - X_t^{(n)}|^p] \leq \begin{cases} C\varepsilon_{2p,n}^{1/2} & \text{if } \alpha = 1/2, \\ C\varepsilon_{1,n}^{\alpha/(2\alpha+1)} & \text{if } \alpha \in (0, 1/2), \\ \dfrac{C}{\sqrt{\log(1/\varepsilon_{1,n})}} & \text{if } \alpha = 0. \end{cases}$$

The next proposition shows that there exist the sequences $(b_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ satisfying Assumption 5.3.1.

Proposition 5.3.5. *(i) Assume $\sup_{x \in \mathbb{R}} |b(x)| \leq K$. If the set of discontinuity points of b is a null set with respect to the Lebesgue measure, then there exists a differentiable and bounded sequence $(b_n)_{n \in \mathbb{N}}$ such that for any $p \geq 1$,*

$$\int_{\mathbb{R}} |b(x) - b_n(x)|^p e^{-\frac{|x-x_0|^2}{2c_* T}} \mathrm{d}x \to 0 \tag{5.17}$$

as $n \to \infty$. Moreover, if b is a one-sided Lipschitz function, we can construct an explicit sequence $(b_n)_{n\in\mathbb{N}}$ which satisfies a one-sided Lipschitz condition.

(ii) If the diffusion coefficient σ satisfies A'-(ii) and A'-(iii), then there exists a differentiable sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$, σ_n satisfies A'-(iii), A'-(iv) and for any $p \geq 1$,

$$\int_{\mathbb{R}} |\sigma(x) - \sigma_n(x)|^{2p} e^{-\frac{|x-x_0|^2}{2c_*T}} \mathrm{d}x \leq \frac{K^{2p}\sqrt{2\pi c_* T}}{n^{2p\eta}}.$$

Proof. Let $\rho(x) := \mu e^{-1/(1-|x|^2)}\mathbf{1}(|x|<1)$ with $\mu^{-1} = \int_{|x|<1} e^{-1/(1-|x|^2)}\mathrm{d}x$ and a sequence $(\rho_n)_{n\in\mathbb{N}}$ be defined by $\rho_n(x) := n\rho(nx)$. We set $b_n(x) := \int_{\mathbb{R}} b(y)\rho_n(x-y)\mathrm{d}y$ and $\sigma_n(x) := \int_{\mathbb{R}} \sigma(y)\rho_n(x-y)\mathrm{d}y$. Then for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have $|b_n(x)| \leq K$ and $\lambda^{-1} \leq a_n(x) := \sigma_n^2(x) \leq \lambda$, b_n and σ_n are differentiable.

Proof of (i). From Jensen's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}} |b(x) - b_n(x)|^p e^{-\frac{|x-x_0|^2}{2c_*T}} \mathrm{d}x &\leq \int_{\mathbb{R}} \mathrm{d}x \left(\int_{\mathbb{R}} \mathrm{d}y |b(x) - b(y)|\rho_n(x-y) \right)^p e^{-\frac{|x-x_0|^2}{2c_*T}} \\
&= \int_{\mathbb{R}} \mathrm{d}x \left(\int_{|z|<1} \mathrm{d}z |b(x) - b(x-z/n)|\rho(z) \right)^p e^{-\frac{|x-x_0|^2}{2c_*T}} \\
&\leq \int_{|z|<1} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}x |b(x) - b(x-z/n)|^p e^{-\frac{|x-x_0|^2}{2c_*T}} \rho(z).
\end{aligned}$$

Since b is bounded, we have

$$\int_{\mathbb{R}} |b(x) - b(x-z/n)|^p e^{-\frac{|x-x_0|^2}{2c_*T}} \mathrm{d}x \leq (2K)^p \int_{\mathbb{R}} e^{-\frac{|x-x_0|^2}{2c_*T}} \mathrm{d}x = (2K)^p \sqrt{2\pi c_* T}. \tag{5.18}$$

On the other hand, since the set of discontinuity points of b is a null set with respect to the Lebesgue measure, b is continuous almost everywhere. From (5.18), using the dominated convergence theorem, we have

$$\int_{\mathbb{R}} |b(x) - b(x-z/n)|^p e^{-\frac{|x-x_0|^2}{2c_*T}} \mathrm{d}x \to 0$$

as $n \to \infty$. From this fact and the dominated convergence theorem, $(b_n)_{n\in\mathbb{N}}$ satisfies (5.17).

Let b be a one-sided Lipschitz function. Then, we have

$$\begin{aligned}
(x-y)(b_n(x) - b_n(y)) &= \int_{\mathbb{R}} (x-y)(b(x-z) - b(y-z))\rho_n(z)\mathrm{d}z \\
&= \int_{\mathbb{R}} \{(x-z) - (z-y)\}(b(x-z) - b(y-z))\rho_n(z)\mathrm{d}z \\
&\leq L|x-y|^2,
\end{aligned}$$

which implies that $(b_n)_{n\in\mathbb{N}}$ satisfies the one-sided Lipschitz condition.

Proof of (ii). In the same way as in the proof of (i), we have from Hölder continuity of σ

$$\int_{\mathbb{R}} |\sigma(x) - \sigma_n(x)|^{2p} e^{-\frac{|x-x_0|^2}{2c_*T}} \mathrm{d}x \leq \int_{|z|<1} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}x |\sigma(x) - \sigma(x-z/n)|^{2p} e^{-\frac{|x-x_0|^2}{2c_*T}} \rho(z)$$

$$\leq \frac{K^{2p}}{n^{2p\eta}} \int_{|z|<1} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}x e^{-\frac{|x-x_0|^2}{2c_* T}} \rho(z) = \frac{K^{2p}\sqrt{2\pi c_* T}}{n^{2p\eta}}.$$

Finally, we show that σ_n is η-Hölder continuous. For any $x, y \in \mathbb{R}$,

$$|\sigma_n(x) - \sigma_n(y)| \leq \int_{\mathbb{R}} |\sigma(x-z) - \sigma(y-z)| \rho_n(z) \mathrm{d}z \leq K|x-y|^{\eta},$$

which implies that σ_n is η-Hölder continuous. This concludes that $(\sigma_n)_{n\in\mathbb{N}}$ satisfies (ii). $\square$

Chapter 6

The parametrix method for skew diffusions

6.1 Introduction

A skew diffusion is the unique solution of the following one dimensional stochastic differential equation (SDE) with local time:

$$X_t(x) = x + \int_0^t b(X_s(x))\mathrm{d}s + \int_0^t \sigma(X_s(x))\mathrm{d}W_s + (2\alpha - 1)L_t^0(X),\ t \geq 0, \alpha \in (0,1), \tag{6.1}$$

where $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion and $L^0(X) = (L_t^0(X))_{t\geq 0}$ is the symmetric local time of X at the origin. Here, we will assume that b is bounded and measurable and σ is uniformly elliptic, bounded and $a^2 = \sigma$ is a Hölder continuous function.

Suppose that $b = 0$ and $\sigma = 1$, then the solution of (6.1) is called the skew Brownian motion. Harrison and Shepp [43] prove that if $|2\alpha - 1| \leq 1$ then there is a unique strong solution and if $|2\alpha - 1| > 1$, there is no solution. The idea of the proof is a transformation technique to relate (6.1) with another stochastic differential equation without local time.

The equation (6.1) is linked with various applications as can be seen in Lejay [78] and the references therein. For example, Lejay and Martinez [79] introduce a numerical scheme for a skew diffusion, which is based on the simulation of skew Brownian motion. Martinez and Talay [86] prove that the expectation of a skew diffusion is a solution to a parabolic type partial differential equation with interface conditions at zero. They also introduce a transformed Euler scheme and provide the weak convergence rate for their numerical scheme. Another approximation scheme for a skew diffusion is introduced by Étoré [22] using a random walk approach. In [23], Étoré and Martinez introduce an exact simulation scheme for skew diffusions when the diffusion coefficient is constant.

Gairat and Shcherbakov in [30] give explicitly the joint density function of a skew diffusion with constant diffusion coefficient and some of its functionals. They apply their results to a mathematical finance model of stock prices with switching coefficients.

In this chapter, our goal is to show that one can apply the parametrix method for stochastic differential equations of the type (6.1) and provide a probabilistic representation for the density. The other goal of this chapter is to show as one possible application of this result a Gaussian upper bound for the density exists and the differentiability of this density with respect to the initial variable x. The main mathematical difficulty we have to face is the fact that we have to deal with the local time term appearing in the equation (6.1).

The parametrix method (cf. Friedman [28]) is a classical method in order to construct fundamental solution for parabolic type partial differential equations using a "Taylor-like" expansion argument. This method allows for coefficients to be less regular than in the Malliavin Calculus approach for the study of the density. On the other hand, this methodology is restricted to situations where the underlying process is Markov. As a sample of recent developments of this method, we refer the reader to Menozzi [89], Foschi et al [19] and the references therein.

Bally and Kohatsu-Higa [9] introduce the parametrix method using a semigroup approach and obtain the probabilistic representation for the density of the solution to a diffusion equation or for Lévy driven SDEs. They consider two kinds of parametrix methods already considered in [19]: the first one is called "forward parametrix method" and second one is called "backward parametrix method". In order to construct a forward parametrix expansion for diffusion process, we need to assume that the coefficients are C_b^2. On the other hand, a backward parametrix expansion converges if the drift coefficient is bounded and measurable and diffusion coefficient is bounded, uniformly elliptic and Hölder continuous.

To simplify the discussion, we will only consider the backward parametrix method. When one applies the parametrix method for the semigroup of diffusion equations, one uses the Euler scheme with coefficients evaluated at the arrival point of the density as the approximation process, in order to obtain the expansion for the semigroup around this approximation process.

For a skew diffusion case, we will take a generalized version of the skew Brownian motion (see (6.10)) as "approximation process". The reason for this choice instead of the usual Euler scheme is because the latter is probably not suitable for this argument and that the density function of skew Brownian motion can still be written explicitly.

The parametrix expansion leads to a probabilistic representation for the density function of skew diffusions and therefore also provide a representation for the expectation of $f(X_T(x))$ for certain classes of functions f. Such a probabilistic representation can be used for many purposes, notably for Monte Carlo simulation or as an extension of the classical infinite dimensional analysis known as Malliavin Calculus. These and other applications such as lower bounds, differentiability with respect to time will be discussed elsewhere.

Finally, we note that our results for a skew diffusion process can also be extended to a diffusion process with discontinuous coefficients by using the relation between the two processes (see Proposition 6.2.7).

This chapter is divided as follows: In Section 6.2, we give the notation and assumptions used throughout the chapter. We will introduce the definition of the symmetric local time for continuous semi-martingale and the skew Brownian motion. We will also see the relation between skew diffusion and a SDE with discontinuous diffusion coefficient with an explicit construction of the skew diffusion flows. In Section 6.3, we obtain the generator associated with X and its domain of definition. In Section 6.4 we will give some key estimates in order to construct a parametrix expansion for a skew diffusion based on skew Brownian motion. In Section 6.5, we provide the parametrix method for the skew diffusion process using the semigroup approach to prove existence and Gaussian upper bound for its density function. Our main result is given in Theorem 6.5.1. In Section 6.6, we will show a regularity of the density function for a skew diffusion obtained in Section 6.5. This gives our second main result in the form of Theorem 6.6.1. In Section 6.7, we provide our probabilistic representation for the density of a skew diffusion process which based on a parametrix expansion. In a short Appendix, we provide an explicit calculation for beta type integrals.

6.2 Preliminaries

6.2.1 Notations and assumptions

We give some basic notations and definitions used throughout this chapter. For a sequence of operators $(S_i)_{i=1,\ldots,n}$, we define $\prod_{i=1}^{n} S_i = S_1 \ldots S_n$ and $\prod_{i=n}^{1} S_i = S_n \ldots S_1$. S^* will denote the adjoint operator of the operator S. The space of real valued infinitely differentiable functions with compact support contained in $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. Similarly, $L^\infty(\mathbb{R})$ denotes the space of all bounded measurable functions with the norm $||f||_\infty := \text{esssup}_{x\in\mathbb{R}}|f(x)|$. We define $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. We denote by g_t^c the density function of the standard Brownian motion with variance c, i.e., $g_t^c(y) := \frac{e^{-\frac{y^2}{2tc}}}{\sqrt{2\pi tc}}$, $y \in \mathbb{R}$. The associated Hermite polynomials are defined respectively as $H_i(y, ct) := g_t^c(y)^{-1}\partial_y^i g_t^c(y)$, $i \in \mathbb{N}$. We define the Mittag-Leffler function $E_{\alpha,\beta}$ defined as $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$, $z \in \mathbb{R}, \alpha, \beta > 0$. Throughout this chapter, we will use $t_0 := T$ as a fixed time where the densities will be evaluated.

We now state our main hypothesis on the coefficients of the SDE (6.1).

Assumption 6.2.1. *The measurable functions b and $\sigma : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(i) σ is a positive, bounded and uniformly elliptic function. In particular, there exist positive constants $\overline{a}$ and $\underline{a}$, such that for any $x \in \mathbb{R}$, $\underline{a} \le a(x) := \sigma^2(x) \le \overline{a}$.

(ii) b is bounded and $a = \sigma^2$ is η-Hölder continuous for some $\eta \in (0, 1]$, i.e., there exist a positive constant K such that

$$\sup_{x\in\mathbb{R}} |b(x)| + \sup_{x,y\in\mathbb{R},x\neq y} \frac{|a(x) - a(y)|}{|x - y|^\eta} \le K.$$

Remark 6.2.2. If σ is continuous and uniformly elliptic, then σ is either positive or negative. If σ is negative, we can replace the Brownian motion W_t by $-W_t$. Therefore, the assumption that σ is positive is only for convenience.

Through the article, the constants C and c may change from line to line, where C may depend on $(K, \alpha, \underline{a}, \overline{a}, \eta)$ and c may depend on $(\alpha, \overline{a})$. As a particular constant with explicit dependence on time, we use the notation $C_T := C(1 + T^{\frac{1-\eta}{2}})$.

For solutions of SDE's, we write $X_t(x)$ or X_t indistinctly to denote the solution process.

6.2.2 Construction of the solution flow process X

In this section, we give a specific construction of the solution for (6.1) using a transformation method. The arguments of this section have an intersection with parts in Lejay [78] and Kulik [73]. We give them here for the sake of completeness so that the reader may follow easily the arguments.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We first recall the definition of *symmetric local time* for a one-dimensional continuous semi-martingale $Y = (Y_t)_{t\geq 0}$. A stochastic process $(L_t^a(Y))_{t\geq 0}$ is called the symmetric local time of Y at $a \in \mathbb{R}$ if it satisfies

$$|Y_t - a| = |Y_0 - a| + \int_0^t (\mathbf{1}(Y_s > a) - \mathbf{1}(Y_s < a))\mathrm{d}Y_s + L_t^a(Y).$$

By Itô-Tanaka formula, the symmetric local time of Y exists and is unique (see e.g. Karatzas and Shreve [60], Section 3.7).

Let us consider the one-dimensional stochastic differential equation

$$Z_t(z) = z + \int_0^t \rho(Z_s(z))\mathrm{d}W_s, z \in \mathbb{R}, t \geq 0, \tag{6.2}$$

on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this section, we prove that the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$. We will use this in the proof of Proposition 6.5.6.

Theorem 6.2.3. *Assume that ρ is a measurable function and that there exist positive constants $c_0 > 1$ such that for any $z \in \mathbb{R}$, $c_0^{-1} \leq \rho^2(z) \leq c_0$. Then there exists a weak solution for the SDE (6.2) and the uniqueness in the sense of probability law holds. Moreover if ρ is continuous on $\mathbb{R}_0$ then the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$ and $z \in \mathbb{R}$ almost surely.*

In order to prove the above theorem, we first introduce the result of Engelbert and Schmidt.

Lemma 6.2.4 ([60], Theorems 5.5.4 and 5.5.7)**.** *The stochastic differential equation (6.2) has a non-exploding weak solution if and only if*

$$I(\rho) \subseteq \mathcal{Z}(\rho),$$

where

$$I(\rho) := \left\{ z \in \mathbb{R} | \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{d}y}{\rho^2(z+y)} = \infty, \forall \varepsilon > 0 \right\} \text{ and } \mathcal{Z}(\rho) := \{z \in \mathbb{R} | \rho(z) = 0\}.$$

Moreover, the SDE (6.2) has a weak solution and is unique in the sense of probability law if and only if

$$I(\rho) = \mathcal{Z}(\rho).$$

The explicit construction of this unique solution will be used in what follows. For explicit details, we refer the reader to [60], Chapter 5.

Let $B = (B_t, \mathcal{G}_t)_{t\geq 0}$ be a one dimensional Brownian motion on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and assume without loss of generality that the filtration $\mathcal{G} := (\mathcal{G}_t)_{t\geq 0}$ satisfies the usual conditions. For $(s, z) \in [0, \infty) \times \mathbb{R}$, we define

$$T_s(z) := \int_0^s \frac{\mathrm{d}u}{\rho^2(z + B_u)}.$$

Then $T_s(z)$ is strictly increasing and continuous with respect to s. Furthermore, from Problem 3.6.30 [60] (using the uniform ellipticity condition), we have that

$$\lim_{s\uparrow\infty} T_s(z) = \infty,\ a.s.$$

Define $A_\cdot(z)$ as the inverse of $T_\cdot(z)$, i.e.,

$$A_t(z) := \inf\{s \geq 0 | T_s(z) > t\}.$$

From Problem 3.4.5 (v) in [60], $A_t(z)$ is a $\mathcal{G}$-stopping time. We set

$$M_t(z) := B_{A_t(z)},\ Z_t(z) := z + M_t(z),\ \mathcal{F}_t := \mathcal{G}_{A_t},\ 0 \leq t. \tag{6.3}$$

Then there exists a Brownian motion $W = (W_t, \tilde{\mathcal{F}}_t)_{t\geq 0}$ on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$Z_t(z) = z + M_t(z) = z + \int_0^t \rho(Z_s(z))\mathrm{d}W_s,\ 0 \leq t,\ \tilde{\mathbb{P}}\text{-a.s.}$$

This means that $(Z(z), W)$ is a weak solution to the SDE (6.2). Using these notations, we prove Theorem 6.2.3.

Proof of Theorem 6.2.3. Since ρ is bounded and uniformly elliptic, we have from Lemma 6.2.4 that there exists a weak solution and uniqueness in the sense of probability law holds.

Now we prove that the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$ almost surely. From (6.3), it suffices to prove that $z \mapsto A_t(z)$ is continuous for any $t \geq 0$. In order to obtain that result, we first need to prove that $z \mapsto T_t(z)$ is continuous.

Fix $z \in \mathbb{R}$. We take a sequence $(z_k)_{k\in\mathbb{N}}$ which converges to z. Then for given $\delta > 0$, there exists $K \in \mathbb{N}$ such that for any $k \geq K$, $|z_k - z| \leq \delta$. Moreover, for any $\varepsilon > 0$ and $k \geq K$ we have that

$$\begin{aligned}
|T_t(z) - T_t(z_k)| \leq & \frac{1}{c_0}\int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(z + B_u)\mathrm{d}u + \frac{1}{c_0}\int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(z_k + B_u)\mathrm{d}u \\
& + \int_0^t \left| \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z + B_u)}{\rho^2(z + B_u)} - \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z_k + B_u)}{\rho^2(z_k + B_u)} \right| \mathrm{d}u \\
\leq & \frac{2}{c_0}\int_0^t \mathbf{1}_{A_{\varepsilon,\delta}}(B_u)\mathrm{d}u + \int_0^t \left| \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z + B_u)}{\rho^2(z + B_u)} - \frac{\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(z_k + B_u)}{\rho^2(z_k + B_u)} \right| \mathrm{d}u,
\end{aligned} \tag{6.4}$$

where $A_{\varepsilon,\delta} := (-\varepsilon - \delta - z, \varepsilon + \delta - z)$. Since ρ is continuous on $\mathbb{R}_0$, the second term on the right hand side of (6.4) converges to 0 as $k \to \infty$. Therefore, we have

$$\limsup_{k\to\infty} |T_t(z) - T_t(z_k)| \leq \frac{2}{c_0}\int_0^t \mathbf{1}_{A_{\varepsilon,\delta}}(B_u)\mathrm{d}u.$$

Since $\varepsilon, \delta > 0$ are arbitrary, by taking limits as ε, δ tend to 0, we have from the occupation formula (see [60], Problem 3.6.7),

$$\limsup_{k\to\infty} |T_t(z) - T_t(z_k)| \leq \frac{2}{c_0}\int_0^t \mathbf{1}(B_u = -z)\mathrm{d}u = \frac{4}{c_0}\int_{\mathbb{R}} \mathbf{1}(y = -z)L_t^y(B)\mathrm{d}y = 0,\ a.s.$$

Therefore we conclude that $\lim_{k\to\infty} |T_t(z) - T_t(z_k)| = 0$ a.s. Note that the right hand side of (6.4) is increasing with respect to t. Therefore, we also have that $z \mapsto T_t(z)$ is continuous for any $t \geq 0$ a.s.

Next we prove that for fixed $t_0 > 0$, $z \mapsto A_t(z)$ is continuous for any $t \in [0, t_0/c_0]$. We first note that since ρ is uniformly elliptic we have for any $z \in \mathbb{R}$,

$$\frac{t_0}{c_0} \leq T_{t_0}(z).$$

Since $T_t(z)$ is strictly increasing with respect to $t \geq 0$, it holds that for any $(t, z) \in [0, t_0/c_0] \times \mathbb{R}$,

$$A_t(z) \leq A_{t_0/c_0}(z) \leq A_{T_{t_0}(z)}(z) = t_0.$$

Fix $(t, z) \in [0, t_0/c_0] \times \mathbb{R}$. We take a sequence $(z_k)_{k\in\mathbb{N}}$ which converges to z and define $s := A_t(z)$ and $s_k := A_t(z_k)$. Note that $s, s_k \in [0, t_0]$ for any $k \in \mathbb{N}$. Now, we assume by contradiction that s_k does not converge to s. Then there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$, there exists $k_n \geq n$ such that $|s_{k_n} - s| \geq \varepsilon_0$. On the other hand, $(s_{k_n})_{n\in\mathbb{N}}$ is a sequence on the compact set $[0, t_0]$. Therefore there exists a sub-sequence $(s_{k_n(m)})_{m\in\mathbb{N}}$ of $(s_{k_n})_{n\in\mathbb{N}}$ such that $\lim_{m\to\infty} s_{k_n(m)} = s' \neq s$. Since $\lim_{s\to t} \sup_{x\in\mathbb{R}} |T_t(z) - T_s(z)| = 0$ then $T_t(z)$ is continuous in time uniformly for $z \in \mathbb{R}$. This result together with the continuity of $T_t(z)$ with respect to z gives the joint continuity of $T_t(z)$ for $(t, z) \in [0, \infty) \times \mathbb{R}_0$. Therefore we conclude

$$T_s(z) = T_{A_t(z)}(z) = t = T_{A_t(z_{k_n(m)})}(z_{k_n(m)}) = T_{s_{k_n(m)}}(z_{k_n(m)}) = \lim_{m\to\infty} T_{s_{k_n(m)}}(z_{k_n(m)}) = T_{s'}(z).$$

Since $T_t(z)$ is strictly increasing with respect to t, we conclude $s = s'$. This is a contradiction, so $A_t(z)$ is continuous with respect to z for any $t \in [0, t_0/c_0]$ a.s. Furthermore, as t_0 is arbitrary, $A_t(z)$ is continuous with respect to z for any $t \geq 0$ a.s. Therefore the conclusion follows from (6.3). ☐

Now we consider the following one-dimensional stochastic differential equation with drift

$$Z_t(z) = z + \int_0^t \mu(Z_s(z))\mathrm{d}s + \int_0^t \rho(Z_s(z))\mathrm{d}W_s,\ z \in \mathbb{R}. \tag{6.5}$$

We introduce the method of removal of drift coefficient introduced in section 5.5 B in [60]. Assume that

$$\text{(ND)} : \rho^2(z) > 0,\ z \in \mathbb{R},$$

$$\text{(LI)} : \forall z \in \mathbb{R}, \exists \varepsilon > 0 \text{ such that } \int_{z-\varepsilon}^{z+\varepsilon} \frac{|\mu(y)|}{\rho^2(y)}\mathrm{d}y < \infty.$$

For some constant $c \in \mathbb{R}$, we define the scale function

$$p(z) := \int_c^z \exp\left(-2\int_c^y \frac{\mu(r)}{\rho^2(r)}dr\right)\mathrm{d}y,\ z \in \mathbb{R}. \tag{6.6}$$

The function p is continuous with strictly positive derivative and its second derivative, p'', exists and satisfies

$$p''(z) = -\frac{2\mu(z)}{\rho^2(z)}p'(z).$$

Moreover the function $p : \mathbb{R} \to (p(-\infty), p(\infty)))$ has a continuous and differentiable inverse function $q : (p(-\infty), p(\infty))) \to \mathbb{R}$. Then the following proposition holds:

Proposition 6.2.5 ([60], Proposition 5.5.13)**.** *Suppose that (ND) and (LI) hold. A stochastic process $Z = (Z_t, \mathcal{F}_t)_{t\geq 0}$ is a weak (or strong) solution of equation (6.5) if and only if the stochastic process $Y = (Y_t := p(Z_t), \mathcal{F}_t)_{t\geq 0}$ is a weak (or strong) solution of the equation*

$$Y_t = y_0 + \int_0^t \tilde{\rho}(Y_s)\mathrm{d}W_s,$$

where

$$y_0 := p(z) \in (p(-\infty), p(\infty)))$$
$$\tilde{\rho}(y) := \begin{cases} p'(q(y))\rho(q(y)) & \text{if } y \in (p(-\infty), p(\infty))), \\ 0 & \text{otherwise .} \end{cases}$$

Theorem 6.2.6. *Assume that μ is bounded, measurable and that ρ is a measurable function such that there exist positive constants $c_0 > 1$ for which $c_0^{-1} \leq \rho^2(z) \leq c_0$ for any $z \in \mathbb{R}$. Then there exists a unique weak solution of equation (6.5). Moreover if ρ is continuous on $\mathbb{R}_0$ then the mapping $z \mapsto Z_t(z)$ is continuous for any $t \geq 0$ and any $z \in \mathbb{R}$ almost surely.*

Proof. Under the assumption, μ and ρ satisfy the conditions (ND) and (LI). Therefore from Proposition 6.2.5 and Theorem 6.2.3, it suffices to show that $\tilde{\rho}$ is bounded uniformly elliptic and continuous on $\mathbb{R}_0$. Since b is bounded and ρ is bounded and uniformly elliptic, we have $p(-\infty) = -\infty$ and $p(\infty) = \infty$ and p' is bounded and uniformly elliptic. Therefore $\tilde{\rho}$ is bounded and uniformly elliptic. If we choose $c = 0$ in (6.6), then $p(0) = q(0) = 0$. Hence $\tilde{\rho}$ is continuous on $\mathbb{R}_0$. $\square$

Clearly, from the above statement one also obtains that the process Z is a Markov process.

We will prove that the skew diffusion (6.1) has a unique (weak or strong) solution by using an equivalent SDE without reflection. To this end, fix $\alpha \in (0, 1)$ and we define the following functions:

$$\begin{aligned}
s_\alpha(x) &:= (1-\alpha)x\mathbf{1}(x \geq 0) + \alpha x\mathbf{1}(x < 0), \\
r_\alpha(x) &:= s_\alpha^{-1}(x) = \frac{x}{(1-\alpha)}\mathbf{1}(x \geq 0) + \frac{x}{\alpha}\mathbf{1}(x < 0), \\
f_\alpha(x) &:= \frac{D_- s_\alpha(x) + D_+ s_\alpha(x)}{2} = (1-\alpha)\mathbf{1}(x > 0) + \alpha\mathbf{1}(x < 0) + \frac{1}{2}\mathbf{1}(x = 0).
\end{aligned}$$

Here D_- and D_+ denote the left and right derivatives, respectively. Note that $f_\alpha \circ r_\alpha(x) = f_\alpha \circ s_\alpha(x) = f_\alpha(x)$. Using these notations, we have the following result.

Proposition 6.2.7. *Suppose that $\alpha \in (0, 1)$, and that Assumption 6.2.1 is satisfied. Define the coefficients*

$$\mu(z) := f_\alpha(z)b(r_\alpha(z)) = (1-\alpha)b\left(\frac{z}{1-\alpha}\right)\mathbf{1}(z > 0) + \alpha b\left(\frac{z}{\alpha}\right)\mathbf{1}(z < 0) + \frac{b(0)}{2}\mathbf{1}(z = 0).$$

Similarly, $\rho(z) := f_\alpha(z)\sigma(r_\alpha(z))$. Then there exists a unique solution for the equation

$$Z_t(z) = z + \int_0^t \mu(Z_s(z))\mathrm{d}s + \int_0^t \rho(Z_s(z))\mathrm{d}W_s, \ z \in \mathbb{R}. \tag{6.7}$$

Furthermore this solution defines an a.s. continuous flow $z \to Z_t(z)$ for all $t \geq 0$. Define $X_t(x) := r_\alpha(Z_t(z))$ with $z = s_\alpha(x)$. Then $X = (X_t(x))_{t\geq 0}$ is a solution of the SDE (6.1). Similarly, if X is a solution of (6.1) then $Z = (Z_t)_{t\geq 0} = (s_\alpha(X_t))_{t\geq 0}$ is the solution to (6.7) and the flow $x \to X_t(x)$ is a.s. continuous for all $t \geq 0$. In particular, for any bounded measurable function f,

$$\mathbb{E}[f(X_T(x))] = \mathbb{E}[f \circ r_\alpha(Z_T(z))],\ z = s_\alpha(x).$$

Proof. The proof of the first part is straightforward, we just show that if X is a solution of (6.1) then $Z = s_\alpha(X)$ is the solution of (6.7).

By the symmetric Itô-Tanaka formula (see e.g. (32) of [78]), we have

$$\begin{aligned}
Z_t := s_\alpha(X_t) = s_\alpha(x) + \int_0^t f_\alpha(X_s)\mathrm{d}X_s + \frac{1-2\alpha}{2}L_t^0(X)\\
= &s_\alpha(x) + \int_0^t f_\alpha(X_s)b(X_s)\mathrm{d}s + \int_0^t f_\alpha(X_s)\sigma(X_s)\mathrm{d}W_s + (2\alpha-1)\int_0^t f_\alpha(X_s)\mathrm{d}L_s^0(X) + \frac{1-2\alpha}{2}L_t^0(X)\\
= &s_\alpha(x) + \int_0^t f_\alpha \circ r_\alpha \circ s_\alpha(X_s) b(r_\alpha(Z_s))\mathrm{d}s + \int_0^t f_\alpha \circ r_\alpha \circ s_\alpha(X_s)\sigma(r_\alpha(Z_s))\mathrm{d}W_s\\
&+ (2\alpha-1)f_\alpha(0)L_t^0(X) - \frac{2\alpha-1}{2}L_t^0(X)\\
= &s_\alpha(x) + \int_0^t f_\alpha(Z_s)b(r_\alpha(Z_s))\mathrm{d}s + \int_0^t f_\alpha(Z_s)\sigma(r_\alpha(Z_s))\mathrm{d}W_s\\
= &z + \int_0^t \mu(Z_s)\mathrm{d}s + \int_0^t \rho(Z_s)\mathrm{d}W_s.
\end{aligned}$$

Since $r_\alpha = s_\alpha^{-1}$, Z is a solution of (6.7) if and only if X is a solution of (6.1). The other statements follow from Theorem 6.2.6. □

Remark 6.2.8. Nakao [94] proved that if ρ is positive and of bounded variation function on any compact interval of $\mathbb{R}$, then the pathwise uniqueness holds for SDE (6.7) (see Le Gall [77] for a stronger result). Therefore similar statements can be made about existence and uniqueness of solutions for (6.1).

6.3 The generator of X

Definition 6.3.1. *Let $\alpha \in (0,1)$. Let $D(\alpha)$ be the class of continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ with bounded continuous derivatives f' and f'' on $\mathbb{R}_0$ such that $f'(0+), f'(0-)$ exist and $\alpha f'(0+) = (1-\alpha)f'(0-)$.*

For measurable functions b and σ, we define the differential operator L by

$$Lf(x) := b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x), f \in D(\alpha), x \in \mathbb{R}_0. \tag{6.8}$$

Define for any bounded measurable function f, the semigroup associated to the Markov process X as $P_tf(x) := \mathbb{E}[f(X_t(x))]$. The next proposition shows that the infinitesimal generator of P is L on $D(\alpha)$.

Proposition 6.3.2. *Suppose that b and σ are bounded and continuous functions. Then the infinitesimal generator of $(P_t)_{t\geq 0}$ on $D(\alpha)$ is given by*

$$\lim_{h\to 0}\frac{P_h f(x)-f(x)}{h}=Lf(x), f\in D(\alpha), x\in\mathbb{R}_0.$$

Proof. If $f\in D(\alpha)$, then f has a generalized second order derivative given by

$$\mu(\mathrm{d}x)=f''(x)\mathrm{d}x-\frac{2\alpha-1}{\alpha}f'(0-)\delta_0(\mathrm{d}x),$$

where δ_a is a point mass measure at $a\in\mathbb{R}$. From here, it also follows that f is the difference of two convex functions. Therefore, from the symmetric Itô-Tanaka formula, the fact that $f'(0+)=\frac{(1-\alpha)}{\alpha}f'(0-)$ and the occupation time formula, we have for any $x\in\mathbb{R}$,

$$\begin{aligned}
f(X_t)=&f(x)+\int_0^t f'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_{\mathbb{R}}L_t^x(X)\mu(\mathrm{d}x)\\
=&f(x)+\int_0^t f'(X_s)\sigma(X_s)\mathrm{d}W_s+\int_0^t f'(X_s)b(X_s)\mathrm{d}s+(2\alpha-1)\int_0^t f'(X_s)\mathrm{d}L_s^0(X)\\
&+\frac{1}{2}\int_{\mathbb{R}}L_t^x(X)f''(x)\mathrm{d}x-\frac{(2\alpha-1)f'(0-)}{2\alpha}L_t^0(X)\\
=&f(x)+\int_0^t f'(X_s)\sigma(X_s)\mathrm{d}W_s+\int_0^t f'(X_s)b(X_s)\mathrm{d}s\\
&+(2\alpha-1)\frac{f'(0+)+f'(0-)}{2}L_t^0(X)+\frac{1}{2}\int_{\mathbb{R}}L_t^x(X)f''(x)\mathrm{d}x-\frac{(2\alpha-1)f'(0-)}{2\alpha}L_t^0(X)\\
=&f(x)+\int_0^t f'(X_s)\sigma(X_s)\mathrm{d}W_s+\int_0^t f'(X_s)b(X_s)\mathrm{d}s+\frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle X\rangle_s\\
=&f(x)+\int_0^t f'(X_s)\sigma(X_s)\mathrm{d}W_s+\int_0^t f'(X_s)b(X_s)\mathrm{d}s+\frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\mathrm{d}s.
\end{aligned}$$

Since f' and σ are bounded, we have that the above stochastic integral has expectation zero and therefore

$$P_t f(x)=f(x)+\int_0^t\mathbb{E}\left[b(X_s)f'(X_s)+\frac{f''(X_s)\sigma^2(X_s)}{2}\right]\mathrm{d}s.$$

Hence we get

$$\begin{aligned}
\frac{P_t f(x)-f(x)}{t}=&\frac{1}{t}\int_0^t\mathbb{E}\left[b(X_s)f'(X_s)+\frac{f''(X_s)\sigma^2(X_s)}{2}\right]\mathrm{d}s\\
=&\frac{1}{t}\int_0^t P_sLf(x)\mathrm{d}s.
\end{aligned}$$

Therefore we have by continuity of X with respect to the time variable that for $x\neq 0$,

$$\left|\frac{P_t f(x)-f(x)}{t}-Lf(x)\right|\leq\frac{1}{t}\int_0^t|P_sLf(x)-Lf(x)|\,\mathrm{d}s\to 0,\ \text{as } t\to 0,$$

Hence we conclude the proof. $\square$

Remark 6.3.3. Note that the above proof also gives that

$$\frac{\mathrm{d}P_t f}{\mathrm{d}t}(x) = P_t L f(x), f \in D(\alpha),\ x \in \mathbb{R}_0,\ t \geq 0. \tag{6.9}$$

6.4 Skew Brownian motion as the approximation process

We now define the approximation process that will be used in order to construct the parametrix in the next section. This is a slightly generalized version of the skew Brownian motion. We refer to [78] for general information about skew Brownian motion.

Proposition 6.4.1. *Assume that σ is a measurable function, $z \in \mathbb{R}$ and $\alpha \in (0,1)$. There exists an adapted stochastic process $X^{\alpha,z}$ which is the strong unique solution of the stochastic equation:*

$$X_t^{\alpha,z} = x + \sigma(z) W_t + (2\alpha - 1) L_t^0(X^{\alpha,z}). \tag{6.10}$$

As $\sigma(z) > 0$, then the density of $X_t^{\alpha,z}$, denoted by $p_t^z(x,y)$, exists and can be explicitly written in separate cases as:
Case A: For $x \geq 0$,

$$p_t^z(x,y) = \left(g_t^{a(z)}(y-x) + (2\alpha-1) g_t^{a(z)}(y+x)\right) \mathbf{1}(y \geq 0) + 2(1-\alpha) g_t^{a(z)}(y-x)\, \mathbf{1}(y < 0).$$

Case B: For $x < 0$

$$p_t^z(x,y) = \left(g_t^{a(z)}(y-x) + (1-2\alpha) g_t^{a(z)}(y+x)\right) \mathbf{1}(y < 0) + 2\alpha g_t^{a(z)}(y-x)\, \mathbf{1}(y \geq 0).$$

Furthermore $p_t^z(x,y)$ satisfies the following properties

- *For any fixed $x, y, z \in \mathbb{R}$, $p_\cdot^z(x,y)$ is a continuous function on $(0,\infty)$.*
- *$p_t^z(\cdot, y)$ is a Lipschitz continuous function whose derivative is continuous everywhere except at $x = 0$ where we have $\alpha \partial_x p_t^z(0+, y) = (1-\alpha)\partial_x p_t^z(0-, y)$.*
- *If $\alpha \neq 1/2$, then for any $(t,x) \in (0,\infty) \times \mathbb{R}$, the mapping $y \mapsto p_t^z(x,y)$ is not continuous at $y = 0$. Indeed, we have $p_t^z(x, 0+) = 2\alpha g_t^{a(z)}(x)$ and $p_t^z(x, 0-) = 2(1-\alpha) g_t^{a(z)}(x)$.*

Define the semigroup associated to $X^{\alpha,z}(x)$ as $P_t^z f(x) \equiv P_t^{\alpha,z} f(x) := \mathbb{E}[f(X_t^{\alpha,z}(x))]$. Then the infinitesimal generator of P_t^z on $D(\alpha)$ is given as

$$L^z f(x) \equiv L^{\alpha,z} f(x) = \frac{\sigma^2(z)}{2} f''(x), f \in D(\alpha), x \in \mathbb{R}_0.$$

Proof. We have that

$$\frac{X_t^{\alpha,z}}{\sigma(z)} = \frac{x}{\sigma(z)} + W_t + \frac{(2\alpha-1)L_t^0(X^{\alpha,z})}{\sigma(z)} = \frac{x}{\sigma(z)} + W_t + (2\alpha-1) L_t^0\left(\frac{X^{\alpha,z}}{\sigma(z)}\right).$$

This equation is a particular case of skew Brownian motion. That is,

$$X_t^\alpha = x + W_t + (2\alpha - 1)L_t^0(X^\alpha). \tag{6.11}$$

The solution of (6.11) is called the skew Brownian motion with parameter α. Harrison and Shepp [43] showed that there is no solution if $|2\alpha - 1| > 1$. By Revuz and Yor ([103], Chapter 3, Exercise 1.16), the density of X_t^α with $X_0^\alpha = x$ is given explicitly. Then the density $p_t^z(x, y)$ is obtained by doing a change of variables. The properties of $p_t^z(x, y)$ are obtained from the explicit formula obtained and the statement about the generator is a particular case of Proposition 6.3.2. □

Remark 6.4.2. Assume that $\sigma(w) > \lambda_0 > 0$ for any $w \in \mathbb{R}$. Let $(X_t^{\alpha,w})_{t\geq 0}$ be the unique solution of the stochastic equation (6.10). Define $Z_t^{\alpha,w} := s_\alpha(X_t^{\alpha,w})$. Then from Proposition 6.2.7, $(Z_t^{\alpha,w})_{t\geq 0}$ is the solution to the following SDE:

$$Z_t^{\alpha,w} = z + \int_0^t \rho_w(Z_s^{\alpha,w}) \mathrm{d}W_s,$$

where $z = s_\alpha(x)$ and

$$\rho_w(x) := \sigma(w) f_\alpha(x) = \sigma(w)\left((1-\alpha)\mathbf{1}(x > 0) + \alpha \mathbf{1}(x < 0) + \frac{1}{2}\mathbf{1}(x = 0)\right).$$

Then since $X_t^{\alpha,w}$ has the density function $p_t^w(x, \cdot)$, using the change of variables theorem for densities of random variables, we can obtain the density of $Z_t^{\alpha,w}$ explicitly. This gives

$$p^w_{Z_t}(z, u) := \frac{p_t^w(r_\alpha(z), r_\alpha(u))}{1-\alpha}\mathbf{1}(u \geq 0) + \frac{p_t^w(r_\alpha(z), r_\alpha(u))}{\alpha}\mathbf{1}(u < 0).$$

Therefore, choosing $w = r_\alpha(u)$ on the "backward parametrix method", we can also get a parametrix expansion for Z.

6.4.1 Some auxiliary estimates

In this section, we introduce some key estimates (Lemmas 6.4.4 and 6.4.5) in order to construct a parametrix expansion for the skew diffusion (6.1).

We define $\Phi(t, x, y) := (L - L^y)\phi_t^y(x)$ and $\overline{p}(t, x, y) \equiv \phi_t^y(x) := p_t^y(x, y)$. Here, we need to explain why we need to use these three notations for the same mathematical object: The first $\overline{p}(t, x, y)$ is used to make clear how the time-space convolutions are taken. For more on this, see Section 6.5. The second is used in order to know to which variable the derivative operators are acting on. Finally, the third is used in order to note that the density is just a variant of the skew Brownian motion density.

Lemma 6.4.3. *Let $\alpha \in (0, 1)$. Then $\phi_t^y \in D(\alpha)$ for any $(t, y) \in (0, \infty) \times \mathbb{R}$.*

The proof of the above statement follows directly from Proposition 6.4.1. Moreover the function ϕ_t^y satisfies the following Gaussian estimate.

Lemma 6.4.4. *Under Assumption 6.2.1, there exist positive constants C and c such that for any $x, y \in \mathbb{R}$, $t > 0$,*

$$\phi_t^y(x) \leq C g_t^c(y - x).$$

Proof. Since σ is bounded and uniformly elliptic, there exist positive constants C and c such that

$$\phi_t^y(x) \leq \begin{cases} Cg_t^c(y) & \text{if } x = 0, \\ Cg_t^c(y-x) & \text{if } y < 0 < x \text{ or } x < 0 \leq y, \\ C\left(g_t^c(y-x) + g_t^c(y+x)\right) & \text{if } y \geq 0, x > 0 \text{ or } y < 0, x < 0. \end{cases}$$

If the signs of y and x are the same, we have $g_t^c(y+x) \leq g_t^c(y-x)$, hence the proof is finished. □

We now give the essential estimate that will be used in order to prove the convergence of the parametrix method.

Lemma 6.4.5. *Under Assumption 6.2.1, there exist positive constants $C_T = C(1+T^{(1-\eta)/2})$ and c such that for any $x, y \in \mathbb{R}_0$, $t > 0$,*

$$|\Phi(t,x,y)| \leq \frac{C_T}{t^{1-\eta/2}} g_t^c(y-x).$$

Proof. We first compute the action of the operators on the function ϕ_t^y explicitly. As before, we need to separate the study in various cases:

Case 1: If $x, y > 0$, then

$$\begin{aligned}(L - L^y)\phi_t^y(x) =& b(x)\left(-H_1(y-x, ta(y))g_t^{a(y)}(y-x) + (2\alpha-1)H_1(y+x, ta(y))g_t^{a(y)}(y+x)\right) \\ &+ \frac{a(x)-a(y)}{2}\left(H_2(y-x, ta(y))g_t^{a(y)}(y-x) + H_2(y+x, ta(y))(2\alpha-1)g_t^{a(y)}(y+x)\right).\end{aligned}$$

Case 2: If $x > 0 > y$, then

$$(L - L^y)\phi_t^y(x) = 2(1-\alpha)\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right) g_t^{a(y)}(y-x).$$

Case 3: If $x < 0 < y$, then

$$(L - L^y)\phi_t^y(x) = 2\alpha\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right) g_t^{a(y)}(y-x).$$

Case 4: If $x, y < 0$, then

$$\begin{aligned}(L - L^y)\phi_t^y(x) =& b(x)\left(-H_1(y-x, ta(y))g_t^{a(y)}(y-x) + (1-2\alpha)H_1(y+x, ta(y))g_t^{a(y)}(y+x)\right) \\ &+ \frac{a(x)-a(y)}{2}\left(H_2(y-x, ta(y))g_t^{a(y)}(y-x) + H_2(y+x, ta(y))(1-2\alpha)g_t^{a(y)}(y+x)\right).\end{aligned}$$

As all the cases follow similarly, we only consider the case $x, y > 0$.

From Assumption 6.2.1 and the inequality $|x|^p e^{-qx^2} \leq (p/(2qe))^{p/2}$ for any $p, q > 0$ and $x \in \mathbb{R}$, we have

$$|(L - L^y)\phi_t^y(x)| \leq \frac{C}{t^{1/2}}\left(g_t^c(y-x) + g_t^c(y+x)\right)$$

$$+\frac{C|a(x)-a(y)|}{t}\left(g_t^c(y-x)+g_t^c(y+x)\right).$$

Note that for $x, y > 0$, we have $g_t^c(y+x) \leq g_t^c(y-x)$. Hence, using the Hölder continuity of a and $t > 0$, we have

$$\begin{aligned}|(L-L^y)\phi_t^y(x)| &\leq \frac{C}{t^{1/2}}g_t^c(y-x)+\frac{C|x-y|^\eta}{t}g_t^c(y-x)\\ &\leq \frac{C_T}{t^{1-\eta/2}}g_t^c(y-x).\end{aligned}$$

This concludes the proof. □

6.5 Parametrix for skew diffusion

In this section, we prove the existence of the density of the skew diffusion (6.1), using the parametrix method for the semigroup P.

We first define the following time-space convolutions $\circledast$ for functions $f, g : [0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and time dependent operators A and B

$$\begin{aligned}f\circledast g(t,x,y) &:= \int_0^t \mathrm{d}s\int_{\mathbb{R}}\mathrm{d}z f(s,x,z)g(t-s,z,y),\\ (A\circ B)_t f &:= \int_0^t A_s B_{t-s} f\mathrm{d}s.\end{aligned}$$

We denote $f^{\circledast 1}=f$, $f^{\circledast k}=f\circledast f^{\circledast(k-1)}$ and $f\circledast g^{\circledast 0}=f$ and similarly for the time convolution of operators. That is, $A^{\circ k}=A\circ A^{\circ(k-1)}$ for $k\in\mathbb{N}$.

Now we introduce the following operators for $f\in L^\infty(\mathbb{R})$ and $y\in\mathbb{R}_0$,

$$\begin{aligned}\hat{Q}_t f(y) &:= (P_t^y)^* f(y) = \int_{\mathbb{R}} f(x)p_t^y(x,y)\mathrm{d}x = \int_{\mathbb{R}} f(x)\overline{p}(t,x,y)\mathrm{d}x,\\ \hat{S}_t f(y) &:= \int_{\mathbb{R}} f(x)(L-L^y)\phi_t^y(x)\mathrm{d}x = \int_{\mathbb{R}} f(x)\Phi(t,x,y)\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\hat{I}_{t_0}^n(f)(y) &:= \begin{cases}(\hat{S}^{\circ n}\circ\hat{Q})_{t_0}f(y), & \text{if } n\geq 1,\\ \hat{Q}_{t_0}f(y), & \text{if } n=0,\end{cases}\\ &= \int_{\mathbb{R}} f(x)\overline{p}\circledast\Phi^{\circledast n}(t,x,y)\mathrm{d}x.\end{aligned}$$

Moreover, we define $\hat{I}_{t_0}^0(y_0,y_1):=p_{t_0}^{y_0}(y_1,y_0)$ and for $n\geq 1$,

$$\begin{aligned}\hat{I}_{t_0}^n(y_0,y_{n+1}) :=& \overline{p}\circledast\Phi^{\circledast n}(t_0,y_{n+1},y_0)\\ =& \int_0^{t_0}\mathrm{d}t_1\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\int_{\mathbb{R}^n}\mathrm{d}y_1\cdots\mathrm{d}y_n\prod_{i=0}^{n-1}\Phi(t_i-t_{i+1},y_{i+1},y_i)p_{t_n}^{y_n}(y_{n+1},y_n).\end{aligned}$$

Furthermore, we define the adjoint operators for $x \in \mathbb{R}_0$ and $f \in L^\infty(\mathbb{R})$:

$$\begin{aligned}
\hat{Q}_t^* f(x) &:= \int_{\mathbb{R}} f(y) p_t^y(x,y) \mathrm{d}y = \int_{\mathbb{R}} f(y) \overline{p}(t,x,y) \mathrm{d}y, \\
\hat{S}_t^* f(x) &:= \int_{\mathbb{R}} f(y) (L - L^y) \phi_t^y(x) \mathrm{d}y = \int_{\mathbb{R}} f(y) \Phi(t,x,y) \mathrm{d}y, \\
\hat{I}_{t_0}^{n,*}(f)(x) &:= \begin{cases} (\hat{Q}^* \circ (\hat{S}^*)^{\circ n})_{t_0} f(x), & \text{if } n \geq 1, \\ \hat{Q}_{t_0}^* g(x), & \text{if } n = 0, \end{cases} \\
&= \int_{\mathbb{R}} f(y) \overline{p} \circledast \Phi^{\circledast n}(t_0, x, y) \mathrm{d}y.
\end{aligned}$$

We will extend the definition of $\hat{Q}_t^* f(x)$ at $x = 0$ by continuity. Under Assumption 6.2.1 and using Lemmas 6.4.5 and 6.4.4, there exists a positive constant C such that for any $f \in L^\infty(\mathbb{R})$,

$$\sup_{t \geq 0} \max\{t^{1-\eta/2} ||\hat{S}_t f||_\infty, t^{1-\eta/2} ||\hat{S}_t^* f||_\infty, ||\hat{Q}_t f||_\infty, ||\hat{Q}_t^* f||_\infty\} \leq C_T ||f||_\infty. \tag{6.12}$$

Therefore $\hat{S}_t f$, $\hat{S}_t^* f$, $\hat{Q}_t f$ and $\hat{Q}_t^* f$ are well defined. Now, we will prove that $\hat{I}_{t_0}^n(f)$, $\hat{I}_{t_0}^{n,*}(f)$ and $\hat{I}_{t_0}(y_0, y_{n+1})$ are well-defined. Indeed, from (6.12) and Lemma 6.4.5,

$$\left|\left| \left(\prod_{i=0}^{n-1} \hat{S}_{t_i - t_{i-1}} \right) \hat{Q}_{t_n} f \right|\right|_\infty \leq ||f||_\infty \prod_{i=0}^{n-1} \frac{C_T}{(t_i - t_{i+1})^{1-\eta/2}} = C_T^n ||f||_\infty \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}},$$

hence $\hat{I}_{t_0}^n(f)(y)$ is well defined due to Lemma 6.8.1. In fact,

$$\begin{aligned}
\sum_{n=0}^\infty \left|\left| \hat{I}_{t_0}^n(f) \right|\right|_\infty &\leq \sum_{n=0}^\infty C_T^n ||f||_\infty \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}} \\
&= ||f||_\infty \sum_{n=0}^\infty t_0^{n(1-\eta/2)} C_T^n \prod_{i=0}^{n-1} B(1 + i\eta/2, \eta/2) \\
&= ||f||_\infty \sum_{n=0}^\infty \left(t_0^{(1-\eta/2)} C_T \Gamma(\eta/2) \right)^n \frac{1}{\Gamma(1 + n\eta/2)} \\
&= ||f||_\infty E_{\eta/2,1}(t_0^{(1-\eta/2)} C_T \Gamma(\eta/2)) < \infty.
\end{aligned}$$

So $\sum_{n=0}^\infty \hat{I}_{t_0}^n(g)(y)$ converges absolutely and uniformly for $(t, y) \in (0, T] \times \mathbb{R}$. Due to a similar argument, we have

$$\left|\left| \hat{Q}_{t_n}^* \left(\prod_{i=n-1}^{0} \hat{S}_{t_i - t_{i-1}}^* \right) f \right|\right|_\infty \leq C_T^n ||f||_\infty \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}}. \tag{6.13}$$

So $\hat{I}_{t_0}^{n,*}(f)(x)$ is well defined and

$$\sum_{n=0}^\infty \left|\left| \hat{I}_{t_0}^{n,*}(f) \right|\right|_\infty < \infty. \tag{6.14}$$

From here, we conclude that $\sum_{n=0}^{\infty} \hat{I}_t^{n,*}(f)(x)$ converges absolutely and uniformly for $(t,x) \in (0,T] \times \mathbb{R}$.

Now we state our main results for the parametrix expansion of skew diffusions.

Theorem 6.5.1. *Suppose that Assumption 6.2.1 holds and that the drift coefficient b is continuous. Define*

$$p_T(x,y) := \sum_{n=0}^{\infty} \hat{I}_T^n(y,x),\ x \in \mathbb{R}, y \in \mathbb{R}_0.$$

Then $p_T(x,y)$ converges absolutely and uniformly for $x \in \mathbb{R}, y \in \mathbb{R}_0$ and it is continuous at any $x \in \mathbb{R}$, and it has Gaussian upper bounds. That is, there exists positive constants C and c such that for any $x \in \mathbb{R}$, $y \in \mathbb{R}_0$

$$p_T(x,y) \le E_{\eta/2,1}(C(1 \vee T)^{(3-2\eta)/2}) g_T^c(y-x).$$

Moreover, for any bounded measurable function f and $x \in \mathbb{R}$,

$$\mathbb{E}[f(X_T(x))] = \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f)(x) = \int_{\mathbb{R}} f(y) p_T(x,y) \mathrm{d}y. \tag{6.15}$$

Therefore, $p_T(x,\cdot)$ is the probability density function of $X_T(x)$ for any $(T,x) \in (0,\infty) \times \mathbb{R}$.

Proof. The idea of the proof requires two steps: In the first step, done in Proposition 6.5.2, we use the parametrix method for the semigroup associated to X. We first prove that (6.15) holds for any $f \in C_c^\infty(\mathbb{R})$ and almost every $x \in \mathbb{R}_0$. So $p_T(x,\cdot)$ is the density function of $X_T(x)$ for almost every $x \in \mathbb{R}_0$.

This weakness in the argument is due to the duality that is used in order to apply the backward parametrix method. For more details, see the proof of Proposition 6.5.2.

In the second step, done in Proposition 6.5.3, using the continuity of flows of $X_t(\cdot)$ and the continuity of $p_T(\cdot,y)$, we will obtain the density of $X_t(x)$ for all $x \in \mathbb{R}$.

As a consequence, we also get similar results for $Z_t(z)$ in Corollary 6.5.6. □

Proposition 6.5.2. *Assume that Assumption 6.2.1 holds and that the drift coefficient b is a continuous function. Then for any $f \in C_c^\infty(\mathbb{R})$, $\sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f)(x)$ converges absolutely and uniformly for $x \in \mathbb{R}_0$ and the following expansion holds:*

$$\mathbb{E}[f(X_T(x))] = \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f)(x), \quad a.e.\ x \in \mathbb{R}_0.$$

Proposition 6.5.3. *Assume that Assumption 6.2.1 holds and that the drift coefficient b is a continuous function. Then $p_T(x,y)$ converges absolutely and uniformly for $x,y \in \mathbb{R}_0$ and is continuous at any $x \in \mathbb{R}$, and it has a Gaussian upper bound. That is, there exists positive constants C and c such that*

$$p_T(x,y) \le E_{\eta/2,1}(C(1 \vee T)^{(3-2\eta)/2}) g_T^c(y-x).$$

Furthermore the density of $X_T(x)$ is given by $p_T(x,\cdot)$ for all $x \in \mathbb{R}$.

We first prove the following continuity lemma.

Lemma 6.5.4. *Let f be a bounded and continuous function on $\mathbb{R}$. Then for any $x \in \mathbb{R}$, we have*

$$\lim_{t\to 0+} \hat{Q}_t^* f(x) = \lim_{t\to 0+} \int_{\mathbb{R}} f(y)\phi_t^y(x)\mathrm{d}y = f(x).$$

Proof. We first consider $(t, x) \in [0, T] \times \mathbb{R}_0$. Then we have

$$\begin{aligned}
\hat{Q}_t^* f(x) =& \int_{\mathbb{R}} f(y)\phi_t^y(x)\mathrm{d}y \\
=& \int_{-\infty}^{0} f(y)\phi_t^y(x)\mathrm{d}y\mathbf{1}_{(-\infty,0)}(x) + \int_{0}^{\infty} f(y)\phi_t^y(x)\mathrm{d}y\mathbf{1}_{(-\infty,0)}(x) \\
&+ \int_{-\infty}^{0} f(y)\phi_t^y(x)\mathrm{d}y\mathbf{1}_{(0,\infty)}(x) + \int_{0}^{\infty} f(y)\phi_t^y(x)\mathrm{d}y\mathbf{1}_{(0,\infty)}(x) \\
=:& J_1(t,x) + J_2(t,x) + J_3(t,x) + J_4(t,x).
\end{aligned}$$

First, we consider the limit of $J_1(t, x)$. From the definition of $\phi_t^y(x) = p_t^y(x, y)$, we have

$$\begin{aligned}
J_1(t,x) =& \int_{-\infty}^{0} f(y)g_t^{a(y)}\,(y-x)\,\mathrm{d}y\mathbf{1}_{(-\infty,0)}(x) + (1-2\alpha)\int_{-\infty}^{0} f(y)g_t^{a(y)}\,(y+x)\,\mathrm{d}y\mathbf{1}_{(-\infty,0)}(x) \\
=:& J_{1,1}(t,x) + J_{1,2}(t,x).
\end{aligned}$$

By a similar proof of Theorem 1 of [28], $\lim_{t\to 0} J_{1,1}(t, x) = f(x)\mathbf{1}_{(-\infty,0)}(x)$. Since σ is bounded and uniformly elliptic, using the change of variables $z = (y + x)/\sqrt{t}$, we have

$$J_{1,2}(t,x) \le C||f||_\infty \int_{-\infty}^{0} g_t^c(x+y)\mathrm{d}y\mathbf{1}_{(-\infty,0)}(x) = C||f||_\infty \int_{-\infty}^{\frac{x}{\sqrt{t}}} g_1^c(y)\mathrm{d}y\mathbf{1}_{(-\infty,0)}(x) \to 0, \text{ as } t \to 0+.$$

So we conclude $J_1(t, x) + J_2(t, x) \to f(x)\mathbf{1}_{(-\infty,0)}(x)$ as $t \to 0+$. In the same way, $J_3(t, x) + J_4(t, x) \to f(x)\mathbf{1}_{(0,\infty)}(x)$. Therefore, we have $\lim_{t\to 0+} \hat{Q}_t^* f(x) = f(x)$ for any $x \in \mathbb{R}_0$.

In a similar fashion, one deals with the particular case $x = 0$. □

Proof of Proposition 6.5.2. Let $t_0 = T$ and $x \neq 0$. We first prove that

$$\partial_t(P_t P_{T-t}^{y_2})\phi_\varepsilon^{y_0}(x) = P_t(L - L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(x). \tag{6.16}$$

Since X_t^z is a time homogeneous Markov process, by the Chapman-Kolmogorov equation, we have

$$P_t^z\phi_\varepsilon^z(x) = \int_{\mathbb{R}} \phi_\varepsilon^z(y)p_t^z(x,y)\mathrm{d}y = \int_{\mathbb{R}} p_\varepsilon^z(y,z)p_t^z(x,y)\mathrm{d}y = p_{\varepsilon+t}^z(x,z) = \phi_{\varepsilon+t}^z(x). \tag{6.17}$$

From equation (6.17), using Lemma 6.4.3 and (6.9), we have for $x \neq 0$,

$$\begin{aligned}
\partial_t(P_t P_{T-t}^{y_1})\phi_\varepsilon^{y_1}(x) =& (\partial_t P_t)P_{T-t}^{y_1}\phi_\varepsilon^{y_1}(x) - P_t\partial_t P_{T-t}^{y_1}\phi_\varepsilon^{y_1}(x) \\
=& (\partial_t P_t)\phi_{T-t+\varepsilon}^{y_1}(x) - P_t L^{y_1} P_{T-t}^{y_1}\phi_\varepsilon^{y_1}(x) \\
=& P_t(L - L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(x),
\end{aligned}$$

which gives (6.16).

We will now consider an argument by duality. Therefore we need to introduce the following inner product notation $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)\mathrm{d}x$, for two measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $fg \in L^1(\mathbb{R})$.

Fix for the moment, $f, g \in C_c^\infty(\mathbb{R})$. Using (6.16), we have

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)\left(\langle g, P_T\phi_\varepsilon^{y_1}\rangle - \langle g, P_T^{y_1}\phi_\varepsilon^{y_1}\rangle\right) &= \int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)\int_{\mathbb{R}} \mathrm{d}x g(x)\left(P_T - P_T^{y_1}\right)\phi_\varepsilon^{y_1}(x)\\ &= \int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)\int_{\mathbb{R}} \mathrm{d}x g(x)\int_0^T \mathrm{d}t\partial_t(P_t P_{T-t}^{y_1})\phi_\varepsilon^{y_1}(x)\\ &= \int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)\int_{\mathbb{R}} \mathrm{d}x g(x)\int_0^T \mathrm{d}t P_t\left((L - L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(\cdot)\right)(x).\end{aligned} \tag{6.18}$$

Now we consider the limit of both sides of (6.18). From Lemma 6.4.5

$$\begin{aligned}&\int_{\mathbb{R}} \mathrm{d}x|g(x)|P_t\left(\int_{\mathbb{R}} \mathrm{d}y_1|f(y_1)|\left|(L - L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(\cdot)\right|\right)(x)\\ &\le \frac{C_T||f||_\infty}{(T-t+\varepsilon)^{1-\eta/2}}\int_{\mathbb{R}} \mathrm{d}x|g(x)|P_t\left(\int_{\mathbb{R}} \mathrm{d}y_1\left(g_{T-t+\varepsilon}^c(y_1 - \cdot)\right)\right)(x)\\ &= \frac{C_T||f||_\infty}{(T-t+\varepsilon)^{1-\eta/2}}\int_{\mathbb{R}} \mathrm{d}x|g(x)|P_t\mathbf{1}_{\mathbb{R}}(x) \le \frac{C_T||f||_\infty||g||_{L^1}}{(T-t)^{1-\eta/2}}.\end{aligned}$$

Since $1 - \eta/2 \in [1/2, 1)$, the above expression is integrable on $[0, T]$ and by the dominated convergence theorem, the right hand side of (6.18) converges to

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^T \mathrm{d}t\int_{\mathbb{R}} \mathrm{d}x g(x)P_t\left(\int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)(L - L^{y_1})\phi_{T-t+\varepsilon}^{y_1}(\cdot)\right)(x)\\ &= \int_0^T \mathrm{d}t\int_{\mathbb{R}} \mathrm{d}x g(x)P_t\left(\int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)(L - L^{y_1})\phi_{T-t}^{y_1}(\cdot)\right)(x).\end{aligned} \tag{6.19}$$

Next, we consider the left hand side of (6.18). We define $P_t(x, A) := \mathbb{P}(X_t(x) \in A)$ for any $A \in \mathcal{B}(\mathbb{R})$ and $t \ge 0$. Then from Fubini's theorem, Lemma 6.5.4, (6.12) and dominated convergence theorem, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)\langle g, P_T\phi_\varepsilon^{y_1}\rangle \mathrm{d}y_1 = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)\int_{\mathbb{R}} \mathrm{d}x g(x)P_T\phi_\varepsilon^{y_1}(x)\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)\int_{\mathbb{R}} \mathrm{d}x g(x)\int_{\mathbb{R}} P_T(x, \mathrm{d}w)\phi_\varepsilon^{y_1}(w) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathrm{d}x g(x)\int_{\mathbb{R}} P_T(x, \mathrm{d}w)\hat{Q}_\varepsilon^* f(w)\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} g(x)P_T\hat{Q}_\varepsilon^* f(x)\mathrm{d}x = \int_{\mathbb{R}} g(x)P_T f(x)\mathrm{d}x = \langle P_T f, g\rangle.\end{aligned} \tag{6.20}$$

Finally, we consider the second term on the left hand side of (6.18). Before doing that note that

$$\lim_{\varepsilon\to 0}(P_{T+\varepsilon}^y)^* g(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} g(z)p_{T+\varepsilon}^{(y)}(z, x)\mathrm{d}z.$$

Therefore using Lemma 6.4.4 and the dominated convergence theorem we have that for any $x \in \mathbb{R}_0$, $\lim_{\varepsilon\to 0}(P^y_{T+\varepsilon})^* g(x) = (P^y_T)^* g(x)$. Applying this result, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)\langle g, P^{y_1}_T \phi^{y_1}_\varepsilon\rangle \mathrm{d}y_1 = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)\langle (P^{y_1}_T)^* g, \phi^{y_1}_\varepsilon\rangle \mathrm{d}y_1 \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)\int_{\mathbb{R}} \mathrm{d}x (P^{y_1}_T)^* g(x)\phi^{y_1}_\varepsilon(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)(P^{y_1}_\varepsilon)^*(P^{y_1}_T)^* g(y_1)\mathrm{d}y_1 \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(y_1)(P^{y_1}_{T+\varepsilon})^* g(y_1)\mathrm{d}y_1 = \langle \hat{Q}^*_T f, g\rangle. \qquad (6.21)
\end{aligned}$$

Therefore from (6.18), (6.19), (6.20) and (6.21), we conclude

$$\begin{aligned}
\langle P_T f, g\rangle =& \langle \hat{Q}^*_T f, g\rangle + \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x g(x) P_t \left(\int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)(L - L^{y_1})\phi^{y_1}_{T-t}(\cdot)\right)(x) \\
=& \langle \hat{Q}^*_T f, g\rangle + \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x g(x) \int_{\mathbb{R}} P_t(x, \mathrm{d}y_2) \int_{\mathbb{R}} \mathrm{d}y_1 f(y_1)(L - L^{y_1})\phi^{y_1}_{T-t}(y_2) \\
=& \langle \hat{Q}^*_T f, g\rangle + \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x g(x) \int_{\mathbb{R}} P_t(x, \mathrm{d}y_2)\hat{S}^*_{T-t} f(y_2) \\
=& \langle \hat{Q}^*_T f, g\rangle + \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x g(x) P_t \hat{S}^*_{T-t} f(x) \\
=& \langle \hat{Q}^*_T f, g\rangle + \int_0^T \langle g, P_t \hat{S}^*_{T-t} f\rangle \mathrm{d}t. \qquad (6.22)
\end{aligned}$$

Note that, by taking an appropriate sequence of smooth bounded functions, one can claim that the equation (6.22) holds for any $f \in L^\infty(\mathbb{R})$ and $g \in C^\infty_c(\mathbb{R})$.

Hence by replacing f by $\hat{S}^*_{T-t} f$ in the equation (6.22) and iterating, we have

$$\begin{aligned}
\langle P_T f, g\rangle =& \langle \hat{Q}^*_T f, g\rangle + \sum_{n=1}^{N-1} \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \langle \hat{Q}^*_{t_n} \hat{S}^*_{t_{n-1}-t_n} \cdots \hat{S}^*_{t_0-t_1} f, g\rangle \\
&+ \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{N-1}} \mathrm{d}t_N \langle P_{t_N} \hat{S}^*_{t_{N-1}-t_N} \cdots \hat{S}^*_{t_0-t_1} f, g\rangle.
\end{aligned}$$

Now, we apply Fubini's theorem in order to exchange time and space integrals which is assured by the hypothesis that $f \in C^\infty_c(\mathbb{R})$ and the estimates in Lemma 6.4.5, so that

$$\langle P_T f, g\rangle = \langle \hat{Q}^*_T f, g\rangle + \sum_{n=1}^{N-1} \langle (\hat{Q}^* \circ (\hat{S}^*)^{\circ n})_{t_0} f, g\rangle + \langle \hat{R}^N_{t_0}(f), g\rangle,$$

where

$$\hat{R}^N_{t_0}(f)(z) := (P \circ (\hat{S}^*)^{\circ N})_{t_0} f(z) = \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{N-1}} \mathrm{d}t_N P_{t_N} \hat{S}^*_{t_{N-1}-t_N} \cdots \hat{S}^*_{t_0-t_1} f(z).$$

From (6.13) and Lemma 6.8.1 with $a = 1 - \eta/2$ and $b = 0$, we have

$$||\hat{R}^N_{t_0}(f)||_\infty \leq C_T^N \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{N-1}} \mathrm{d}t_N \prod_{i=0}^{N} \frac{||f||_\infty}{(t_i - t_{i+1})^{1-\eta/2}} \to 0, \text{ as } N \to \infty.$$

Hence we have

$$|\langle \hat{R}^N_{t_0}(f), g\rangle| \leq ||\hat{R}^N_{t_0}(f)||_\infty ||g||_{L^1} \to 0, \text{ as } N \to \infty.$$

Therefore, using (6.14), we get the infinite sum expansion

$$\begin{aligned}\langle P_T f, g\rangle, =& \langle \hat{Q}^*_T f, g\rangle + \sum_{n=1}^{\infty} \langle (\hat{Q}^* \circ (\hat{S}^*)^{\circ n})_{t_0} f, g\rangle \\ =& \langle \hat{Q}^*_T f, g\rangle + \left\langle \sum_{n=1}^{\infty} \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \hat{Q}^*_{t_n} \hat{S}^*_{t_{n-1}-t_n} \cdots \hat{S}^*_{t_0 - t_1} f, g \right\rangle \\ =& \left\langle \sum_{n=0}^{\infty} \hat{I}^{n,*}_T(f), g \right\rangle.\end{aligned}$$

This implies that

$$P_T f = \sum_{n=0}^{\infty} \hat{I}^{n,*}_T(f), \text{ a.e. } x \in \mathbb{R}_0.$$

This concludes the statement. □

Proof of Proposition 6.5.3. We first prove that $p_T(x, y)$ is well defined. Let $y_0 = y$, $t_0 = T$. Define for $x \in \mathbb{R}$ and $y \in \mathbb{R}_0$,

$$\hat{K}^n_{t_0}(y, x) := \bar{p} \circledast |\Phi|^{\circledast n}(t_0, x, y).$$

Then from Lemma 6.4.5 and 6.4.4, we have

$$\hat{K}^n_{t_0}(y, x) \leq \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \int_{\mathbb{R}^n} \mathrm{d}y_1 \cdots \mathrm{d}y_n \prod_{i=0}^{n-1} \frac{C_T}{(t_i - t_{i+1})^{1-\eta/2}} g^c_{t_i - t_{i+1}}(y_i - y_{i+1})\, g^c_{t_n}(y_n - x).$$

By the Chapman-Kolmogorov property, we get

$$\hat{K}^n_{t_0}(y, x) \leq C_T^n \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}} g^c_{t_0}(y - x). \tag{6.23}$$

Since $1 - \eta/2 \in [1/2, 1)$, we have

$$\sum_{n=1}^{\infty} C_T^n \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}} = \sum_{n=1}^{\infty} t_0^{n(1-\eta/2)} C_T^n \prod_{i=0}^{n-1} B(1 + i\eta/2, \eta/2)$$

$$= \sum_{n=1}^{\infty} \frac{\left(t_0^{(1-\eta/2)} C_T \Gamma(\eta/2)\right)^n}{\Gamma(1+n\eta/2)}$$
$$< E_{\eta/2,1}(t_0^{(1-\eta/2)} C_T \Gamma(\eta/2)) < \infty.$$

Therefore, $p_T(x,y)$ is well defined and (6.23) gives

$$p_T(x,y) \leq E_{\eta/2,1}(C(1 \vee T)^{(3-2\eta)/2}))g_T^c(y-x).$$

Since for any $n \geq 0$ and $f \in C_c^\infty(\mathbb{R})$, $\hat{I}_T^{n,*}(f)(x) = \int_{\mathbb{R}} f(y)\hat{I}_T^n(y,x)\mathrm{d}y$ is satisfied we obtain that

$$P_T f(x) = \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f)(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}} f(y)\hat{I}_t^n(y,x)\mathrm{d}y = \int_{\mathbb{R}} f(y)p_T(x,y)\mathrm{d}y, \quad \text{a.e. } x \in \mathbb{R}_0,$$

which implies that $p_T(x,y)$ is a density of $X_T(x)$ for almost every $x \in \mathbb{R}_0$. As $p_{t_n}^{y_n}(x, y_n)$ is continuous at $x \in \mathbb{R}$ (see Proposition 6.4.1), then $p_T(x,y)$ is also continuous for $x \in \mathbb{R}$.

Moreover, the law of $X_T(x)$ is absolutely continuous with respect to the Lebesgue measure and for almost every $x \in \mathbb{R}_0$, $p_T(x,\cdot)$ is its corresponding probability density function. Therefore, we conclude that for any bounded measurable function f,

$$\mathbb{E}[f(X_T(x))] = \sum_{n=0}^{\infty} \hat{I}_T^{n,*}(f)(x) = \int_{\mathbb{R}} f(y)p_T(x,y)\mathrm{d}y, \quad \text{a.e. } x \in \mathbb{R}_0.$$

Next, we use Proposition 6.2.7 which gives the continuity of $\mathbb{E}[f(X_T(x))]$ with respect to $x \in \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R})$ and then finally leads to the conclusion. □

By using appropriate approximation arguments, we can extend the statement of Theorem 6.5.1 for bounded measurable drift coefficients.

Corollary 6.5.5. *Under Assumption 6.2.1, all the statements of Theorem 6.5.1 hold.*

Proof. By Theorem 174 of Kestelman [63], page 111, there exists a sequence of continuous functions $(b_N)_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} b_N = b, \text{ a.e.,} \tag{6.24}$$
$$\sup_{N\in\mathbb{N}} ||b_N||_\infty \leq ||b||_\infty. \tag{6.25}$$

Let $X^{(N)} = (X_t^{(N)})_{t\geq 0}$ be the unique weak solution to the following SDE

$$X_t^{(N)} = x + \int_0^t b_N(X_s^{(N)})\mathrm{d}s + \int_0^t \sigma(X_s^{(N)})\mathrm{d}W_s + (2\alpha-1)L_t^0(X^{(N)}), \ t \geq 0, \alpha \in (0,1).$$

Let $T > 0$. We first prove that there exists a subsequence $(X_T^{(N_k)})_{k\in\mathbb{N}}$ of the sequence $(X_T^{(N)})_{N\in\mathbb{N}}$ such that for any $f \in C_c^\infty(\mathbb{R})$,

$$\lim_{k\to\infty} \mathbb{E}[f(X_T^{(N_k)})] = \mathbb{E}[f(X_T)]. \tag{6.26}$$

Let $Z_t := s_\alpha(X_t)$ and $Z_t^{(N)} := s_\alpha(X_t^{(N)})$, $t \in [0,T]$. Note that from Lemma 6.2.7, the drift coefficients of Z and $Z^{(N)}$ are bounded measurable functions and the diffusion coefficient $\rho(z) = f_\alpha(z)\sigma(r_\alpha(z))$ of Z and $Z^{(N)}$ is a bounded, uniformly elliptic function. The proof of Theorem 2 of Krylov [71] shows that there exists a subsequence $(Z_T^{(N_k)})_{k\in\mathbb{N}}$ of the sequence $(Z_T^{(N)})_{N\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathbb{E}[|Z_T^{(N_k)} - Z_T|^2] = 0.$$

Since r_α is a Lipschitz continuous function, by using Jensen's inequality, it holds that

$$|\mathbb{E}[f(X_T^{(N_k)})] - \mathbb{E}[f(X_T)]| \leq ||f'||_\infty \mathbb{E}[|X_T^{(N_k)} - X_T|^2]^{1/2} \leq C\mathbb{E}[|Z_T^{(N_k)} - Z_T|^2]^{1/2} \to 0,$$

as $k \to \infty$ for some constant C. This implies (6.26).
Let $p_T^{(N)}(x,\cdot)$ be the density function of $X_T^{(N)}$. Then from Theorem 6.5.1, it holds that

$$p_T^{(N)}(x,y) = \sum_{n=0}^{\infty} \hat{I}_T^{n,N}(y,x),\ x \in \mathbb{R}, y \in \mathbb{R}_0, \tag{6.27}$$

$$p_T^{(N)}(x,y) \leq E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}) g_T^c(y-x). \tag{6.28}$$

Here $\hat{I}_{t_0}^{0,N}(y_0,y_1) := p_{t_0}^{y_0}(y_1,y_0)$ and for $n \geq 1$,

$$\begin{aligned}\hat{I}_{t_0}^{n,N}(y_0,y_{n+1}) :=& \overline{p} \circledast \Phi_N^{\circledast n}(t_0,y_{n+1},y_0)\\ =& \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \int_{\mathbb{R}^n} \mathrm{d}y_1 \cdots \mathrm{d}y_n \prod_{i=0}^{n-1} \Phi_N(t_i - t_{i+1}, y_{i+1}, y_i) p_{t_n}^{y_n}(y_{n+1},y_n),\end{aligned}$$

$\Phi_N(t,x,y) := (L_N - L^y)\phi_t^y(x)$ and L_N is the differentiable operator defined on (6.8) with drift coefficient b_N. Note that from (6.25), the constants C and c in (6.28) do not depend on N. From (6.24) and (6.28), by using dominated convergence theorem, we have

$$p_T(x,y) := \lim_{N\to\infty} p_T^{(N)}(x,y) = \sum_{n=0}^{\infty} \lim_{N\to\infty} \hat{I}_T^{n,N}(y,x) = \sum_{n=0}^{\infty} \hat{I}_T^{n}(y,x),\ x \in \mathbb{R}, y \in \mathbb{R}_0.$$

This fact and the equation (6.26) imply that $p_T(x,\cdot)$ is the density function of X_T. □

Now, due to the functional relation between X and Z, we can also present similar results for the density of Z.

Corollary 6.5.6. *Suppose that Assumption 6.2.1 holds. Let $Z_T(z) := s_\alpha(X_T(x))$ where $z = s_\alpha(x)$. Then $Z_T = Z_T(z)$ has the probability density function, $p_{Z_T}(z,\cdot)$ for any $z \in \mathbb{R}_0$, and its given by*

$$p_{Z_T}(z,u) = \frac{p_T(r_\alpha(z), r_\alpha(u))}{1-\alpha}\mathbf{1}(u \geq 0) + \frac{p_T(r_\alpha(z), r_\alpha(u))}{\alpha}\mathbf{1}(u < 0).$$

Moreover $p_{Z_T}(z,\cdot)$ satisfies the following Gaussian upper bound:

$$p_{Z_T}(z,u) \leq \max\left\{\frac{1}{1-\alpha}, \frac{1}{\alpha}\right\} E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2})) g_T^c(u-z).$$

for some $C, c > 0$.

Proof. We prove that $p_{Z_T}(z,\cdot)$ is the density function of $Z_T(z)$ for every $z\in\mathbb{R}$. For any $f\in C_c^\infty(\mathbb{R})$, by the change of variables $u=s_\alpha(y)$ we have for every $z\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}[f(Z_T(z))] =&\mathbb{E}[f(s_\alpha(X_T(x)))]\\ =&\int_0^\infty \frac{f(u)}{1-\alpha}p_T(r_\alpha(x),r_\alpha(u))\mathrm{d}u+\int_{-\infty}^0\frac{f(u)}{\alpha}p_T(r_\alpha(x),r_\alpha(u))\mathrm{d}u\\ =&\int_{\mathbb{R}} f(u)p_{Z_T}(z,u)\mathrm{d}u.\end{aligned}$$

Therefore, $p_{Z_T}(z,\cdot)$ is the density function of $Z_T(z)$ for all $z\in\mathbb{R}$. Now we prove the Gaussian upper bound for $p_{Z_T}(z,\cdot)$. From Proposition 6.5.3, we have for some $C,c>0$,

$$\begin{aligned}p_{Z_T}(z,u)&\le \max\left\{\frac{1}{1-\alpha},\frac{1}{\alpha}\right\}E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}))g_T^c\left(r_\alpha(u)-r_\alpha(z)\right)\\ &\le \max\left\{\frac{1}{1-\alpha},\frac{1}{\alpha}\right\}E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}))\begin{cases} g_T^c\left(u-z\right), & \text{if } z,u>0,\\ g_T^c\left((1-\alpha)u-\alpha z\right), & \text{if } z>0>u,\\ g_T^c\left(\alpha u-(1-\alpha)z\right), & \text{if } z<0<u,\\ g_T^c\left(u-z\right), & \text{if } z,u<0.\end{cases}\end{aligned}$$

Let $z>0>u$. If $\alpha/(1-\alpha)\ge 1$, then $-\frac{(1-\alpha)u-\alpha z}{1-\alpha}\ge z-u>0$ and if $\alpha/(1-\alpha)<1$, then $-\frac{(1-\alpha)u-\alpha z}{\alpha}>z-u>0$. Let $z<0<u$. If $\alpha/(1-\alpha)\ge 1$, then $\frac{\alpha u-(1-\alpha)z}{1-\alpha}\ge u-z>0$ and if $\alpha/(1-\alpha)<1$, then $\frac{\alpha u-(1-\alpha)z}{\alpha}>u-z>0$. Therefore, we obtain that

$$p_{Z_T}(z,u)\le \max\left\{\frac{1}{1-\alpha},\frac{1}{\alpha}\right\}E_{\eta/2,1}(C(1\vee T)^{(3-2\eta)/2}))g_T^c\left(u-z\right),$$

which concludes the statement. □

6.6 Regularity of the density for a skew diffusion

In this section, we prove that $x\mapsto p_T(x,y)$ is differentiable function on $\mathbb{R}_0$.

Theorem 6.6.1. *Suppose Assumption 6.2.1 holds. For any $y\in\mathbb{R}$, $p_T(x,y)$ is differentiable with respect to $x\in\mathbb{R}_0$ and we have*

$$\partial_x p_T(x,y)=\sum_{n=0}^\infty \partial_x \hat{I}^n_{t_0}(y,x)=\sum_{n=0}^\infty(\partial_x\bar{p})\circledast\Phi^{\circledast n}(t_0,x,y)$$

where $t_0=T$. Moreover, we have

$$|\partial_x p_T(x,y)|\le \frac{E_{\eta/2,1/2}(C(1\vee T)^{1+\eta/2})}{T^{1/2}}g_T^c\left(y-x\right) \tag{6.29}$$

and for any $x,y\in\mathbb{R}_0$, we have

$$\alpha\partial_x p_T(0+,y)=(1-\alpha)\partial_x p_T(0-,y) \tag{6.30}$$

Proof of Theorem 6.6.1. We first prove that for $y_0 := y$,

$$\partial_x \hat{I}^n_{t_0}(y,x) = \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \int_{\mathbb{R}^n} \mathrm{d}y_1 \cdots \mathrm{d}y_n \prod_{i=0}^{n-1} \Phi(t_i - t_{i+1}, y_{i+1}, y_i) \partial_x p^{y_n}_{t_n}(x, y_n). \tag{6.31}$$

For any $x, y \in \mathbb{R}_0$,

$$\partial_x p^y_t(x,y) = \begin{cases} -H_1(y-x, ta(y)) g^{a(y)}_t(y-x) + (2\alpha - 1) H_1(y+x, ta(y)) g^{a(y)}_t(y+x), & \text{if } x, y > 0, \\ -2(1-\alpha) H_1(y-x, ta(y)) g^{a(y)}_t(y-x), & \text{if } x > 0 > y, \\ -2\alpha H_1(y-x, ta(y)) g^{a(y)}_t(y-x), & \text{if } x < 0 < y, \\ -H_1(y-x, ta(y)) g^{a(y)}_t(y-x) + (1-2\alpha) H_1(y+x, ta(y)) g^{a(y)}_t(y+x), & \text{if } x, y < 0. \end{cases}$$

Therefore, from $|x|^p e^{-qx^2} \le (p/(2qe))^{p/2}$ for any $p, q > 0$ and $x \in \mathbb{R}$, we have for any $x, y \in \mathbb{R}_0$,

$$|\partial_x p^y_t(x,y)| \le \frac{C}{t^{1/2}} g^c_t(y-x).$$

As in the proof of Proposition 6.5.3, from Lemma 6.4.4 and Chapman-Kolmogorov's equation, we have

$$\begin{aligned}
&\int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \int_{\mathbb{R}^n} \mathrm{d}y_1 \cdots \mathrm{d}y_n \prod_{i=0}^{n-1} |\Phi(t_i - t_{i+1}, y_{i+1}, y_i)| |\partial_x p^{y_n}_{t_n}(x, y_n)| \\
&\le \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \int_{\mathbb{R}^n} \mathrm{d}y_1 \cdots \mathrm{d}y_n \prod_{i=0}^{n-1} \frac{C_T}{(t_i - t_{i+1})^{1-\eta/2}} g^c_{t_i - t_{i+1}}(y_i - y_{i+1}) \frac{C}{t_n^{1/2}} g^c_{t_n}(y_n - x) \\
&\le C_T^n \int_0^{t_0} \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i=0}^{n-1} \frac{1}{(t_i - t_{i+1})^{1-\eta/2}} \frac{1}{t_n^{1/2}} g^c_{t_0}(y_0 - x) \\
&= \frac{C_T^n t_0^{n\eta/2} \Gamma^n(\eta/2) \Gamma(1/2)}{t_0^{1/2} \Gamma(1/2 + n\eta/2)} g^c_{t_0}(y_0 - x) < \infty,
\end{aligned}$$

where we used Lemma 6.8.1 with $b = -1/2$ and $a = 1 - \eta/2$ in the last equation. Hence the, right hand side of (6.31) is well defined. Then from the dominated convergence theorem, we obtain (6.31) and

$$|\partial_x \hat{I}^n_{t_0}(y,x)| \le \frac{C_T^n t_0^{n\eta/2} \Gamma^n(\eta/2)}{t_0^{1/2}} \frac{\Gamma(1/2)}{\Gamma(1/2 + n\eta/2)} g^c_{t_0}(y_0 - x) < \infty.$$

Therefore,

$$\sum_{n=0}^{\infty} \sup_{x,y \in \mathbb{R}_0} |\partial_x \hat{I}^n_{t_0}(y,x)| < \infty.$$

From here, we conclude the first two statements of the Theorem. Finally, since for any $y \in \mathbb{R}_0$ and $t > 0$, $p^y_t(\cdot, y) \in D(\alpha)$, then one obtains (6.30). $\square$

From Theorem 6.6.1, we have the following gradient estimate for the semigroup.

Corollary 6.6.2. *Suppose Assumption 6.2.1 holds. We assume that f is a measurable function such that $\int_{\mathbb{R}} |f(y)|g_T^c(y-x)\mathrm{d}y < \infty$ for any $c > 0$. Then $\mathbb{E}[f(X_T(x))]$ is differentiable with respect to $x \in \mathbb{R}_0$ and it holds that*

$$\partial_x P_T f(x) = \int_{\mathbb{R}} f(y)\partial_x p_T(x,y)\mathrm{d}y.$$

Proof. For $x \in \mathbb{R}_0$ and $h \in (-1,1)$ with $x \pm h \neq 0$, it follows from the mean value theorem and the upper bound for $\partial_x p_T(x,y)$ in (6.29) that

$$\begin{aligned} |p_T(x+h,y) - p_T(x,y)| &= |h| \left| \int_0^1 \partial_x p_T(x+\theta h, y)\mathrm{d}\theta \right| \\ &\leq |h| \frac{E_{\eta/2,1/2}(C(1\vee T)^{1+\eta/2})}{T^{1/2}} \int_0^1 \frac{e^{-\frac{(y-x-h\theta)^2}{2cT}}}{\sqrt{2\pi T}}\mathrm{d}\theta. \end{aligned}$$

Using the inequality $(a-b)^2 \geq a^2/2 - b^2$, we have

$$\left| \frac{p_T(x+h,y) - p_T(x,y)}{h} \right| \leq \frac{E_{\eta/2,1/2}(C(1\vee T)^{1+\eta/2})}{T^{1/2}} \frac{e^{-\frac{(y-x)^2}{4cT}}}{\sqrt{2\pi T}}.$$

Therefore, from the dominated convergence theorem, we have

$$\lim_{h\to 0} \frac{\mathbb{E}[f(X_T(x+h))] - \mathbb{E}[f(X_T(x))]}{h} = \int_{\mathbb{R}} f(y) \lim_{h\to 0} \frac{p_T(x+h,y) - p_T(x,y)}{h}\mathrm{d}y = \int_{\mathbb{R}} f(y)\partial_x p_T(x,y)\mathrm{d}y,$$

which concludes the statement. $\square$

Theorem 6.6.3. *Suppose that Assumption* 6.2.1 *holds. Let $Z(z)$ be the solution to SDE (6.7). For any $(z,u) \in \mathbb{R}_0 \times \mathbb{R}$, $p_{Z_T}(z,u)$ is differentiable as a function of $z \in \mathbb{R}_0$ and we have*

$$\partial_z p_{Z_T}(z,u) = \begin{cases} \dfrac{1}{1-\alpha}\partial_x p_T(z/(1-\alpha), r_\alpha(u)) \left(\dfrac{\mathbf{1}(u \geq 0)}{1-\alpha} + \dfrac{\mathbf{1}(u<0)}{\alpha} \right), & \text{if } z > 0, \\ \dfrac{1}{\alpha}\partial_x p_T(z/\alpha, r_\alpha(u)) \left(\dfrac{\mathbf{1}(u \geq 0)}{1-\alpha} + \dfrac{\mathbf{1}(u<0)}{\alpha} \right), & \text{if } z < 0. \end{cases}$$

As a consequence of the above result and 6.29 one can obtain a similar result as Corollary 6.5.6.

6.7 Probabilistic representation

In this section, we introduce a probabilistic representation of the density function of the skew diffusion $X_T(x)$ and $\mathbb{E}[f(X_T(x))]$.

We first define the following counting process.

Definition 6.7.1. *Let $R_t := \sum_{n=1}^{\infty} \mathbf{1}(\tau_n \leq t)$ where $(\tau_n - \tau_{n-1})_{n\in\mathbb{N}}$ with $\tau_0 = 0$ are independent and identically distributed random variables with density function ζ. Then $R = (R_t)_{t\geq 0}$ is called the counting process with jump times $(\tau_n)_{n\in\mathbb{N}}$.*

Remark 6.7.2. Usual choices for the density function ζ are: $\zeta(t) = \lambda e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t)$ then, $R = (R_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda > 0$. Another choice is $\zeta(t) := \frac{A}{t^\beta}\mathbf{1}_{[0,2T]}(t)$ where $A := (1-\beta)/(2T)^{1-\beta}$ and $\beta \in (0,1)$. For more on this, see Andersson and Kohatsu [4].

Lemma 6.7.3. *Let $R = (R_t)_{t\geq 0}$ be the counting process with jumps times $(\tau_n)_{n\in\mathbb{N}}$. Then for any $t > 0$, $n \in \mathbb{N}$ and any measurable bounded function $V_n : \mathbb{R}^n \to \mathbb{R}$,*

$$
\begin{aligned}
&\mathbb{E}[\mathbf{1}(R_t = n)V_n(\tau_1,\dots,\tau_n)] \\
&= \int_0^t \mathrm{d}s_n \int_0^{s_n} \mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1 V_n(s_1,\dots,s_n)(1-F_\zeta(t-s_n))\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i),
\end{aligned}
$$

where $F_\zeta(x) := \int_{-\infty}^x \zeta(y)\mathrm{d}y$ and $s_0 = 0$. In particular, if R is a Poisson process with parameter $\lambda > 0$, then we have

$$
\mathbb{E}[\mathbf{1}(R_t = n)V_n(\tau_1,\dots,\tau_n)] = \lambda^n e^{-\lambda t}\int_0^t \mathrm{d}s_n \int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1 V_n(s_1,\dots,s_n).
$$

Proof. We first prove by induction that the joint probability density function of $(\tau_1,\dots,\tau_n)$ is given by

$$
\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i)\mathbf{1}(0 < s_1 < \cdots < s_n). \tag{6.32}
$$

If $n = 1$, the statement holds by the definition of ζ. Assume that (6.32) holds for $n > 1$. Then since $\tau_{n+1} - \tau_n$ is independent from τ_i, for any $i = 1,...,n$, we have for any $x_1,...,x_{n+1} > 0$,

$$
\begin{aligned}
&\mathbb{P}(\tau_1 < x_1,\dots,\tau_{n+1} < x_{n+1}) = \mathbb{P}(\tau_1 < x_1,\dots,\tau_n + (\tau_{n+1}-\tau_n) < x_{n+1}) \\
&= \int_0^{x_1}\mathrm{d}s_1\cdots\int_0^{x_n}\mathrm{d}s_n\int_0^\infty \mathrm{d}t_{n+1}\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i)\mathbf{1}(0 < s_1 < \cdots < s_n)\zeta(t_{n+1})\mathbf{1}(s_n + t_{n+1} < x_{n+1}) \\
&= \int_0^{x_1}\mathrm{d}s_1\cdots\int_0^{x_{n+1}}\mathrm{d}s_{n+1}\prod_{i=0}^{n}\zeta(s_{i+1}-s_i),
\end{aligned}
$$

where in the last equation we use the change of variable $s_{n+1} = s_n + t_{n+1}$. Hence (6.32) holds for any $n \in \mathbb{N}$.

From (6.32) and the Fubini theorem, we have

$$
\begin{aligned}
&\mathbb{E}[\mathbf{1}(R_t = n)V_n(\tau_1,\dots,\tau_n)] = \mathbb{E}[\mathbf{1}(\tau_n \leq t < \tau_{n+1})V_n(\tau_1,\dots,\tau_n)] \\
&= \int_t^\infty \mathrm{d}s_{n+1}\int_0^t \mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1 V_n(s_1,\dots,s_n)\prod_{i=0}^{n}\zeta(s_{i+1}-s_i) \\
&= \int_0^t \mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1\int_t^\infty \mathrm{d}s_{n+1}\zeta(s_{n+1}-s_n)V_n(s_1,\dots,s_n)\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i) \\
&= \int_0^t \mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1 V_n(s_1,\dots,s_n)(1-F_\zeta(t-s_n))\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i),
\end{aligned}
$$

which concludes the proof of Lemma 6.7.3. ∎

Let $\varphi_t^z(x)$ be a strictly positive density function of a time-homogeneous Markov process Y_t^z with $Y_0^z = z$. We define a function $\hat{\theta}_t(x,z)$ as

$$\hat{\theta}_t(x,z) := \begin{cases} \dfrac{(L-L^z)\phi_t^z(x)}{\varphi_t^z(x)} & \text{if } x \text{ and } z \neq 0, \\ 0 & \text{if } x \text{ or } z = 0. \end{cases}$$

Note that when $\alpha = 1/2$ and $\varphi_t^z(x) := \phi_t^z(x)$, i.e., the diffusion case, it holds that

$$\hat{\theta}_t(x,z)\phi_t^z(x) = (L-L^z)\phi_t^z(x).$$

However, in the general skew diffusion case, this kind of property does not hold.

Let $\pi_0 = (s_i \wedge T)_{i\in\mathbb{N}}$ with $0 =: s_0 \leq s_1 < \cdots < s_n < \cdots$. Suppose that for any partition π_0 there exists a time continuous Markov chain $Y^{*,\pi_0}(y_0)$ such that

$$\begin{aligned} Y_0^{*,s_0}(y_0) &= y_0, \\ \mathbb{P}(Y_{s_i}^{*,\pi_0}(y_0) \in dy_{i+1} | Y_{s_{i-1}}^{*,\pi_0}(y_0) = y_i) &= \varphi_{s_i - s_{i-1}}^{y_i}(y_{i+1}) dy_{i+1}. \end{aligned} \tag{6.33}$$

Let $y_0 := y$ and $t_0 := T$. From the definition of $p_T(x,y)$ and using the change of variables $s_n = t_0 - t_n$, we have

$$p_{t_0}(x,y) = \sum_{n=0}^{\infty} \int_0^{t_0} \mathrm{d}s_n \int_0^{s_n} \mathrm{d}s_{n-1} \cdots \int_0^{s_2} \mathrm{d}s_1 h_n(s_1,\ldots,s_n,y,x). \tag{6.34}$$

Here,

$$h_n(s_1,\ldots,s_n,y,x) := \int_{\mathbb{R}^n} \mathrm{d}y_1 \cdots \mathrm{d}y_n \prod_{i=0}^{n-1} \hat{\theta}_{s_{i+1}-s_i}(y_{i+1},y_i)\varphi_{s_{i+1}-s_i}^{y_i}(y_{i+1}) p_{t_0-s_n}^{y_n}(x,y_n).$$

Then from Lemmas 6.4.4, 6.4.5 and the Chapman-Kolmogorov's equation, we have

$$|h_n(s_1,\ldots,s_n,y,x)| \leq \prod_{i=0}^{n-1} \frac{C_T}{(s_{i+1}-s_i)^{1-\eta/2}} g_{t_0}^c(y-x). \tag{6.35}$$

This gives the needed integrability properties for the convergence of the sum (6.34). In the probabilistic representation to follow this condition will imply the $L^1(\Omega)$-integrability of the probabilistic representation.

Theorem 6.7.4. *Assume that Assumption 6.2.1 holds. Let $R = (R_t)_{t\geq 0}$ be the counting process with jump times $\pi := (\tau_n)_{n\in\mathbb{N}}$ independent of $(Y^{*,\pi_0})_{\pi_0}$. Let $f : \mathbb{R} \to \mathbb{R}$ and define $D := \operatorname{supp} f \subseteq \mathbb{R}$. Also, let Z be a D-valued random variable independent of R and $(Y^{*,\pi_0})_{\pi_0}$. Assume that g is the density function for Z such that $g > 0$ on D. Suppose that $\int_{\mathbb{R}} |f(y)| g_T^c(y)\mathrm{d}y < \infty$ for any $c > 0$.*

Then for every $x \in \mathbb{R}$, we have

$$\mathbb{E}[f(X_T(x))] = \mathbb{E}\left[\frac{f(Z)}{g(Z)} \frac{p_{T-\tau_T}^{Y_{\tau_T}^{*,\pi}(Z)}(x, Y_{\tau_T}^{*,\pi}(Z))}{1 - F_\zeta(T-\tau_T)} \prod_{i=0}^{R_T-1} \frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y_{\tau_{i+1}}^{*,\pi}(Z), Y_{\tau_i}^{*,\pi}(Z))}{\zeta(\tau_{i+1}-\tau_i)}\right].$$

Here, $\tau_T := \tau_{R_T}$. Furthermore the density of X_T can be represented as

$$p_T(x,y) = \mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x, Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)} \prod_{i=0}^{R_T-1} \frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y), Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right].$$

Proof. First, note that since the density $p_T(x,\cdot)$ of $X_T(x)$ satisfies a Gaussian upper bound then using Young's inequality, we get that $\mathbb{E}[|f(X_T(x))|] < \infty$.

We will obtain a representation formula for h_n. For $n=1$, since $\varphi^{y_0}_{s_1}$ is a density function of $Y^{*,\pi_0}_{s_1}(y_0)$, we have

$$\begin{aligned}
h_1(s_1,y,x) =& \int_{\mathbb{R}} \hat{\theta}_{s_1}(y_1,y_0)\varphi^{y_0}_{s_1}(y_1) p^{y_1}_{t_0-s_1}(x,y_1)\mathrm{d}y_1 \\
=&\mathbb{E}\left[p_{t_0-s_1}^{Y^{*,\pi_0}_{s_1}(y)}(x, Y^{*,\pi_0}_{s_1}(y))\hat{\theta}_{s_1}(Y^{*,\pi_0}_{s_1}(y), y)\right].
\end{aligned}$$

For $n=2$, we have

$$\begin{aligned}
h_2(s_1,s_2,y,x) =& \int_{\mathbb{R}} \hat{\theta}_{s_1}(y_1,y_0)\varphi^{y_0}_{s_1}(y_1)\int_{\mathbb{R}} \hat{\theta}_{s_2-s_1}(y_2,y_1)\varphi^{y_1}_{s_2-s_1}(y_2)p^{y_2}_{t_1-s_2}(x,y_2)\mathrm{d}y_2 \\
=& \int_{\mathbb{R}} \hat{\theta}_{s_1}(y_1,y_0)\varphi^{y_0}_{s_1}(y_1)\mathbb{E}\left[\hat{\theta}_{s_2-s_1}(Y^{*,\pi_0}_{s_2}(y),y_1)p_{t_1-s_2}^{Y^{*,\pi_0}_{s_2}(y)}(x,Y^{*,\pi_0}_{s_2}(y))\middle| Y^{*,\pi_0}_{s_1}(y)=y_1\right] \\
=&\mathbb{E}\left[p_{t_1-s_2}^{Y^{*,\pi_0}_{s_2}(y)}(x,Y^{*,\pi_0}_{s_2}(y))\hat{\theta}_{s_1}(Y^{*,\pi_0}_{s_2}(y),y_0)\hat{\theta}_{s_2-s_1}(Y^{*,\pi_0}_{s_2}(y),Y^{*,\pi_0}_{s_1}(y))\right].
\end{aligned}$$

In the same way as in the case $n=2$, we obtain that

$$h_n(s_1,\ldots,s_n,y,x) =\mathbb{E}\left[p_{t_0-s_n}^{Y^{*,\pi_0}_{s_n}(y)}(x,Y^{*,\pi_0}_{s_n}(y))\prod_{i=0}^{n-1}\hat{\theta}_{s_{i+1}-s_i}(Y^{*,\pi_0}_{s_{i+1}}(y),Y^{*,\pi_0}_{s_i}(y))\right].$$

Let $R=(R_t)_{t\geq 0}$ be the counting process with jump times $(\tau_n)_{n\in\mathbb{N}}$ independent from Y^{*,π_0}. From Lemma 6.7.3, we get for every $x \in \mathbb{R}$

$$p_{t_0}(x,y) = \sum_{n=0}^{\infty}\int_0^{t_0}\mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1 V_n(s_1,\ldots,s_n,y,x)(1-F_\zeta(T-s_n))\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i),$$

where

$$V_n(s_1,\ldots,s_n,y,x) := \frac{h_n(s_1,\ldots,s_n,y,x)}{(1-F_\zeta(T-s_n))\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i)}.$$

Therefore, we have

$$\begin{aligned}
p_T(x,y) =& \sum_{n=0}^{\infty}\mathbb{E}[\mathbf{1}(R_T=n)V_n(\tau_1,\ldots,\tau_n,y,x)] = \sum_{n=0}^{\infty}\mathbb{E}[\mathbf{1}(R_T=n)V_n(\tau_1,\ldots,\tau_n,y,x)] && (6.36)\\
=&\mathbb{E}[V_{R_T}(\tau_1,\ldots,\tau_T,y,x)] && (6.37)
\end{aligned}$$

$$=\mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right]. \tag{6.38}$$

Since $X_T(x)$ has a density $p_T(x,\cdot)$ and Z is independent from R and $Y^{*,\pi}$, we have

$$\begin{aligned}\mathbb{E}[f(X_T(x))] &= \int_D \frac{f(y)}{g(y)}g(y)p_T(x,y)\mathrm{d}y = \mathbb{E}\left[\frac{f(Z)}{g(Z)}p_T(x,Z)\right]\\ &=\mathbb{E}\left[\frac{f(Z)}{g(Z)}\mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right]\Bigg|_{y=Z}\right]\\ &=\mathbb{E}\left[\frac{f(Z)}{g(Z)}\mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{s_n}(Z)}(x,Y^{*,\pi}_{\tau_T}(Z))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(Z),Y^{*,\pi}_{\tau_i}(Z))}{\zeta(\tau_{i+1}-\tau_i)}\Bigg|\sigma(Z)\right]\right].\end{aligned}$$

This implies the statement. □

For example, if we choose $\zeta(t)=\lambda e^{-\lambda t}\mathbf{1}_{[0,\infty)(t)}$, then we have

$$p_T(x,y)=e^{\lambda T}\mathbb{E}\left[\lambda^{-R_T}p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))\prod_{i=0}^{R_T-1}\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))\right],$$

We also have the following probabilistic representation for the derivative of the density of X_T.

Theorem 6.7.5. *Assume that Assumption 6.2.1 holds. Let $R=(R_t)_{t\geq 0}$ be the counting process with $\pi:=(\tau_n)_{n\in\mathbb{N}}$ independent of $(Y^{*,\pi_0})_{\pi_0}$. Then for any $x\in\mathbb{R}_0$,*

$$\partial_x p_T(x,y)=\mathbb{E}\left[\frac{\partial_x p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right].$$

Moreover, if we assume the same hypothesis as in Theorem 6.7.4 then we have the following probabilistic representation:

$$\partial_x\mathbb{E}[f(X_T(x))]=\mathbb{E}\left[\frac{f(Z)}{g(Z)}\frac{\partial_x p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(y)}(x,Y^{*,\pi}_{\tau_T}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(y),Y^{*,\pi}_{\tau_i}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right]. \tag{6.39}$$

In the same way the following probabilistic representation for the density of Z_t holds.

Theorem 6.7.6. *Assume that Assumption* 6.2.1 *holds and also the same hypothesis as in Theorem 6.7.4. Then for any $x\in\mathbb{R}_0$,*

$$\begin{aligned}p_{Z_T}(z,u)=&\mathbb{E}\left[\frac{p_{T-\tau_T}^{Y^{*,\pi}_{\tau_T}(r_\alpha(u))}(r_\alpha(z),Y^{*,\pi}_{\tau_T}(r_\alpha(u)))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\hat{\theta}_{\tau_{i+1}-\tau_i}(Y^{*,\pi}_{\tau_{i+1}}(r_\alpha(u)),Y^{*,\pi}_{\tau_i}(r_\alpha(u)))\right]\\ &\times\left(\frac{\mathbf{1}(u\geq 0)}{1-\alpha}+\frac{\mathbf{1}(u<0)}{\alpha}\right).\end{aligned}$$

Example 6.7.7. One example of a Markov process Y^z satisfying the assumptions in this section is

$$Y_t^z = z + \sigma(z) W_t$$

and its density is given by

$$\varphi_t^z(x) = g_t^{a(z)}(x-z) = \frac{e^{-\frac{(x-z)^2}{2t\sigma(z)^2}}}{\sqrt{2\pi\sigma^2(z)t}}.$$

In this case, the Markov chain $Y^{*,\pi_0}(y_0)$ is given by

$$Y_0^{*,\pi_0}(y_0) = y_0 \text{ and } Y_{s_i}^{*,\pi_0}(y_0) = Y_{s_{i-1}}^{*,\pi_0}(y_0) + \sigma(Y_{s_{i-1}}^{*,\pi_0}(y_0))(W_{s_i} - W_{s_{i-1}})$$

and we have the following cases:
Case 1: If $x, y > 0$,

$$\begin{aligned}\hat{\theta}_t(x,y) =& b(x)\left(-H_1(y-x, ta(y)) + (2\alpha-1)H_1(y+x, ta(y))\exp\left(-\frac{2xy}{a(y)t}\right)\right)\\ &+ \frac{a(x)-a(y)}{2}\left(H_2(y-x, ta(y)) + (2\alpha-1)H_2(y+x, ta(y))\exp\left(-\frac{2xy}{a(y)t}\right)\right).\end{aligned}$$

Case 2: If $x > 0 > y$,

$$\hat{\theta}_t(x,y) = 2(1-\alpha)\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right).$$

Case 3: If $x < 0 < y$,

$$\hat{\theta}_t(x,y) = 2\alpha\left(-b(x)H_1(y-x, ta(y)) + \frac{a(x)-a(y)}{2}H_2(y-x, ta(y))\right).$$

Case 4: If $x, y < 0$,

$$\begin{aligned}\hat{\theta}_t(x,y) =& b(x)\left(-H_1(y-x, ta(y)) + (1-2\alpha)H_1(y+x, ta(y))\exp\left(-\frac{2xy}{a(y)t}\right)\right)\\ &+ \frac{a(x)-a(y)}{2}\left(H_2(y-x, ta(y)) + (1-2\alpha)H_2(y+x, ta(y))\right).\end{aligned}$$

6.7.1 Finite variance method

In this subsection, we introduce an example of finite variation probabilistic representation.

Theorem 6.7.8. *Suppose that Assumption 6.2.1 holds. Let $p \geq 2$. Define $\zeta(t) := \frac{A}{t^\beta}\mathbf{1}_{[0,2T]}(t)$ where $A := (1-\beta)/(2T)^{1-\beta}$ and $\beta \in (p(1-\eta/2)-1)/(p-1), 1)$. Let $R_t := \sum_{n=1}^\infty \mathbf{1}(\tau_n \leq t)$ where $(\tau_n - \tau_{n-1})_{n\in\mathbb{N}}$ with $\tau_0 = 0$ are independent and identically distributed random variables with density function ζ. Then it holds that*

$$\mathbb{E}\left[\left|\frac{p_{T-\tau_T}^{Y_{\tau_T}^{*,\pi}(y)}(x, Y_{\tau_T}^{*,\pi}(y))}{1-F_\zeta(T-\tau_T)}\prod_{i=0}^{R_T-1}\frac{\hat{\theta}_{\tau_{i+1}-\tau_i}(Y_{\tau_{i+1}}^{*,\pi}(y), Y_{\tau_i}^{*,\pi}(y))}{\zeta(\tau_{i+1}-\tau_i)}\right|^p\right] < \infty.$$

Proof. It suffices to prove that $\mathbb{E}[|V_{R_T}(\tau_1,\ldots,\tau_T,y,x)|^p]<\infty$. Using Lemma 6.7.3, we have

$$\begin{aligned}
&\mathbb{E}[|V_{R_T}(\tau_1,\ldots,\tau_T,y,x)|^p]=\sum_{n=0}^{\infty}\mathbb{E}[\mathbf{1}(R_T=n)|V_n(\tau_1,\cdots,\tau_T,y,x)|^p]\\
&=\sum_{n=0}^{\infty}\int_0^T\mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1|V_n(s_1,\cdots,s_n,y,x)|^p(1-F_\zeta(T-s_n))\prod_{i=0}^{n-1}\zeta(s_{i+1}-s_i)\\
&=\sum_{n=0}^{\infty}\int_0^T\mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1\frac{|h(s_1,\cdots,s_n,y,x)|^p}{(1-F_\zeta(T-s_n))^{p-1}\prod_{i=0}^{n-1}\zeta^{p-1}(s_{i+1}-s_i)}.
\end{aligned}$$

Since for any $s_n\in[0,T]$, $1-F_\zeta(T-s_n)\geq A((2T)^{1-\beta}-T^{1-\beta})/(1-\beta)$, it follows from (6.35) that,

$$\begin{aligned}
\mathbb{E}[|V_{R_T}(\tau_1,\cdots,\tau_T,y,x)|^p]\leq& g_T^p\left(\frac{y-x}{c}\right)\left(\frac{1-\beta}{A((2T)^{1-\beta}-T^{1-\beta})}\right)^{p-1}\\
&\times\sum_{n=0}^{\infty}C^{pn}\int_0^T\mathrm{d}s_n\int_0^{s_n}\mathrm{d}s_{n-1}\cdots\int_0^{s_2}\mathrm{d}s_1\prod_{i=0}^{n-1}(s_{i+1}-s_i)^{(p-1)\beta-p+p\eta/2}.
\end{aligned}$$

Since $(p(1-\eta/2)-1)/(p-1)\in[0,1)$, by taking $\beta\in(p(1-\eta/2)-1)/(p-1),1)$, we conclude the statement by using Lemma 6.8.1. □

6.8 Appendix

6.8.1 On some Beta type integral

Lemma 6.8.1. *Let $b>-1$ and $a\in[0,1)$. Then for any $t_0>0$,*

$$\int_0^{t_0}\mathrm{d}t_1\cdots\int_0^{t_{n-1}}\mathrm{d}t_n t_n^b\prod_{j=0}^{n-1}(t_j-t_{j+1})^{-a}=\frac{t_0^{b+n(1-a)}\Gamma^n(1-a)\Gamma(1+b)}{\Gamma(1+b+n(1-a))}.$$

Proof. Let $b>-1$ and $a\in[0,1)$. Using the change of variables $s=ut$, we have

$$\int_0^t s^b(t-s)^{-a}\mathrm{d}s=t^{b+1-a}\int_0^1 u^b(1-u)^{-a}\mathrm{d}u=t^{b+1-a}B(1+b,1-a),$$

where $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}$ is the standard Beta function. Using this repeatedly, we obtain the statement. □

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