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Essays on stochastic calculus in relation to number theory and representation theory

(整数論および表現論に関連する
確率解析についての考察)

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Chapter 1

Introduction

The theory of stochastic calculus is one of the main mathematical subjects established in the 20th century. It certainly evolved probability theory and mathematical statistics, and made great efforts in many other fields. Stochastic processes having independent and stationary increments such as Wiener processes and Poisson processes are usually called Lévy processes. The marginal distributions of Lévy processes are always infinitely divisible in a certain sense. Conversely, every infinite divisible distribution induces a Lévy process and they are often studied analytically in terms of characteristic functions. In this thesis, we pick up such processes and discuss some prospects of stochastic calculus through characteristic functions in relation to number theory and representation theory.

This thesis consists of three parts. Two of them are based on papers accepted for publications in mathematical journals and the rest is of original results. The first part is taken from Aoyama and Yoshikawa [7] with some modifications. Several definitions and propositions are added and the proof of one theorem which was omitted in Aoyama and Yoshikawa [7] is also given. Chapter 2 is the first part, where we discuss multivariable and multiple zeta functions and their definable multidimensional discrete distributions. Zeta functions are one of the rich classes of functions in mathematics. The Riemann zeta function is regarded as the prototype and now variously extended. Historically, it is well-known that the Riemann zeta function generates a one dimensional infinitely divisible discrete distribution in the region of absolute convergence. As a generalization of the Riemann zeta function, the Hurwitz zeta function is also well-known. In Hu, Iksanov, Lin and Zakusylo [14], the Hurwitz zeta distributions are introduced, and their infinite divisibilities are studied. In recent years, multidimensional Shintani zeta functions are introduced by Aoyama and Nakamura

[5] which are of multivariable and multiple zeta functions. These functions enable us to define a new class of multidimensional discrete distributions called multidimensional Shintani zeta distributions which is also introduced by Aoyama and Nakamura [5]. In this chapter, we show that this class includes many kinds of multidimensional discrete distributions. In fact, multinomial or negative binomial distributions are of the multidimensional Shintani zeta class, which allows us to define some classes regarded as their generalizations in view of zeta distributions. We draw exact outlines of these classes by giving the necessary and sufficient conditions for some cases of multidimensional Shintani zeta functions to generate probability distributions. We also consider their infinite divisibilities.

The second part consists of original researches. In chapter 3, we show some results of them, which are focused on Euler products. It is well-known that the Riemann zeta function has the Euler product in the region of absolute convergence. This is usually regarded as a key to show the prime number theorem. However, the infinite divisibility of the Riemann zeta distribution also can be shown by this fact. As a generalization of the Euler product, Aoyama and Nakamura [3, 4] introduced multidimensional polynomial Euler products which were generalized to be multivariable and multiple infinite products. Furthermore, they gave the necessary and sufficient conditions for those products to generate some infinitely divisible characteristic functions. In their cases, non-principal Dirichlet L-functions, which is one of the well-known zeta function, are not included. As a new result, we show that these functions can generate infinitely divisible characteristic functions in some cases. We also treat some products of two multidimensional Shintani zeta functions and consider their possibilities to generate characteristic functions. The necessary and sufficient condition for a product of a real-valued Dirichlet L-function and the Riemann zeta function to generate an infinitely divisible characteristic function is given as a main result.

The last part is based on Yoshikawa [26] in addition to several fundamental facts. Chapter 4 is the part, which consists of a study of a Fermion Fock space on Wiener functionals and its applications. As is well-known as the Wiener chaos expansion, all Wiener functionals with finite second moment can be expanded by using Hermite polynomials and an orthonormal Gaussian random basis. This expansion induces a representation of the Heisenberg algebra with some symmetric structure, which is a basis of Malliavin calculus. On the other hand, in Akahori et al. [2], an anti-symmetric calculus is studied by constructing a representation of a Clifford algebra on Wiener functionals. First, in chapter 4, we obtain that all Wiener functionals in the Fermion Fock space, which generated by the Clifford algebra acting on the vacuum, can be expressed as a polynomial of first order integrals and second order anti-symmetric integrals. The second order antisymmetric integrals are called (generalized) stochastic areas, which have some relations with soliton solutions of the KdV equation (see, Aihara et al. [1]). Since Lévy found that the characteristic function of the stochastic area was explicitly given by trigonometric functions, there have been many studies related to the formula (see, e.g. Helmes and Schwane [13]). Secondly, in chapter 4, we see explicit forms of the characteristic functions of some joint distributions with stochastic areas. As an application of the first and second results, we propose an approximation scheme based on the anti-symmetric calculus over Wiener space.

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Chapter 2

Multinomial distributions in Shintani zeta class

Multidimensional stochastic models in mathematical finance and so on are now well-studied. As to obtain more properties of them, we focus on some multidimensional discrete distributions in relation to a class of multiple zeta functions. The class of multiple zeta functions called "multidimensional Shintani zeta functions" was first introduced in Aoyama and Nakamura [5], where a class of probability distributions called "multidimensional Shintani zeta distributions" associated with these zeta functions is definable. In this chapter, we show that this class includes many kinds of multidimensional discrete distributions. We pick up some cases of multidimensional Shintani zeta functions and introduce some classes of probability distributions which contain generalized multinomial and negative multinomial distributions. More precisely, we give some necessary and sufficient conditions for the functions to generate probability distributions in view of zeta functions and consider their infinite divisibilities as well.

2.1 Infinitely divisible distributions on $\mathbb{R}^d$

Infinitely divisible distributions are known as one of the most important class of probability distributions. They correspond to some essential stochastic processes such as Wiener processes and Poisson processes. In 1930's, such stochastic processes were well-studied by P. Lévy and now we usually call them Lévy processes. We can find the detail of Lévy processes in Sato [25]. First, we mention some known properties of infinitely divisible distributions.

Definition 2.1 (Infinitely divisible distribution (see, e.g. Sato [25])). A probability measure μ on $\mathbb{R}^d$ is infinitely divisible if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that

$$\mu = \mu_n^{n*},$$

where μ_n^{n*} is the n-fold convolution of μ_n.

Example 2.2. Normal, degenerate and Poisson distributions are infinitely divisible.

Let $\widehat{\mu}(z) := \int_{\mathbb{R}^d} e^{\mathrm{i}\langle z,x\rangle}\mu(dx),\ z \in \mathbb{R}^d$, be the characteristic function of a distribution μ on $\mathbb{R}^d$, where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^d$.

The following is well known.

Proposition 2.3 (Lévy-Khintchine representation (see, e.g. Sato [25])). (i) *If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then*

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle\gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \frac{\mathrm{i}\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\},\quad z \in \mathbb{R}^d, \tag{2.1}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d}\min\{|x|^2, 1\}\,\nu(dx) < \infty, \tag{2.2}$$

and $\gamma \in \mathbb{R}^d$.

(ii) *The representation of $\widehat{\mu}(z)$ in* (i) *by A, ν and γ is unique.*

(iii) *Conversely, if A is a symmetric nonnegative-definite $d\times d$ matrix, ν is a measure satisfying* (2.2)*, and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by* (2.1)*.*

The measure ν and (A, ν, γ) is called the Lévy measure and the Lévy-Khintchine triplet of an infinite divisible distribution μ, respectively. In chapter 2 and 3, we treat the following infinitely divisible distributions called compound Poisson distributions.

Definition 2.4 (Compound Poisson distribution (see, e.g. Sato [25])). A probability measure μ on $\mathbb{R}^d$ is called compound Poisson if its characteristic function can be written by

$$\widehat{\mu}(\vec{t}) = \exp\{c\,(\widehat{\rho}(\vec{t}) - 1)\},\quad \vec{t} \in \mathbb{R}^d,$$

for some $c > 0$ and some probability measure ρ on $\mathbb{R}^d$ with $\rho(\{0\}) = 0$.

Here the measure $c\rho$ is the (finite) Lévy measure of a compound Poisson distribution μ. The Poisson distribution is a special case when $d = 1$ and $\rho = \delta_1$, where δ_x is a delta measure at x.

Remark 2.5. Note that any infinitely divisible distribution can be expressed as the weak limit of a certain sequence of compound Poisson distributions.

Next, we mention Lévy processes.

Definition 2.6 (Lévy process (see, e.g. Sato [25]))**.** A stochastic precess $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is a Lévy process (in law) if the following conditions are satisfied.

(1) For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent (independent increments property).

(2) $Pr[X_0 = 0] = 1$.

(3) The distribution of $X_{s+t} - X_s$ does not depend on s (stationary increments property).

(4) It is stochastically continuous, that is $\lim_{s \to t} Pr[|X_s - X_t| > \epsilon] = 0$ for any $\epsilon > 0$.

Remark 2.7. We note that if μ is an infinitely divisible distribution, then μ^{t*} is definable for every $t \geq 0$ and is also infinitely divisible. Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ and μ be the distribution of X_1. Then, for every $t \geq 0$, the distribution of X_t is infinitely divisible and is given by μ^{t*}. Conversely, for any infinitely divisible distribution μ, there is a Lévy process whose distribution at time 1 is μ. This one-to-one correspondence shows the importance of the class of infinitely divisible distributions in the studies of Lévy processes.

Example 2.8. Wiener processes and Poisson processes are Lévy processes.

We use the following Lévy precesses in section 2.5.

Definition 2.9 (Subordinator)**.** An increasing Lévy process on $\mathbb{R}$ is called a subordinator.

Definition 2.10. Let $\{X_t : t \geq 0\}$ be a Lévy precess on $\mathbb{R}^d$ and $\{T(t) : t \geq 0\}$ be a subordinator. Suppose that $\{X_t : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ are independent processes with right-continuities and left-limits. Then, a transformation $\{Y_t : t \geq 0\}$ of $\{X_t : t \geq 0\}$ defined by $Y_t = X_{T(t)}$ is called a subordination by the subordinator $\{T(t) : t \geq 0\}$.

Subordinators and subordinations are often utilized. We can find more details of the following two propositions in Sato [25].

Proposition 2.11. *Let $\{T(t) : t \geq 0\}$ be a subordinator. Then, there exist a Lévy measure ρ and a real number β such that*

$$\beta \geq 0 \quad \textit{and} \quad \int_{(0,\infty)} \min\{1, s\}\, \rho(ds) < \infty, \tag{2.3}$$

and we have $E[e^{-uT(t)}] = e^{t\Psi(-u)}$, t, $u \geq 0$, where

$$\Psi(w) = \beta w + \int_{(0,\infty)} (e^{ws} - 1)\, \rho(ds), \;\; w \in \mathbb{C} \textit{ with } \operatorname{Re} w \leq 0. \tag{2.4}$$

Proposition 2.12. *Let $\{Y_t : t \geq 0\}$ be a subordination of a Lévy process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ by a subordinator $\{T(t) : t \geq 0\}$ with a Lévy measure ρ, a real number β and a function Ψ satisfying* (2.3) *and* (2.4)*. Then, $\{Y_t : t \geq 0\}$ is a Lévy process on $\mathbb{R}^d$, and we have*

$$E[e^{\mathrm{i}\langle z, Y_t\rangle}] = e^{t\Psi(\log \widehat{\mu}(z))}, \;\; t \geq 0, \;\; z \in \mathbb{R}^d.$$

2.2 Zeta functions and distributions

Zeta functions are one of valuable functions in mathematics and some other related fields. In mathematical statistics, they appear in several objects. One is that discrete distributions on $\mathbb{R}$ are definable by them. In this section, we introduce one variable zeta functions and their definable discrete probability distributions on $\mathbb{R}$. Then, we also introduce multivariable zeta functions and corresponding discrete probability distributions on $\mathbb{R}^d$. They include multidimensional discrete distributions with infinitely many mass points which, we may say, is the case not treatable enough compared to finitely many or continuous cases. By applying the infinite products representations of zeta functions, the infinite divisibilities of them are focused as well.

First we introduce the Riemann zeta function and the Euler product.

Definition 2.13 (Riemann zeta function, Euler product (see, e.g. Apostol [9])). Let $\zeta(s)$ be a function of a complex variable $s = \sigma + \mathrm{i}t \in \mathbb{C}$, for $\sigma > 1$, $t \in \mathbb{R}$, given by

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} && (2.5) \\ &= \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, && (2.6) \end{aligned}$$

where $\mathbb{P}$ is the set of all prime numbers. The function $\zeta(s)$ given by (2.5) and the infinite product (2.6) are called the Riemann zeta function and the Euler product, respectively.

It is well-known that the series in (2.5) and the infinite product (2.6) converge absolutely in the region $\sigma > 1$. The Riemann zeta function $\zeta(s)$ is analytically continuable to the whole complex plane as a meromorphic function by applying the Riemann's functional equation. We can find the basic properties of zeta functions in Apostol [9].

Next, we introduce the following probability distribution on $\mathbb{R}$ associating with the Riemann zeta function.

Definition 2.14 (Riemann zeta distribution). For each $\sigma > 1$, a probability measure μ_σ on $\mathbb{R}$ is called a Riemann zeta distribution, if

$$\mu_\sigma(\{-\log n\}) = \frac{n^{-\sigma}}{\zeta(\sigma)}, \quad n \in \mathbb{N}.$$

Then we have its characteristic function f_σ as follows:

$$f_\sigma(t) = \int_{\mathbb{R}} e^{\mathrm{i}tx} \, \mu_\sigma(dx) = \frac{\zeta(\sigma + \mathrm{i}t)}{\zeta(\sigma)}, \quad t \in \mathbb{R}.$$

This class of distributions is introduced by Jessen and Wintner [15] without normalization as to give an example in the studies of infinitely many times convolutions. As a probability distribution, it is first appeared in Khinchine [16].

Proposition 2.15 (See, e.g. Gnedenko and Kolmogorov [12]). *The characteristic function $f_\sigma(t)$, $t \in \mathbb{R}$, is a compound Poisson with a finite Lévy measure N_σ on $\mathbb{R}$:*

$$\log f_\sigma(t) = \int_0^\infty (e^{-\mathrm{i}tx} - 1) \, N_\sigma(dx),$$

where

$$N_\sigma(dx) = \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \, \delta_{r \log p}(dx).$$

This proposition implies that the Riemann zeta distribution is infinitely divisible. As a generalization of the Riemann zeta distribution, the following zeta function can also generate a probability distribution on $\mathbb{R}$.

Definition 2.16 (Hurwitz zeta function (see, e.g. Apostol [9])). For $0 < u \leq 1$ and $\sigma > 1$, the Hurwitz zeta function $\zeta(s,u)$ is defined by

$$\zeta(s,u) = \sum_{n=0}^{\infty} \frac{1}{(n+u)^s}, \quad s = \sigma + \mathrm{i}t, \ t \in \mathbb{R}.$$

We note that $\zeta(s,1)$ is the Riemann zeta function. For $0 < u \leq 1$ and $\sigma > 1$, put

$$f_{\sigma,u}(t) := \frac{\zeta(\sigma + \mathrm{i}t, u)}{\zeta(\sigma, u)}, \quad t \in \mathbb{R}.$$

Then $f_{\sigma,u}$ is a characteristic function of a probability distribution $\mu_{\sigma,u}$ on $\mathbb{R}$ which is called the Hurwitz zeta distribution. This class of distribution is introduced by Hu, Iksanov, Lin and Zakusylo [14] and its infinite divisibility is studied as well.

Proposition 2.17 (Hu, Iksanov, Lin and Zakusylo [14]). *The Hurwitz zeta distribution $\mu_{\sigma,u}$ is infinitely divisible if and only if*

$$u = \frac{1}{2} \qquad or \qquad u = 1.$$

This proposition comes from the fact that the Hurwitz zeta function has the Euler product only when $u = 1/2$ or 1.

The Riemann zeta function is now variously extended. Let $m, r \in \mathbb{N}$ and $\vec{s} \in \mathbb{C}^m$. For $\lambda_{lj}, u_j > 0$, where $1 \leq j \leq r$ and $1 \leq l \leq m$, a function

$$\zeta_S(\vec{s}) := \sum_{n_1,\dots,n_r=0}^{\infty} \prod_{l=1}^{m} \big(\lambda_{l1}(n_1+u_1) + \cdots + \lambda_{lr}(n_r+u_r)\big)^{-s_l}$$

is a generalized Barnes multiple zeta function called the Shintani zeta function. Aoyama and Nakamura [5] introduced the following functions.

Definition 2.18 (Multidimensional Shintani zeta function, Aoyama and Nakamura [5]). Let $d, m, r \in \mathbb{N}$, $\vec{s} \in \mathbb{C}^d$ and $(n_1, \dots, n_r) \in \mathbb{Z}^r_{\geq 0}$. For $\lambda_{lj}, u_j > 0$, $\vec{c_l} \in \mathbb{R}^d$, where $1 \leq j \leq r$ and $1 \leq l \leq m$, and a function $\theta(n_1, \dots, n_r) \in \mathbb{C}$ satisfying $|\theta(n_1, \dots, n_r)| = O((n_1 + \cdots + n_r)^{\varepsilon})$, for any $\varepsilon > 0$, we define a multidimensional Shintani zeta function by

$$Z_S(\vec{s}) := \sum_{n_1,\dots,n_r=0}^{\infty} \frac{\theta(n_1,\dots,n_r)}{\prod_{l=1}^{m}(\lambda_{l1}(n_1+u_1) + \cdots + \lambda_{lr}(n_r+u_r))^{\langle \vec{c_l}, \vec{s} \rangle}}. \tag{2.7}$$

Here we write $\langle \vec{c}, \vec{s} \rangle := \langle \vec{c}, \vec{\sigma} \rangle + \mathrm{i}\langle \vec{c}, \vec{t} \rangle$ for $\vec{c} \in \mathbb{R}^d$ and $\vec{s} \in \mathbb{C}^d$, where $\vec{\sigma}, \vec{t} \in \mathbb{R}^d$ and $\vec{s} = \vec{\sigma} + \mathrm{i}\vec{t}$. We call the function $\theta(n_1, \ldots, n_r)$ a character of the multidimensional Shintani zeta function, which is derived from Dirichlet characters (see Definition 3.1).

The absolute convergence of $Z_S(\vec{s})$ is also given.

Proposition 2.19 (Aoyama and Nakamura [5])**.** *The series $Z_S(\vec{s})$ defined by* (2.7) *converges absolutely in the region* $\min_{1 \le l \le m} \langle \vec{c}_l, \vec{\sigma} \rangle > r/m$.

We denote by $\mathrm{D_S}$ the region $\min_{1 \le l \le m} \langle \vec{c}_l, \vec{\sigma} \rangle > r/m$ of absolute convergence of the series $Z_S(\vec{s})$. Suppose that $\theta(n_1, \ldots, n_r)$ is non-negative or non-positive definite, then we can define the following class of distributions on $\mathbb{R}^d$.

Definition 2.20 (Multidimensional Shintani zeta distribution, Aoyama and Nakamura [5])**.** For each $\vec{\sigma} \in \mathrm{D_S}$, a probability measure $\mu_{\vec{\sigma}}$ on $\mathbb{R}^d$ is called a multidimensional Shintani zeta distribution if, for all $(n_1, \ldots, n_r) \in \mathbb{Z}^r_{\ge 0}$,

$$\mu_{\vec{\sigma}}\Bigg(\Big\{ -\sum_{l=1}^{m} c_{l1} \log\Big(\sum_{k=1}^{r} \lambda_{lk}(n_k+u_k)\Big), \ldots, -\sum_{l=1}^{m} c_{ld} \log\Big(\sum_{k=1}^{r} \lambda_{lk}(n_k+u_k)\Big)\Big\}\Bigg)$$
$$= \frac{\theta(n_1, \ldots, n_r)}{Z_S(\vec{\sigma})} \prod_{l=1}^{m} \Big(\sum_{k=1}^{r} \lambda_{lk}(n_k+u_k)\Big)^{-\langle \vec{c}_l, \vec{\sigma} \rangle}.$$

Then, its characteristic function $f_{\vec{\sigma}}$ is given by the normalization of $Z_S(\vec{s})$ as the Riemann zeta case.

Proposition 2.21 (Aoyama and Nakamura [5])**.** *Let $f_{\vec{\sigma}}$ be a characteristic function of a multidimensional Shintani zeta distribution $\mu_{\vec{\sigma}}$. Then $f_{\vec{\sigma}}(\vec{s})$ is given as follows.*

$$f_{\vec{\sigma}}(\vec{t}) = \int_{\mathbb{R}^d} e^{\mathrm{i}\langle \vec{t}, x \rangle} \, \mu_{\vec{\sigma}}(dx) = \frac{Z_S(\vec{\sigma} + \mathrm{i}\vec{t})}{Z_S(\vec{\sigma})}, \quad \vec{t} \in \mathbb{R}^d.$$

Remark 2.22. This class contains both infinitely divisible and non infinitely divisible distributions on $\mathbb{R}^d$. By applying the Euler products, some simple examples of compound Poisson case on $\mathbb{R}^2$ and generalized cases on $\mathbb{R}^d$ are given in Aoyama and Nakamura [3] and Aoyama and Nakamura [4], respectively.

As in this section, by following the history of zeta functions and distributions, Aoyama and Nakamura [5] introduced multidimensional Shintani zeta functions and their definable distributions on $\mathbb{R}^d$. In Aoyama and Nakamura [5], some examples of known distributions which belong to the multidimensional Shintani zeta class are given. Though, it is not sufficient for understanding this new class. In this chapter, we focus on the characters of multidimensional Shintani zeta functions and show the relations with some known discrete distributions on $\mathbb{R}^d$. We also consider the infinite divisibilities of them which were also studied by applying generalized Euler products called the multidimensional polynomial Euler products introduced by Aoyama and Nakamura [3, 4]. Our purpose is to give them and show some new results. We show that multinomial and negative binomial distributions belong to the multidimensional Shintani zeta class in section 2.3 and 2.5, respectively. In section 2.4, we give some necessary and sufficient conditions for certain multivariate functions to be characteristic functions which contain generalized multinomial distributions. Some properties of characters and infinite divisibilities are also studied throughout these three sections.

2.3 Generalized multinomial distributions

Many discrete distributions including the multinomial ones can be represented in terms of multidimensional Shintani zeta functions by choosing suitable characters. In this section, we consider and study some cases of them.

First, we consider a generalization of multinomial distributions by expressing them such as multidimensional Shintani zeta functions.

Definition 2.23. Let $d, m \in \mathbb{N}$, $\vec{\sigma}$, $\vec{c_l} = (c_{lj})_{j=1}^d \in \mathbb{R}^d \backslash \{0\}$, $\phi(l) \in \mathbb{R}$ and $j(l) \in \mathbb{N}\backslash\{1\}$, where $1 \le l \le m$. For each $N \in \mathbb{Z}_{\ge 0}$, we define a character θ_N by

$$\theta_N(n_1,\ldots,n_m) = \begin{cases} N\,! \displaystyle\prod_{l=1}^m \frac{(\phi(l))^{k_l}}{k_l\,!} & \Big(n_l + 1 = (j(l))^{k_l},\ \displaystyle\sum_{l=1}^m k_l = N\Big), \\ 0 & \text{(otherwise)}, \end{cases} \tag{2.8}$$

where $(n_1,\ldots,n_m) \in \mathbb{Z}_{\ge 0}^m$, and a N-multinomial function $Z_{S,N}$ by

$$Z_{S,N}(\vec{s}) = \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{\theta_N(n_1,\ldots,n_m)}{\prod_{l=1}^m (n_l+1)^{\langle \vec{c_l},\vec{s}\rangle}}, \quad \vec{s} = \vec{\sigma} + \mathrm{i}\vec{t} \in \mathbb{C}^d,\ \vec{t} \in \mathbb{R}^d.$$

Here we regard $0\,! = 1$, $0^0 = 1$. We can see that all of these functions are of the multidimensional Shintani zeta class and, by the multinomial theorem, we have

$$Z_{S,N}(\vec{s}) = \sum_{k_1+\cdots+k_m=N} N\,! \prod_{l=1}^m \frac{(\phi(l))^{k_l}(j(l)^{-\langle \vec{c_l},\vec{s}\rangle})^{k_l}}{k_l\,!} = \left(\sum_{l=1}^m \phi(l)(j(l))^{-\langle \vec{c_l},\vec{s}\rangle}\right)^N.$$

Now we put

$$f_{\vec{\sigma},N}(\vec{t}) := \frac{Z_{S,N}(\vec{\sigma}+\mathrm{i}\vec{t})}{Z_{S,N}(\vec{\sigma})}, \quad \vec{t} \in \mathbb{R}^d,$$

and

$$q(l) := \frac{\phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma}\rangle}}{\sum_{l_0=1}^m \phi(l_0)(j(l_0))^{-\langle \vec{c}_{l_0}, \vec{\sigma}\rangle}}, \quad \vec{x}_l := (x_{lk})_{k=1}^d, \ x_{lk} := -c_{lk} \log j(l).$$

Suppose that $\phi(1), \ldots, \phi(m)$ have the same sign. Then, the character θ_N is nonnegative or nonpositive definite, so that $f_{\vec{\sigma},N}$ is the characteristic function of a multidimensional Shintani zeta distribution $\mu_{\vec{\sigma}}$ given by

$$\begin{aligned}
&\mu_{\vec{\sigma}}\Bigg(\Big\{ -\sum_{l=1}^m c_{l1}\log(n_l+1), \ldots, -\sum_{l=1}^m c_{ld}\log(n_l+1)\Big\}\Bigg) \\
&= \frac{\theta_N(n_1, \ldots, n_m)}{Z_{S,N}(\vec{\sigma})} \prod_{l=1}^m (n_l+1)^{-\langle \vec{c}_l, \vec{\sigma}\rangle} \\
&= \begin{cases} N\,! \displaystyle\prod_{l=1}^m \frac{(q(l))^{k_l}}{k_l!} & \Big(n_l + 1 = (j(l))^{k_l}, \ \displaystyle\sum_{l=1}^m k_l = N\Big), \\ 0 & \text{(otherwise)}. \end{cases}
\end{aligned}$$

We note that $\sum_{l=1}^m q(l) = 1$ and $q(l) \geq 0$ for all $1 \leq l \leq m$, therefore we can define the following subclass of the multidimensional Shintani zeta class.

Definition 2.24. Let $N \in \mathbb{N}$, $\vec{x}_1, \ldots, \vec{x}_m \in \mathbb{R}^d$ and $q(1), \ldots, q(m) \geq 0$ such that $\sum_{l=1}^m q(l) = 1$. A probability measure μ_N on $\mathbb{R}^d$ is called a generalized N-multinomial distribution, if

$$\mu_N\Bigg(\Big\{ \sum_{l=1}^m x_{l1}n_l, \ldots, \sum_{l=1}^m x_{ld}n_l\Big\}\Bigg) = N\,! \prod_{l=1}^m \frac{(q(l))^{n_l}}{n_l!}, \quad \Big(\text{when } \sum_{l=1}^m n_l = N\Big).$$

The class of multinomial distributions is a special case of this class above when $m = d$ and $\vec{x}_1, \ldots, \vec{x}_d$ are the standard basis of $\mathbb{R}^d$. We have that the characteristic function of μ_N is $f_{\vec{\sigma},N}$ and, by using $q(l)$, it can be written as follows:

$$f_{\vec{\sigma},N}(\vec{t}) = \left(\sum_{l=1}^m q(l) e^{\mathrm{i}\langle \vec{x}_l, \vec{t}\rangle}\right)^N.$$

Next, we consider a class of compound distributions including the class of generalized N-multinomial distributions.

Definition 2.25. Let $\vec{\sigma}$, $\vec{c_l} \in \mathbb{R}^d \setminus \{0\}$, $\phi(l) \in \mathbb{R}$ and $j(l) \in \mathbb{N} \setminus \{1\}$, where $1 \leq l \leq m$, and θ_N be a character given by (2.8). For each nonnegative integer valued random variable T satisfying

$$Pr(T=0) \neq 1 \ \text{ and } \ \sum_{N=0}^{\infty} Pr(T=N)\Big(\sum_{l=1}^{m} |q(l)|\Big)^N < \infty, \tag{2.9}$$

we define a character θ_T by

$$\theta_T(n_1, \ldots, n_m) = \sum_{N=0}^{\infty} Pr(T=N) \frac{\theta_N(n_1, \ldots, n_m)}{(\sum_{l=1}^{m} \phi(l)(j(l))^{-\langle \vec{c_l}, \vec{\sigma} \rangle})^N}, \tag{2.10}$$

where $(n_1, \ldots, n_m) \in \mathbb{Z}_{\geq 0}^m$, and a T-multinomial function $Z_{S,T}$ by

$$Z_{S,T}(\vec{s}) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{\theta_T(n_1, \ldots, n_m)}{\prod_{l=1}^{m} (n_l + 1)^{\langle \vec{c_l}, \vec{s} \rangle}}, \quad \vec{s} = \vec{\sigma} + \mathrm{i}\vec{t} \in \mathbb{C}^d, \ \vec{t} \in \mathbb{R}^d.$$

We also can see that all of these functions are of the multidimensional Shintani zeta class. The character θ_T is nonnegative or nonpositive definite whenever $\phi(1), \ldots, \phi(m)$ have the same sign. Therefore, T-multinomial functions can generate the following subclass of the multidimensional Shintani zeta class when θ_T is so.

Definition 2.26. Let $\vec{\sigma} \in \mathbb{R}^d$ and nonnegative integer valued random variable T satisfying (2.9). A probability measure $\mu_{\vec{\sigma},T}$ on $\mathbb{R}^d$ is called a T-multinomial distribution if, for all $(n_1, \ldots, n_m) \in \mathbb{Z}_{\geq 0}^m$,

$$\mu_{\vec{\sigma},T}\Bigg(\Big\{ -\sum_{l=1}^{m} c_{l1} \log(n_l+1), \ldots, -\sum_{l=1}^{m} c_{ld} \log(n_l+1) \Big\}\Bigg) = \frac{\theta_T(n_1, \ldots, n_m)}{Z_{S,T}(\vec{\sigma}) \prod_{l=1}^{m} (n_l+1)^{\langle \vec{c_l}, \vec{\sigma} \rangle}}.$$

The characteristic function $f_{\vec{\sigma},T}$ of $\mu_{\vec{\sigma},T}$ is given by

$$f_{\vec{\sigma},T}(\vec{t}) = \frac{Z_{S,T}(\vec{\sigma} + \mathrm{i}\vec{t})}{Z_{S,T}(\vec{\sigma})}, \quad \vec{t} \in \mathbb{R}^d.$$

The series $Z_{S,T}(\vec{\sigma} + \mathrm{i}\vec{t})$ converges absolutely by the condition (2.9), and so that we have

$$\begin{aligned} Z_{S,T}(\vec{\sigma} + \mathrm{i}\vec{t}) &= \sum_{N=0}^{\infty} \frac{Pr(T=N)}{(\sum_{l=1}^{m} \phi(l)(j(l))^{-\langle \vec{c_l}, \vec{\sigma} \rangle})^N} \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{\theta_N(n_1, \ldots, n_m)}{\prod_{l=1}^{m} (n_l+1)^{\langle \vec{c_l}, \vec{\sigma} + \mathrm{i}\vec{t} \rangle}} \\ &= \sum_{N=0}^{\infty} \frac{Pr(T=N)}{(\sum_{l=1}^{m} \phi(l)(j(l))^{-\langle \vec{c_l}, \vec{\sigma} \rangle})^N} \Big(\sum_{l=1}^{m} \phi(l)(j(l))^{-\langle \vec{c_l}, \vec{\sigma} + \mathrm{i}\vec{t} \rangle} \Big)^N \end{aligned}$$

$$= \sum_{N=0}^{\infty} Pr(T = N)\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l, \vec{t}\rangle}\Big)^N, \quad \vec{t} \in \mathbb{R}^d,$$

and $Z_{S,T}(\vec{\sigma}) = \sum_{N=0}^{\infty} Pr(T = N) = 1$. Thus, we obtain

$$f_{\vec{\sigma},T}(\vec{t}) = \sum_{N=0}^{\infty} Pr(T = N)\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l, \vec{t}\rangle}\Big)^N, \quad \vec{t} \in \mathbb{R}^d,$$

which is a characteristic function of some compound distribution.

Next, we show that some important distributions belong to this class.

Proposition 2.27. *The class of T-multinomial distributions includes the following distributions.*

(i) *N-multinomial distributions.*

(ii) *A compound Poisson distribution with a finite Lévy measure $N_{\vec{\sigma}}$ on $\mathbb{R}^d$ given by*

$$N_{\vec{\sigma}}(dx) = \lambda \sum_{l=1}^{m} q(l)\, \delta_{\vec{x}_l}(dx),$$

where $\lambda > 0$ and $q(1), \ldots, q(m) > 0$.

(iii) *Let $K \in \mathbb{N}$. A compound Poisson distribution with a finite Lévy measure $N_{\vec{\sigma},G(\gamma,K)}$ on $\mathbb{R}^d$ given by*

$$N_{\vec{\sigma},G(\gamma,K)}(dx) = \sum_{r=1}^{\infty} \frac{(rK)!\,\gamma^r}{r} \sum_{n_1+\cdots+n_m=rK} \frac{(q(l))^{n_l}}{n_l\,!}\, \delta_{\sum_{l=1}^{m} n_l \vec{x}_l}(dx),$$

where $0 < \gamma < 1$ and $q(1), \ldots, q(m) > 0$ with $\sum_{l=1}^{m} q(l) = 1$. In particular, when $l = 1$, $f_{\vec{\sigma},G(\gamma,1)}$ is the characteristic function of a geometric distribution with a parameter $1 - \gamma$ and a vector $\vec{x}_1$.

In the following, we give the proofs and note that the logarithm is taken as the distinguished one whole through this thesis.

Proof. (i) Let δ_N be a delta measure. If $T = \delta_N$, then we have $\theta_T(n_1, \ldots, n_m) = \theta_N(n_1, \ldots, n_m)$ for all $(n_1, \ldots, n_m) \in \mathbb{Z}_{\geq 0}^m$. Hence, we have $f_{\vec{\sigma},T} = f_{\vec{\sigma},N}$.

(ii) Let $\lambda > 0$ and $Po(\lambda)$ be a Poisson random variable with mean λ. If $T = Po(\lambda)$, then we have

$$f_{\vec{\sigma},Po(\lambda)}(\vec{t}) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N\,!} e^{-\lambda}\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l, \vec{t}\rangle}\Big)^N = \exp\Big\{\lambda\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l, \vec{t}\rangle} - 1\Big)\Big\}, \quad \vec{t} \in \mathbb{R}^d.$$

This is the characteristic function of a compound Poisson distribution with a finite Lévy measure $N_{\vec{\sigma}}$.

(iii) Let $0 < \gamma < 1$, $K \in \mathbb{N}$ and $G(\gamma, K)$ be a nonnegative integer valued random variable whose distribution is given by

$$Pr(G(\gamma, K) = KN) = (1-\gamma)\gamma^N, \quad N \in \mathbb{Z}_{\geq 0}.$$

If $T = G(\gamma, K)$, then we have, for each $\vec{t} \in \mathbb{R}^d$,

$$f_{\vec{\sigma},G(\gamma,K)}(\vec{t}) = \sum_{N=0}^{\infty}(1-\gamma)\gamma^N \Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l,\vec{t}\rangle}\Big)^{KN} = \frac{1-\gamma}{1-\gamma\,(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l,\vec{t}\rangle})^K}.$$

This is the characteristic function of a compound geometric and also a compound Poisson distribution. Its Lévy measure is given by $N_{\vec{\sigma},G(\gamma,K)}$, since we have

$$\begin{aligned}
f_{\vec{\sigma},G(\gamma,K)}(\vec{t}) &= \frac{1-\gamma}{1-\gamma\,(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l,\vec{t}\rangle})^K} \\
&= \exp\Big\{-\log\Big(1-\gamma\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l,\vec{t}\rangle}\Big)^K\Big) - \log(1-\gamma)^{-1}\Big\} \\
&= \exp\Big\{\sum_{r=1}^{\infty}\frac{\gamma^r}{r}\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l,\vec{t}\rangle}\Big)^{rK} - \log(1-\gamma)^{-1}\Big\} \\
&= \exp\Big\{\sum_{r=1}^{\infty}\frac{\gamma^r}{r}\sum_{n_1+\cdots+n_m=rK}(rK)\,!\,\frac{(q(l))^{n_l}}{n_l\,!}\,e^{\mathrm{i}\langle\sum_{l=1}^{m} n_l\vec{x}_l,\vec{t}\rangle} - \log(1-\gamma)^{-1}\Big\} \\
&= \exp\Big\{\int_{\mathbb{R}^d}(e^{\mathrm{i}\langle x,\vec{t}\rangle}-1)\,N_{\vec{\sigma},G(\gamma,K)}(dx)\Big\}, \quad \vec{t} \in \mathbb{R}^d.
\end{aligned}$$

In the equation above, we note that $N_{\vec{\sigma},G(\gamma,K)}(\mathbb{R}^d) = \log(1-\gamma)^{-1}$. □

2.4 Conditions to be characteristic functions

In the previous section, we treated cases with nonnegative or nonpositive definite characters. Therefore, we now consider the case when the characters are not so.

We again use the notations in section 2.3. Under some additional conditions, the following lemma gives a criteria for $f_{\vec{\sigma},T}$ to be a characteristic function.

Lemma 2.28. *Let $j(1),\ldots,j(m)$ be relatively prime. Suppose that $\mathbb{R}^d$-valued vectors $\vec{c}_1,\ldots,\vec{c}_m$ are linearly independent over $\mathbb{R}$ or $\vec{c}_1 = \cdots = \vec{c}_m$ $(\neq 0)$. If there exists $1 \leq l_0 \leq m$ such that $q(l_0) < 0$, then there exists $\vec{t}_0 \in \mathbb{R}^d$ such that* $\mathrm{Re}\, f_{\vec{\sigma},T}(\vec{t}_0) > 1$.

If this lemma holds, then $f_{\vec{\sigma},T}$ is not to be a characteristic function. By applying Lemma 2.28, we have the following result.

Theorem 2.29 (The necessary and sufficient condition for $f_{\vec{\sigma},T}$ to be a characteristic function)**.** *Let $j(1),\ldots,j(m)$ be relatively prime. Suppose that $\mathbb{R}^d$-valued vectors $\vec{c}_1,\ldots,\vec{c}_m$ are linearly independent over $\mathbb{R}$ or $\vec{c}_1=\cdots=\vec{c}_m\ (\neq 0)$. Then, $f_{\vec{\sigma},T}$ is a characteristic function if and only if $\phi(1),\ldots,\phi(m)$ have the same sign.*

For the proof of Lemma 2.28, we use the linear independence of real numbers and the Kronecker's approximation theorem in the same way as in Aoyama and Nakamura [3, 4]. The Baker's theorem given below is useful to discriminate the linear independence of real numbers.

Proposition 2.30 (Baker [11])**.** *The numbers $\gamma_1^{\beta_1}\cdots\gamma_1^{\beta_1}$ are transcendental for any algebraic numbers $\gamma_1,\ldots,\gamma_n$, other than 0 or 1, and any algebraic numbers $\beta_1,\ldots,\beta_n$ with 1, $\beta_1,\ldots,\beta_n$ are linearly independent over the rationals.*

By using this fact, we have the following.

Proposition 2.31 (Nakamura [22])**.** *Let $j(1),\ldots,j(n)$ be relatively prime and ω_1, ω_2, $\ldots,\omega_m$ with $\omega_1=1$ be algebraic real numbers which are linearly independent over the rationals. Then $\{\omega_l\log j(l)\mid 1\leq l\leq m\}$ is also linearly independent over the rationals.*

The (first form of) Kronecker's approximation theorem given below is a key of the proofs in this section.

Proposition 2.32 (See, e.g. Apostol [10])**.** *If $r_1,\ldots,r_n$ are arbitrary real numbers, if real numbers $\theta_1,\ldots,\theta_n$ are linearly independent over the rationals, and if $\epsilon>0$ is arbitrary, then there exist a real number t and integers $h_1,\ldots,h_n$ such that*

$$|t\theta_k-h_k-r_k|<\epsilon,\quad 1\leq k\leq n.$$

Proof of Lemma 2.28. Put

$$L:=\sum_{l\neq l_0}q(l)-q(l_0)>\sum_{l=1}^{m}q(l)=1,$$

and take $n_0\in\mathbb{N}$ and $0<\epsilon<L$ such that

$$Pr(T=n_0)>0,\quad ((L-\epsilon)(1-\epsilon)-1)Pr(T=n_0)-2\epsilon>0. \tag{2.11}$$

Then, we note that

$$L - \epsilon > 1, \quad (L-\epsilon)(1-\epsilon) > 1.$$

By the absolute convergence of the series (2.9), there exists a natural number $N > n_0$ such that

$$\left| \sum_{n=0}^{N} Pr(T=n) - 1 \right| < \epsilon, \tag{2.12}$$

$$\sup_{\vec{t}\in\mathbb{R}^d} \left| \sum_{n=N+1}^{\infty} Pr(T=n) \Big(\sum_{l=1}^{m} q(l) e^{\mathrm{i}\langle c_l, \vec{t}\rangle \log j(l)} \Big)^n \right| \le \sum_{n=N+1}^{\infty} Pr(T=n) \Big(\sum_{l=1}^{m} |q(l)| \Big)^n < \epsilon. \tag{2.13}$$

First we consider the case when $\vec{c}_1, \ldots, \vec{c}_m$ are linearly independent over $\mathbb{R}^d$. Let $\omega_1, \omega_2, \ldots, \omega_m$ with $\omega_1 = 1$ be algebraic real numbers which are linearly independent over the rationals. Then, there exits $\vec{t}_1 \in \mathbb{R}^d$ such that $(\langle \vec{c}_1, \vec{t}_1\rangle, \ldots, \langle \vec{c}_m, \vec{t}_1\rangle) = (\omega_1, \omega_2, \ldots, \omega_m)$, since $\vec{c}_1, \ldots, \vec{c}_m$ are linearly independent over $\mathbb{R}$. In this case, we put

$$\theta_l = \frac{\omega_l \log j(l)}{2\pi} \quad (1 \le l \le m), \quad \cos N\alpha = 1 - \epsilon \quad \Big(-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \Big).$$

Next, we consider the case when $\vec{c}_1 = \cdots = \vec{c}_m$ $(\neq 0)$. In this case, we put

$$\theta_l = \frac{\log j(l)}{2\pi} \quad (1 \le l \le m), \quad \cos N\alpha = 1 - \epsilon \quad \Big(-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \Big).$$

In both cases, we have that $\theta_1, \ldots, \theta_m$ are linearly independent over the rationals. Therefore, by Proposition 2.32, there exists $T_0 \in \mathbb{R}^d$ such that

$$|e^{\mathrm{i}2\pi T_0 \theta_{l_0}} + 1| < \min\{\epsilon, |\sin\alpha|\} \left(\sum_{l=1}^{m} |q(l)| \right)^{-1},$$
$$|e^{\mathrm{i}2\pi T_0 \theta_{l}} - 1| < \min\{\epsilon, |\sin\alpha|\} \left(\sum_{l=1}^{m} |q(l)| \right)^{-1} \quad (l \neq l_0).$$

Now we have

$$\begin{aligned} \left| \sum_{l=1}^{m} q(l) e^{\mathrm{i}2\pi T_0 \theta_l} - L \right| &\le \sum_{l \neq l_0} |q(l)||e^{\mathrm{i}2\pi T_0 \theta_l} - 1| + |q(l_0)||e^{\mathrm{i}2\pi T_0 \theta_{l_0}} + 1| \\ &< \min\{\epsilon, |\sin\alpha|\}. \end{aligned}$$

This implies that

$$\left| \mathrm{Re} \Big(\sum_{l=1}^{m} q(l) e^{\mathrm{i}2\pi T_0 \theta_l} - L \Big) \right| \le \left| \sum_{l=1}^{m} q(l) e^{\mathrm{i}2\pi T_0 \theta_l} - L \right| < \epsilon,$$

$$\left|\mathrm{Im}\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l}\Big)\right| \le \left|\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l} - L\right| < |\sin\alpha|,$$

so that, for each $1 \le n \le N$, we have

$$\begin{aligned}
\mathrm{Re}\bigg(\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l}\Big)^n\bigg) &\ge \bigg(\mathrm{Re}\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l}\Big)\bigg)^n \cos n\alpha \qquad (2.14)\\
&> (L-\epsilon)^n \cos N\alpha\\
&> (L-\epsilon)(1-\epsilon).
\end{aligned}$$

By taking $\vec{t_0} \in \mathbb{R}^d$ such that $\vec{t_0} = T_0\,\vec{t_1}$ when $\vec{c}_1, \dots, \vec{c}_m$ are linearly independent over $\mathbb{R}$, or $T_0 = \langle \vec{c}_1, \vec{t_0}\rangle$ when $\vec{c}_1 = \cdots = \vec{c}_m$, and by the inequalities (2.11), (2.12), (2.13) and (2.14), we obtain

$$\begin{aligned}
\mathrm{Re}\, f_{\vec{\sigma},T}(\vec{t_0}) &= \mathrm{Re}\bigg(\sum_{n=0}^{\infty} Pr(T=n)\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l}\Big)^n\bigg)\\
&> \mathrm{Re}\bigg(\sum_{n=0}^{N} Pr(T=n)\Big(\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l}\Big)^n\bigg) - \epsilon\\
&> (L-\epsilon)(1-\epsilon)\sum_{n=1}^{N} Pr(T=n) + Pr(T=0) - \epsilon\\
&= \sum_{n=0}^{N} Pr(T=n) + ((L-\epsilon)(1-\epsilon)-1)\sum_{n=1}^{N} Pr(T=n) - \epsilon\\
&> 1 + ((L-\epsilon)(1-\epsilon)-1)Pr(T=n_0) - 2\epsilon > 1.
\end{aligned}$$

□

We have treated only the cases with nonnegative or nonpositive definite characters. In the following, we give an example of characteristic functions when the characters are not so.

Example 2.33. We retake $\vec{\sigma}$, $\vec{c}_1, \dots, \vec{c}_m \in \mathbb{R}^d \setminus \{0\}$, $\phi(1), \dots, \phi(m) > 0$ and $j(1), \dots, j(m) \in \mathbb{N} \setminus \{1\}$ such that

$$\sum_{l=1}^{m} \phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma}\rangle} < 1.$$

Let E_n be the Euler numbers which are integers given by

$$\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} \frac{E_n}{n\,!}\, x^n.$$

For each $(n_1,\ldots,n_m)\in\mathbb{Z}_{\geq 0}^m$, we define a new character by

$$\theta_{EN}(n_1,\ldots,n_m)=\sum_{n=0}^{\infty}\left(\frac{\pi}{2}\right)^n\frac{E_n}{n\,!}\,\theta_n(n_1,\ldots,n_m),$$

where θ_n is the character given by (2.8). Then, for each $n\in\mathbb{Z}_{\geq 0}$, $\theta_{EN}(j(1)^{k_1},\ldots,j(m)^{k_m})>0$ if $\sum_{l=1}^m k_l=4n$, and $\theta_{EN}(j(1)^{k_1},\ \ldots,j(m)^{k_m})<0$ if $\sum_{l=1}^m k_l=2(2n+1)$. Therefore, this character θ_{EN} is neither nonnegative nor nonpositive definite.

However, the multidimensional Shintani zeta function $Z_{S,EN}$ corresponding to the character θ_{EN} converges absolutely and we have

$$\begin{aligned}
Z_{S,EN}(\vec{\sigma}+\mathrm{i}\vec{t}) &:= \sum_{n_1,\ldots,n_m=0}^{\infty}\frac{\theta_{EN}(n_1,\ldots,n_m)}{\prod_{l=1}^m (n_l+1)^{\langle\vec{c}_l,\vec{\sigma}+\mathrm{i}\vec{t}\rangle}}\\
&=\sum_{n=0}^{\infty}\left(\frac{\pi}{2}\right)^n\frac{E_n}{n\,!}\sum_{n_1,\ldots,n_m=0}^{\infty}\frac{\theta_n(n_1,\ldots,n_m)}{\prod_{l=1}^m (n_l+1)^{\langle\vec{c}_l,\vec{\sigma}+\mathrm{i}\vec{t}\rangle}}\\
&=\sum_{n=0}^{\infty}\left(\frac{\pi}{2}\right)^n\frac{E_n}{n\,!}\left(\sum_{l=1}^m\phi(l)(j(l))^{-\langle\vec{c}_l,\vec{\sigma}+\mathrm{i}\vec{t}\rangle}\right)^n\\
&=\left(\cosh\left(\frac{\pi}{2}\sum_{l=1}^m\phi(l)j(l)^{-\langle\vec{c}_l,\vec{\sigma}\rangle}e^{\mathrm{i}\langle\vec{x}_l,\vec{t}\rangle}\right)\right)^{-1},\quad \vec{t}\in\mathbb{R}^d.
\end{aligned}$$

We also have the following equation.

$$\frac{1}{\cosh(\pi z/2)}=\exp\left\{\int_{\mathbb{R}}(e^{zx}-1-zx)\,\nu(dx)\right\},\quad z\in\mathbb{C},|z|<1,$$

where ν is a Lévy measure on $\mathbb{R}$ given by

$$\nu(dx)=dx/x(e^x-e^{-x}).$$

Thus, we obtain

$$\begin{aligned}
&Z_{S,EN}(\vec{\sigma}+\mathrm{i}\vec{t})\\
&=\exp\left\{\int_{\mathbb{R}}\left(e^{x\sum_{l=1}^m\phi(l)j(l)^{-\langle\vec{c}_l,\vec{\sigma}\rangle}e^{\mathrm{i}\langle\vec{x}_l,\vec{t}\rangle}}-1-x\sum_{l=1}^m\phi(l)j(l)^{-\langle\vec{c}_l,\vec{\sigma}\rangle}e^{\mathrm{i}\langle\vec{x}_l,\vec{t}\rangle}\right)\nu(dx)\right\}\\
&=\exp\left\{\sum_{j=2}^{\infty}\frac{1}{j\,!}\int_{\mathbb{R}}x^j\left(\sum_{l=1}^m\phi(l)j(l)^{-\langle\vec{c}_l,\vec{\sigma}\rangle}e^{\mathrm{i}\langle\vec{x}_l,\vec{t}\rangle}\right)^j\nu(dx)\right\}\\
&=\exp\left\{\sum_{j=2}^{\infty}\sum_{k_1+\cdots+k_m=j}\left(\int_{\mathbb{R}}x^j\,\nu(dx)\right)\prod_{l=1}^m\frac{(\phi(l)j(l)^{-\langle\vec{c}_l,\vec{\sigma}\rangle})^{k_l}}{k_l\,!}e^{\mathrm{i}\sum_{l=1}^m k_l\langle\vec{x}_l,\vec{t}\rangle}\right\}.
\end{aligned}$$

Then, we can see that

$$
\begin{aligned}
f_{\vec{\sigma},E}(\vec{t}) &:= \frac{Z_{S,EN}(\vec{\sigma}+\mathrm{i}\vec{t})}{Z_{S,EN}(\vec{\sigma})} \\
&= \exp\left\{\sum_{j=2}^{\infty}\sum_{k_1+\cdots+k_m=j}\Big(\int_{\mathbb{R}} x^j\,\nu(dx)\Big)\prod_{l=1}^{m}\frac{(\phi(l)j(l)^{-\langle\vec{c}_l,\vec{\sigma}\rangle})^{k_l}}{k_l\,!}\Big(e^{\mathrm{i}\sum_{l=1}^{m}k_l\langle\vec{x}_l,\vec{t}\rangle}-1\Big)\right\}
\end{aligned}
$$

is a characteristic function of a compound Poisson distribution with a finite Lévy measure $N_{\vec{\sigma},\nu}$ given by

$$
N_{\vec{\sigma},\nu}(dx) = \sum_{j=2}^{\infty}\sum_{k_1+\cdots+k_m=j}\Big(\int_{\mathbb{R}} x^j\,\nu(dx)\Big)\prod_{l=1}^{m}\frac{(\phi(l)j(l)^{-\langle\vec{c}_l,\vec{\sigma}\rangle})^{k_l}}{k_l\,!}\,\delta_{\sum_{l=1}^{m}k_l\vec{x}_l}(dx).
$$

Therefore, some of multidimensional Shintani zeta functions may generate probability distributions even if their characters are neither nonnegative nor nonpositive.

2.5 Generalized negative multinomial distributions

In this section, we treat characteristic functions of negative multinomial distributions in the sense of multidimensional Shintani zeta functions, and consider the infinite divisibilities of them as well.

Definition 2.34. Let $d,m\in\mathbb{N}$, $\vec{\sigma},\ \vec{c}_1,\ldots,\vec{c}_m\in\mathbb{R}^d\setminus\{0\}$, $\phi(0),\ldots,\phi(m)\in\mathbb{R}$ and $j(1),\ldots,j(m)\in\mathbb{N}\setminus\{1\}$ such that

$$
\min_{1\le l\le m}\langle\vec{c}_l,\vec{\sigma}\rangle\ge 1,\quad \sum_{l=1}^{m}|\phi(l)|<\phi(0). \tag{2.15}
$$

For each $c>0$ and a character $\theta_{neg(c)}$ given by

$$
\begin{aligned}
&\theta_{neg(c)}(n_1,\ldots,n_m) \\
&= \begin{cases} \displaystyle\binom{-c}{\sum_{l=1}^{m}k_l}\phi(0)^{-c-\sum_{l=1}^{m}k_l}\,(\sum_{l=1}^{m}k_l)\,!\prod_{l=1}^{m}\frac{(-\phi(l))^{k_l}}{k_l\,!} & \Big(n_l+1=(j(l))^{k_l}\Big), \\ 0 & \text{(otherwise)}, \end{cases}
\end{aligned}
$$

where $(n_1,\ldots,n_m)\in\mathbb{Z}_{\ge0}^m$, we define a c-negative multinomial function by

$$
Z_{S,neg(c)}(\vec{s}) = \sum_{n_1,\ldots,n_m=0}^{\infty}\frac{\theta_{neg(c)}(n_1,\ldots,n_m)}{\prod_{l=1}^{m}(n_l+1)^{\langle\vec{c}_l,\vec{s}\rangle}},\quad \vec{s}=\vec{\sigma}+\mathrm{i}\vec{t}\in\mathbb{C}^d,\ \vec{t}\in\mathbb{R}^d.
$$

Here

$$\binom{m}{0} = 1, \quad \binom{m}{n} = \frac{m(m-1)\cdots(m-(n-1))}{n\,!}, \quad n \in \mathbb{N},\ m \in \mathbb{C}.$$

We can see that all of these functions are of the multidimensional Shintani zeta class. By the condition (2.15), the series $Z_{S,neg(c)}(\vec{s})$ converges absolutely and we have

$$\begin{aligned}
Z_{S,neg(c)}(\vec{s}) &= \sum_{N=0}^{\infty} \binom{-c}{N} \phi(0)^{-c-N} \sum_{k_1+\cdots+k_m=N} N\,! \prod_{l=1}^{m} \frac{(-\phi(l))^{k_l}}{k_l\,!} \cdot \prod_{l=1}^{m} j(l)^{-k_l\langle \vec{c}_l, \vec{s}\rangle} \\
&= \sum_{N=0}^{\infty} \binom{-c}{N} \phi(0)^{-c-N} \sum_{k_1+\cdots+k_m=N} N\,! \prod_{l=1}^{m} \frac{(-\phi(l) j(l)^{-\langle \vec{c}_l, \vec{s}\rangle})^{k_l}}{k_l\,!} \\
&= \sum_{N=0}^{\infty} \binom{-c}{N} \phi(0)^{-c-N} \Big(-\sum_{l=1}^{m} \phi(l) j(l)^{-\langle \vec{c}_l, \vec{s}\rangle} \Big)^{N} \\
&= \left(\phi(0) - \sum_{l=1}^{m} \phi(l) j(l)^{-\langle \vec{c}_l, \vec{s}\rangle} \right)^{-c}, \quad \vec{s} = \vec{\sigma} + \mathrm{i}\vec{t} \in \mathbb{C}^d,\ \vec{t} \in \mathbb{R}^d.
\end{aligned}$$

Note that the last equation above is obtained by the generalized binomial theorem.

For $1 \le l \le m$, we put

$$q(0)^{-1} := \frac{\phi(0)}{\phi(0) - \sum_{l_0=1}^{m} \phi(l_0) j(l_0)^{-\langle \vec{c}_{l_0}, \vec{\sigma}\rangle}}, \quad q(l) := \frac{\phi(l)\, j(l)^{-\langle \vec{c}_l, \vec{\sigma}\rangle}}{\phi(0) - \sum_{l_0=1}^{m} \phi(l_0) j(l_0)^{-\langle \vec{c}_{l_0}, \vec{\sigma}\rangle}}.$$

Then, we have

$$q(0) + q(0) \sum_{l=1}^{m} q(l) = 1 \tag{2.16}$$

and, for $\vec{t} \in \mathbb{R}^d$, we also put

$$f_{\vec{\sigma},neg(c)}(\vec{t}) := \frac{Z_{S,neg(c)}(\vec{\sigma} + \mathrm{i}\vec{t})}{Z_{S,neg(c)}(\vec{\sigma})} = q(0)^c \left(1 - q(0) \sum_{l=1}^{m} q(l) e^{-\mathrm{i}\langle \vec{c}_l, \vec{t}\rangle \log j(l)} \right)^{-c}.$$

Suppose that $\phi(l) \ge 0$ for all $1 \le l \le m$. Then the character $\theta_{neg(c)}$ is nonnegative, so that $f_{\vec{\sigma},neg(c)}$ is a characteristic function. Therefore, c-negative multinomial functions can generate the following subclass of multidimensional Shintani zeta class as in section 2.3.

Definition 2.35. Let $\vec{x}_1, \ldots, \vec{x}_m \in \mathbb{R}^d$ and $q(0), \ldots, q(m) \geq 0$ satisfying (2.16). A probability measure $\mu_{neg(c)}$ on $\mathbb{R}^d$ is called a generalized c-negative multinomial distribution, if

$$\mu_{neg(c)}\left(\left\{\sum_{l=1}^{m} x_{l1}n_l, \ldots, \sum_{l=1}^{m} x_{ld}n_l\right\}\right)$$
$$= \binom{-c}{N} q(0)^{c+N} N\,! \prod_{l=1}^{m} \frac{(-q(l))^{n(l)}}{n(l)\,!} \quad \left(\text{when } \ n_l, \ N \in \mathbb{Z}_{\geq 0} \text{ and } \sum_{l=1}^{m} n_l = N\right).$$

The class of negative binomial distributions is a special case of this class above when $m = d = 1$ and $x_1 = 1$. We have that the characteristic function of $\mu_{neg(c)}$ is $f_{\vec{\sigma},neg(c)}$. Moreover, generalized c-negative distributions are compound Poisson since we have

$$\begin{aligned} f_{\vec{\sigma},neg(c)}(\vec{t}) &= \exp\left\{-c\log\left(1 - q(0)\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l, \vec{t}\rangle}\right) + c\log q(0)\right\} \\ &= \exp\left\{c\sum_{r=1}^{\infty}\frac{1}{r}\left(q(0)\sum_{l=1}^{m} q(l)e^{\mathrm{i}\langle \vec{x}_l, \vec{t}\rangle}\right)^r - c\log q(0)^{-1}\right\} \\ &= \exp\left\{c\left(\sum_{r=1}^{\infty}\sum_{n_1,\ldots,n_m=0}^{\infty} q(r:\{n_l\})\, e^{\mathrm{i}\sum_{l=1}^{m} n_l\langle \vec{x}_l, \vec{t}\rangle} - \log q(0)^{-1}\right)\right\} \\ &= \exp\left\{\int_{\mathbb{R}^d} (e^{\mathrm{i}\langle x, \vec{t}\rangle} - 1)\, N_{neg(c)}(dx)\right\}, \quad \vec{t} \in \mathbb{R}^d, \end{aligned}$$

where $\{n_l\} = (n_1, \ldots, n_m) \in \mathbb{Z}_{\geq 0}^m$ and

$$q(r:\{n_l\}) := \begin{cases} (r-1)\,!\, q(0)^r \displaystyle\prod_{l=1}^{m} \frac{(q(l))^{n_l}}{n_l\,!} & \left(\text{when } \displaystyle\sum_{l=1}^{m} n_l = r\right), \\ 0 & (\text{otherwise}). \end{cases}$$

Therefore, its finite Lévy measure $N_{neg(c)}$ on $\mathbb{R}^d$ is given by

$$N_{neg(c)}(dx) = c\sum_{r=1}^{\infty}\sum_{n_1,\ldots,n_m=0}^{\infty} q(r:\{n_l\})\,\delta_{\sum_{l=1}^{m} n_l\vec{x}_l}(dx).$$

Next, we consider a subordination. Let $\{X_{\vec{\sigma}}(t) : t \geq 0\}$ be a Lévy process whose distribution at time 1 is a generalized 1-negative multinomial distribution $\mu_{neg(1)}$ generated by a c-negative function $Z_{S,neg(1)}$. We take $\{T(t) : t \geq 0\}$ as a subordinator independent of $\{X_{\vec{\sigma}}(t) : t \geq 0\}$ and satisfies

$$\mathbb{E}\left[q(0)^T\left(1 - q(0)\sum_{l=1}^{m}|q(l)|\right)^{-T}\right] < \infty. \tag{2.17}$$

We also put $T := T(1)$ and $\alpha := \phi_0 - \sum_{l=1}^{m} \phi(l) j(l)^{-\langle \vec{c}_l, \vec{\sigma} \rangle}$.

Definition 2.36. For any subordinator $\{T(t) : t \geq 0\}$ satisfying (2.17), we define a new character Θ_T by

$$\Theta_T(n_1, \ldots, n_m) = \mathbb{E}\Big[\theta_{neg(T)}(n_1, \ldots, n_m)/\alpha^T\Big],$$

where $(n_1, \ldots, n_m) \in \mathbb{Z}_{\geq 0}^m$, and a T-negative multinomial function $Z_{S,neg(T)}$ by

$$Z_{S,neg(T)}(\vec{s}) = \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{\Theta_T(n_1, \ldots, n_m)}{\prod_{l=1}^{m} (n_l + 1)^{\langle \vec{c}_l, \vec{s} \rangle}}, \quad \vec{s} = \vec{\sigma} + \mathrm{i}\vec{t} \in \mathbb{C}^d, \ \vec{t} \in \mathbb{R}^d.$$

Under the condition that $\phi(l) \geq 0$ for all $1 \leq l \leq m$, we put

$$f_{\vec{\sigma},neg(T)}(\vec{t}) := \frac{Z_{S,neg(T)}(\vec{\sigma} + \mathrm{i}\vec{t})}{Z_{S,neg(T)}(\vec{\sigma})}.$$

We can see that $f_{\vec{\sigma},neg(T)}$ is the characteristic function of a multidimensional Shintani zeta distribution with a character Θ_T. Therefore, we also can define the following subclass of multidimensional Shintani zeta class.

Definition 2.37. Let $\vec{\sigma} \in \mathbb{R}^d \setminus \{0\}$ and $\{T(t)\}_{t \geq 0}$ be a subordinator satisfying (2.17). A probability measure $\mu_{\vec{\sigma},neg(T)}$ on $\mathbb{R}^d$ is called a T-negative multinomial distribution if, for $(n_1, \ldots, n_m) \in \mathbb{Z}_{\geq 0}^m$,

$$\begin{aligned} &\mu_{\vec{\sigma},neg(T)}\Bigg(\Big\{ -\sum_{l=1}^{m} c_{l1} \log(n_l + 1), \ldots, -\sum_{l=1}^{m} c_{ld} \log(n_l + 1) \Big\}\Bigg) \\ =& \frac{\Theta_T(n_1, \ldots, n_m)}{Z_{S,neg(T)}(\vec{\sigma})} \prod_{l=1}^{m} (n_l + 1)^{-\langle \vec{c}_l, \vec{\sigma} \rangle}. \end{aligned}$$

Since $Z_{S,neg(T)}(\vec{\sigma}) = 1$, for $\vec{t} \in \mathbb{R}^d$, we have

$$\begin{aligned} f_{\vec{\sigma},neg(T)}(\vec{t}) &= \mathbb{E}\left[\frac{1}{\alpha^T} \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{\Theta_T(n_1, \ldots, n_m)}{\prod_{l=1}^{m} (n_l + 1)^{\langle \vec{c}_l, \vec{\sigma} + \mathrm{i}\vec{t} \rangle}}\right] \\ &= \mathbb{E}\left[q(0)^T \Big(1 - q(0) \sum_{l=1}^{m} q(l)\, e^{-\mathrm{i}\langle \vec{c}_l, \vec{t} \rangle \log j(l)}\Big)^{-T}\right] \\ &= \mathbb{E}\left[\exp\Big\{T \Big(\sum_{r=1}^{\infty} \sum_{n_1,\ldots,n_m=0}^{\infty} q(r : \{n_l\})(e^{\mathrm{i}\sum_{l=1}^{m} n_l \langle \vec{c}_l, \vec{t} \rangle \log j(l)} - 1)\Big)\Big\}\right]. \end{aligned}$$

This implies that $f_{\vec{\sigma},neg(T)}$ is the characteristic function of the distribution of $X_{\vec{\sigma}}(T(1))$ which is compound Poisson. Moreover, similar to Theorem 2.29, we also have a necessary and sufficient condition for $f_{\vec{\sigma},neg(T)}$ to be a characteristic function.

Theorem 2.38. *Let $j(1), \dots, j(m)$ be relatively prime. Suppose that $\mathbb{R}^d$-valued vectors $\vec{c}_1, \dots, \vec{c}_m$ are linearly independent over $\mathbb{R}$ or $\vec{c}_1 = \cdots = \vec{c}_m$ $(\neq 0)$. Then, for any subordinator $\{T(t) : t \geq 0\}$ satisfying* (2.17)*, $f_{\vec{\sigma},neg(T)}$ is a characteristic function if and only if $\phi(l) \geq 0$ for all $1 \leq l \leq m$. Moreover, it is compound Poisson when $\phi(l) \geq 0$ for all $1 \leq l \leq m$.*

The following is the proof of theorem 2.38 which was omitted in Aoyama and Yoshikawa [7].

Proof. Suppose that there exists $1 \leq l_0 \leq m$ such that $\phi(l_0) < 0$. Then, we have

$$q(0) + q(0)\sum_{l \neq l_0} q(l) - q(0)q(l_0) > q(0) + q(0)\sum_{l=1}^{m} q(l) = 1.$$

Since

$$\begin{aligned} q(0)\sum_{l \neq l_0} q(l) - q(0)q(l_0) &= \phi(0)^{-1}\left(\sum_{l \neq l_0} \phi(l)j(l)^{-\langle \vec{c}_l, \vec{\sigma}\rangle} - \phi(l_0)j(l_0)^{-\langle \vec{c}_{l_0}, \vec{\sigma}\rangle}\right) \\ &\leq \phi(0)^{-1}\left(\sum_{l=1}^{m} |\phi(l)|j(l)^{-\langle \vec{c}_l, \vec{\sigma}\rangle}\right) < 1, \end{aligned}$$

we obtain

$$0 < 1 - q(0)\sum_{l \neq l_0} q(l) + q(0)q(l_0) < q(0).$$

Put

$$L := 1 - q(0)\sum_{l \neq l_0} q(l) + q(0)q(l_0)$$

and take $n_0 \in \mathbb{N}$ and $0 < \epsilon < q(0)L^{-1}$ such that

$$Pr(1 \leq T \leq n_0) > 0, \quad \Big((q(0)L^{-1} - \epsilon)(1-\epsilon) - 1\Big)Pr(1 \leq T \leq n_0) - 3\epsilon > 0. \quad (2.18)$$

Then, we note that

$$q(0)L^{-1} - \epsilon > 1, \quad (q(0)L^{-1} - \epsilon)(1-\epsilon) > 1.$$

It follows from (2.17) that there exists a natural number $N > n_0$ such that

$$\Big|Pr(T \leq N) - 1\Big| < \epsilon, \quad (2.19)$$

$$\sup_{\vec{t}\in\mathbb{R}^d}\left|\int_{(N,\infty)} q(0)^u\Big(1-q(0)\sum_{l=1}^{m} q(l)e^{-\mathrm{i}\langle\vec{c}_l,\vec{t}\rangle\log j(l)}\Big)^{-u}\rho_T(du)\right| \tag{2.20}$$
$$\leq \int_{(N,\infty)} q(0)^u\Big(1-q(0)\sum_{l=1}^{m}|q(l)|\Big)^{-u}\rho_T(du) < \epsilon,$$

where ρ_T is the distribution of T. We can take $-\pi/2 < \alpha < \pi/2$ such that $\cos N\alpha = 1-\epsilon$, and $\delta > 0$ such that

$$|q(0)z^{-1} - q(0)L^{-1}| < \min\{\epsilon, |\sin\alpha|\} \quad \text{whenever } |z-L| < \delta,\ z \in \mathbb{C}.$$

Now we follow the proof of Lemma 2.28. In the case when $\vec{c}_1, \ldots, \vec{c}_m$ are linearly independent over $\mathbb{R}$, let $\omega_1, \omega_2, \ldots, \omega_m$ with $\omega_1 = 1$ be algebraic real numbers which are linearly independent over the rationals. Then, there exits $\vec{t_1} \in \mathbb{R}^d$ such that $(\langle\vec{c}_1,\vec{t_1}\rangle, \ldots, \langle\vec{c}_m,\vec{t_1}\rangle) = (\omega_1, \omega_2, \ldots, \omega_m)$, so that we put

$$\theta_l := \frac{\omega_l \log j(l)}{2\pi} \quad (1 \leq l \leq m).$$

In other case when $\vec{c}_1 = \cdots = \vec{c}_m\ (\neq 0)$, we put

$$\theta_l := \frac{\log j(l)}{2\pi} \quad (1 \leq l \leq m).$$

In both cases, we have that $\theta_1, \ldots, \theta_m$ are linearly independent over the rationals. Therefore, by Proposition 2.32, there exists $T_0 \in \mathbb{R}$ such that

$$|e^{\mathrm{i}2\pi T_0\theta_{l_0}} + 1| < \delta\left(q(0)\sum_{l=1}^{m}|q(l)|\right)^{-1},$$
$$|e^{\mathrm{i}2\pi T_0\theta_{l}} - 1| < \delta\left(q(0)\sum_{l=1}^{m}|q(l)|\right)^{-1} \quad (l \neq l_0).$$

Now we have

$$\left|1-q(0)\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l} - L\right| \leq q(0)\left(\sum_{l\neq l_0}|q(l)||e^{\mathrm{i}2\pi T_0\theta_l}-1| + |q(l_0)||e^{\mathrm{i}2\pi T_0\theta_{l_0}}+1|\right)$$
$$< \delta.$$

This implies that

$$\left|\mathrm{Re}\left(q(0)\left(1-q(0)\sum_{l=1}^{m} q(l)e^{\mathrm{i}2\pi T_0\theta_l}\right)^{-1} - q(0)L^{-1}\right)\right| < \epsilon,$$

$$\left|\operatorname{Im}\left(q(0)\left(1-q(0)\sum_{l=1}^{m}q(l)e^{\mathrm{i}2\pi T_0\theta_l}\right)^{-1}\right)\right| < |\sin\alpha|,$$

so that, for each $0 < c \leq N$, we have

$$\begin{aligned}
&\operatorname{Re}\left(q(0)^c\left(1-q(0)\sum_{l=1}^{m}q(l)e^{\mathrm{i}2\pi T_0\theta_l}\right)^{-c}\right) \\
&\geq \left(\operatorname{Re}\left(q(0)\left(1-q(0)\sum_{l=1}^{m}q(l)e^{\mathrm{i}2\pi T_0\theta_l}\right)^{-1}\right)\right)^c\cos N\alpha \\
&> (1-\epsilon)\big(q(0)L^{-1}-\epsilon\big)^c.
\end{aligned} \tag{2.21}$$

By taking $\vec{t_0} \in \mathbb{R}^d$ such that $\vec{t_0} = T_0\,\vec{t_1}$ when $\vec{c}_1, \ldots, \vec{c}_m$ are linearly independent over $\mathbb{R}$, or $T_0 = \langle \vec{c}_1, \vec{t_0}\rangle$ when $\vec{c}_1 = \cdots = \vec{c}_m$, and by the inequalities (2.18), (2.19), (2.20) and (2.21), we obtain

$$\begin{aligned}
&\operatorname{Re} f_{\vec{\sigma},T}(\vec{t_0}) \\
&= \operatorname{Re}\mathbb{E}\left[q(0)^T\left(1-q(0)\sum_{l=1}^{m}q(l)e^{-\mathrm{i}\langle\vec{c}_l,\vec{t_0}\rangle\log j(l)}\right)^{-T}\right] \\
&> \operatorname{Re}\left(\int_0^N q(0)^c\Big(1-q(0)\sum_{l=1}^{m}q(l)e^{2\pi T_0\theta_l}\Big)^{-c}\rho_T(dc)\right)-\epsilon \\
&> (1-\epsilon)Pr(T<1)+(q(0)L^{-1}-\epsilon)(1-\epsilon)Pr(1\leq T\leq N)-\epsilon \\
&= Pr(T\leq N)+\Big((q(0)L^{-1}-\epsilon)(1-\epsilon)-1\Big)Pr(1\leq T\leq N)-\epsilon\,(1+Pr(T<1)) \\
&> 1+\Big((q(0)L^{-1}-\epsilon)\,(1-\epsilon)-1\Big)Pr(1\leq T\leq n_0)-3\epsilon > 1.
\end{aligned}$$

□

Chapter 3

Infinite products in Shintani zeta class

In the previous chapter, we have treated some multiple zeta functions which are of multiple series. Similar to the fact that the Riemann zeta function has the Euler product, some of them can be written by infinite products. We have mentioned that the Hurwitz zeta distribution is compound Poisson if and only if the Hurwitz zeta function has the Euler product. Infinite divisibilities of some of distributions associated with the generalized Euler products are studied by Aoyama and Nakamura [3, 4], so that we treat several cases of multidimensional Shintani zeta distributions which can be written by those products in the present chapter.

3.1 Generalized Euler products

In this section, we give a brief introduction to generalized Euler products. First, we mention Dirichlet characters and Dirichlet L-functions.

Definition 3.1 (Dirichlet character (see, e.g. Apostol [9]))**.** Let q be a positive integer. A function χ is called a Dirichlet character mod q if it is a non-vanishing group homomorphism from the group $(\mathbb{Z}/q\mathbb{Z})^*$ of prime residue classes modulo q to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The character which is identically one is called the principal. By setting $\chi(n) = \chi(a)$ for $n \equiv a \bmod q$, $\chi(q) = 0$, we can extend the character to a completely multiplicative arithmetic function on $\mathbb{Z}$. We also note that $|\chi| = 1$, which follows from its group homomorphism.

Definition 3.2 (Dirichlet L-function (see, e.g. Apostol [9])). For $s = \sigma + \mathrm{i}t \in \mathbb{C}$, $\sigma > 1$, $t \in \mathbb{R}$, the Dirichlet L-function $L(s,\chi)$ attached to a character χ mod q is given by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

The Riemann zeta function is the case when its character is the principal. As in the definition, the Dirichlet L-functions $L(s,\chi)$ have forms of infinite products and also converge absolutely in the same region $\sigma > 1$ as the Riemann zeta function. It is well-known that we can prove the prime number theory by Dirichlet L-functions.

As a generalization of the Euler product, the following multidimensional polynomial Euler products are introduced by Aoyama and Nakamura [4].

Definition 3.3 (Multidimensional Euler Product, Aoyama and Nakamura [4]). Let $d, m \in \mathbb{N}$ and $\vec{s} \in \mathbb{C}^d$. For $-1 \leq \alpha_l(p) \leq 1$ and $\vec{c}_l \in \mathbb{R}^d$, $1 \leq l \leq m$ and $p \in \mathbb{P}$, we define a multidimensional polynomial Euler product given by

$$Z_E(\vec{s}) = \prod_{p \in \mathbb{P}} \prod_{l=1}^{m} \left(1 - \alpha_l(p) p^{-\langle \vec{c}_l, \vec{s} \rangle}\right)^{-1}. \tag{3.1}$$

The absolute convergence of Z_E is also given.

Proposition 3.4 (Aoyama and Nakamura [4]). *The product Z_E converges absolutely and has no zeros in the region* $\min_{1 \leq l \leq m} \langle \vec{c}_l, \vec{\sigma} \rangle > 1$.

For each $\vec{\sigma} \in \mathbb{R}^d$ with $\min_{1 \leq l \leq m} \langle \vec{c}_l, \vec{\sigma} \rangle > 1$, put

$$f_{\vec{\sigma},E}(\vec{t}) := \frac{Z_E(\vec{\sigma} + \mathrm{i}\vec{t})}{Z_E(\vec{\sigma})}.$$

Note that every $f_{\vec{\sigma},E}$ is not always to be a characteristic function. Aoyama and Nakamura [4] showed that there are several necessary and sufficient conditions for some $f_{\vec{\sigma},E}$ to be so.

Now we consider the following two conditions.

(LI) $\mathbb{R}^d$-valued vectors $\vec{c}_1, \ldots, \vec{c}_m$ are linearly independent over $\mathbb{R}$.

(LR) $\mathbb{R}^d$-valued vectors $\vec{c}_1, \ldots, \vec{c}_m$ are linearly dependent but linearly independent over the rationals : $\vec{c}_l = \psi_l \vec{c}$, $1 \leq l \leq m$, where ψ_l are algebraic real numbers and linearly independent over the rationals.

Then, the following is also known.

Proposition 3.5 (Aoyama and Nakamura [4]). *Suppose that $\mathbb{R}^d$-valued vectors $\vec{c}_1$, $\ldots, \vec{c}_m$ satisfy the condition (LI) or (LR) in* (3.1). *Then $f_{\vec{\sigma},E}$ is a characteristic function if and only if $\alpha_l(p) \geq 0$ for all $1 \leq l \leq m$, $p \in \mathbb{P}$. Moreover, when $\alpha_l(p) \geq 0$ for all $1 \leq l \leq m$, $p \in \mathbb{P}$, $f_{\vec{\sigma},E}$ is a compound Poisson characteristic function with its finite Lévy measure $N_{\vec{\sigma}}$ on $\mathbb{R}^d$ given by*

$$N_{\vec{\sigma}}(dx) = \sum_{p\in\mathbb{P}}\sum_{r=1}^{\infty}\sum_{l=1}^{m}\frac{1}{r}\alpha_l(p)^r p^{-r\langle\vec{c}_l,\vec{\sigma}\rangle}\delta_{-r\log p\,\vec{c}_l}(dx).$$

3.2 Products of multidimensional Shintani zeta functions

In this section, we see products of multidimensional Shintani zeta functions. One of interesting objects is a class of the products which can generate characteristic functions even if their characters are neither nonnegative nor nonpositive.

First, we consider a product of the Riemman zeta function and a Dirichlet L-function with multivariable. Let N be a positive integer and χ be a real valued non-principal Dirichlet character mod N. For each $\vec{\sigma}, \vec{\sigma}', \vec{c} \in \mathbb{R}^d$ with $\langle\vec{c},\vec{\sigma}\rangle > 1$, $\langle\vec{c}_l,\vec{\sigma}'\rangle > 1$, $\vec{u} = \vec{\sigma} + \mathrm{i}\vec{t}$, $\vec{v} = \vec{\sigma}' + \mathrm{i}\vec{t}$, $\vec{t} \in \mathbb{R}^d$, we define the following functions:

$$\zeta_{(\vec{c})}(\vec{v}) := \sum_{n=1}^{\infty}\frac{1}{n^{\langle\vec{c},\vec{v}\rangle}} = \prod_{p\in\mathbb{P}}\Big(1 - \frac{1}{p^{\langle\vec{c},\vec{v}\rangle}}\Big)^{-1}, \tag{3.2}$$

$$L_{(\vec{c})}(\vec{u},\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^{\langle\vec{c},\vec{u}\rangle}} = \prod_{p\in\mathbb{P}}\Big(1 - \frac{\chi(p)}{p^{\langle\vec{c},\vec{u}\rangle}}\Big)^{-1}, \tag{3.3}$$

$$f_{\vec{\sigma},\vec{\sigma}'}(\vec{t}) := \frac{L_{(\vec{c})}(\vec{\sigma}+\mathrm{i}\vec{t},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}'+\mathrm{i}\vec{t})}{L_{(\vec{c})}(\vec{\sigma},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}')}.$$

These functions are of multidimensional Shintani zeta class. We have that $\zeta_{(\vec{c})}$ induces a multidimensional Shintani zeta distribution, but, by Proposition 3.5, $L_{(\vec{c})}$ does not. However, the product of $\zeta_{(\vec{c})}$ and $L_{(\vec{c})}$ may generate characteristic functions under some conditions. The following example is given in Aoyama and Nakamura [4].

Example 3.6. We consider the case when $d = 1$. Define a character χ_4 mod 4 by

$$\chi_4(n) := \begin{cases} 1 & n \equiv 1 \bmod 4, \\ -1 & n \equiv 3 \bmod 4, \\ 0 & n \equiv 0, 2 \bmod 4. \end{cases}$$

For $s=\sigma+\mathrm{i}t, \sigma>1, t\in\mathbb{R}$, put

$$\zeta_{\mathbb{Q}(\mathrm{i})}(s) := L(s,\chi_4)\zeta(s), \quad f_\sigma(t) := \zeta_{\mathbb{Q}(\mathrm{i})}(\sigma+\mathrm{i}t)/\zeta_{\mathbb{Q}(\mathrm{i})}(\sigma)$$

Then, $\zeta_{\mathbb{Q}(\mathrm{i})}$ is the Dedekind zeta function of a quadratic field $\mathbb{Q}(\mathrm{i})$ of discriminant -1. The Dedekind zeta function is defined by a sum of the absolute norms of ideals of an algebraic number fields, and is also a product taken over all prime ideals. This function is also well-known as the Riemann zeta function in number theory.

In view of the probability theory, Aoyama and Nakamura [4] showed that $f_\sigma(t)$ was a compound Poisson characteristic function with a finite Lévy measure N_σ on $\mathbb{R}$ given by

$$N_\sigma(dx) = \sum_{r=1}^{\infty} \frac{2^{-r\sigma}}{r}\,\delta_{r\log 2}(dx) + \sum_{p\in\mathbb{P}\setminus\{2\}} \sum_{r=1}^{\infty} \Big(1+(-1)^{\frac{r(p-1)}{2}}\Big)\frac{p^{-r\sigma}}{r}\,\delta_{r\log p}(dx).$$

As a new result, we have the following.

Theorem 3.7. *We have that $f_{\vec{\sigma},\vec{\sigma}'}$ is a characteristic function if and only if $\langle\vec{c},\vec{\sigma}-\vec{\sigma}'\rangle \ge 0$. Moreover, when $\langle\vec{c},\vec{\sigma}-\vec{\sigma}'\rangle \ge 0$, $f_{\vec{\sigma},\vec{\sigma}'}$ is a compound Poisson characteristic function with a finite Lévy measure $N_{\vec{\sigma},\vec{\sigma}'}$ on $\mathbb{R}^d$ given by*

$$N_{\vec{\sigma},\vec{\sigma}'}(dx) = \sum_{p\in\mathbb{P}} \sum_{r=1}^{\infty} \frac{1}{r}\big(p^{-r\langle\vec{c},\vec{\sigma}'\rangle} + \chi(p)\,p^{-r\langle\vec{c},\vec{\sigma}\rangle}\big)\,\delta_{-r\log p\,\vec{c}}(dx).$$

Proof. First, we show that $f_{\vec{\sigma},\vec{\sigma}'}$ is a compound Poisson characteristic function when $\langle\vec{c},\vec{\sigma}-\vec{\sigma}'\rangle \ge 0$. Suppose that $\langle\vec{c},\vec{\sigma}-\vec{\sigma}'\rangle \ge 0$. Then, $N_{\vec{\sigma},\vec{\sigma}'}$ is a measure on $\mathbb{R}^d$ since

$$p^{-r\langle\vec{c},\vec{\sigma}'\rangle} + \chi(p)\,p^{-r\langle\vec{c},\vec{\sigma}\rangle} \ge p^{-r\langle\vec{c},\vec{\sigma}'\rangle} - p^{-r\langle\vec{c},\vec{\sigma}\rangle} = p^{-r\langle\vec{c},\vec{\sigma}'\rangle}\big(1-p^{-r\langle\vec{c},\vec{\sigma}-\vec{\sigma}'\rangle}\big) \ge 0$$

for all $r\in\mathbb{N}$. Moreover, we have

$$\begin{aligned}
N_{\vec{\sigma},\vec{\sigma}'}(\mathbb{R}^d) &= \sum_{p\in\mathbb{P}} \sum_{r=1}^{\infty} \frac{1}{r}\,\big(p^{-r\langle\vec{c},\vec{\sigma}'\rangle} + \chi(p)\,p^{-r\langle\vec{c},\vec{\sigma}\rangle}\big)\\
&\le \sum_{p\in\mathbb{P}} \sum_{r=1}^{\infty} \frac{1}{r}\,\big(p^{-r\langle\vec{c},\vec{\sigma}'\rangle} + p^{-r\langle\vec{c},\vec{\sigma}\rangle}\big) \le \sum_{n=2}^{\infty} \sum_{r=1}^{\infty} \big(n^{-r\langle\vec{c},\vec{\sigma}'\rangle} + n^{-r\langle\vec{c},\vec{\sigma}\rangle}\big)\\
&= \sum_{n=2}^{\infty} \Big(\frac{n^{\langle\vec{c},\vec{\sigma}'\rangle}}{1-n^{\langle\vec{c},\vec{\sigma}'\rangle}} + \frac{n^{\langle\vec{c},\vec{\sigma}\rangle}}{1-n^{\langle\vec{c},\vec{\sigma}\rangle}}\Big) = 2\sum_{n=2}^{\infty} \big(n^{\langle\vec{c},\vec{\sigma}'\rangle} + n^{\langle\vec{c},\vec{\sigma}\rangle}\big)\\
&= 2\big(\zeta(\langle\vec{c},\vec{\sigma}'\rangle) + \zeta(\langle\vec{c},\vec{\sigma}\rangle) - 2\big) < \infty,
\end{aligned}$$

so that $N_{\vec{\sigma},\vec{\sigma}'}$ is a finite measure on $\mathbb{R}^d$.

If $\langle \vec{c}, \vec{\sigma} \rangle > 1$ and $\langle \vec{c}, \vec{\sigma} \rangle > 1$, both $\zeta_{(\vec{c})}(\vec{v})$ and $L_{(\vec{c})}(\vec{u}, \chi)$ converge absolutely and they can be written by Euler products as in (3.2) and (3.3). Therefore, we have

$$\begin{aligned}
\log f_{\vec{\sigma},\vec{\sigma}'}(\vec{t}) &= \log \frac{\zeta_{(\vec{c})}(\vec{\sigma}' + \mathrm{i}\vec{t})}{\zeta_{(\vec{c})}(\vec{\sigma}')} \frac{L_{(\vec{c})}(\vec{\sigma} + \mathrm{i}\vec{t}, \chi)}{L_{(\vec{c})}(\vec{\sigma}, \chi)} \\
&= \sum_{p \in \mathbb{P}} \log \frac{(1 - p^{-\langle \vec{c}, \vec{\sigma}' \rangle})(1 - \chi(p)\, p^{-\langle \vec{c}, \vec{\sigma} \rangle})}{(1 - p^{-\langle \vec{c}, \vec{\sigma}' + \mathrm{i}\vec{t} \rangle})(1 - \chi(p)\, p^{-\langle \vec{c}, \vec{\sigma} + \mathrm{i}\vec{t} \rangle})} \\
&= \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{1}{r} (p^{-r\langle \vec{c}, \vec{\sigma}' \rangle} + \chi(p)\, p^{-r\langle \vec{c}, \vec{\sigma} \rangle})(p^{-\mathrm{i}r\langle \vec{c}, \vec{t} \rangle} - 1) \\
&= \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle x, \vec{t} \rangle} - 1)\, N_{\vec{\sigma},\vec{\sigma}'}(dx). \qquad (3.4)
\end{aligned}$$

Thus $f_{\vec{\sigma},\vec{\sigma}'}$ is a compound Poisson characteristic function with a finite Lévy measure $N_{\vec{\sigma},\vec{\sigma}'}$ on $\mathbb{R}^d$.

Next, we show that there exists $\vec{t_0} \in \mathbb{R}^d$ such that $|f_{\vec{\sigma},\vec{\sigma}'}(\vec{t_0})| > 1$ when $\langle \vec{c}, \vec{\sigma} - \vec{\sigma}' \rangle < 0$. This implies that $f_{\vec{\sigma},\vec{\sigma}'}$ is not a characteristic function. Suppose that $\langle \vec{c}, \vec{\sigma} - \vec{\sigma}' \rangle < 0$. Put

$$\begin{aligned}
D(\vec{t}) &:= \log |f_{\vec{\sigma},\vec{\sigma}'}(\vec{t})| \\
&= \frac{1}{2} \big(\log f_{\vec{\sigma},\vec{\sigma}'}(\vec{t}) + \log f_{\vec{\sigma},\vec{\sigma}'}(-\vec{t}) \big) \\
&= \frac{1}{2} \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \phi(p, r)(p^{-\mathrm{i}r\langle \vec{c}, \vec{t} \rangle} + p^{\mathrm{i}r\langle \vec{c}, \vec{t} \rangle} - 2),
\end{aligned}$$

where $\phi(p, r) := (p^{-r\langle \vec{c}, \vec{\sigma}' \rangle} + \chi(p)\, p^{-r\langle \vec{c}, \vec{\sigma} \rangle})/r$, and $\mathbb{P}^- := \{p \in \mathbb{P} \mid \chi(p) = -1\}$. If $\langle \vec{c}, \vec{\sigma} - \vec{\sigma}' \rangle < 0$, then we have $\phi(p, r) < 0$ for all $p \in \mathbb{P}^-$ and $r \in \mathbb{N}$. Note that $\mathbb{P}^-$ is not an empty set since χ is a real valued non-principal Dirichlet character. We can take $0 < \epsilon < -\phi(p_0, 1)/2$ and $K \in \mathbb{N}$ such that $K > p_0$ and

$$\frac{1}{2} \sum_{l=1}^{m} \sum_{p,r > 2K} |\phi(p, r)| < \epsilon/4, \qquad (3.5)$$

where $\sum_{p,r>2K}$ is a sum taken over all $r \in \mathbb{N}$ and $p \in \mathbb{P}$ with $r > K$ or $p > K$. Now we put

$$\begin{aligned}
\mathbb{P}_{2K}^{+} &:= \{p \in \mathbb{P} \mid 2 \le p \le 2K,\ \chi(p) = 1\}, \\
\mathbb{P}_{2K}^{-} &:= \{p \in \mathbb{P} \mid 2 \le p \le 2K,\ \chi(p) = -1\},
\end{aligned}$$

$$C_1 := \frac{1}{2}\sum_{p\in\mathbb{P}_{2K}^{+}}\sum_{r=1}^{2K}\phi(p,r) > 0, \quad C_2 := \frac{1}{2}\sum_{p\in\mathbb{P}_{2K}^{-}}\sum_{r=1}^{K}\phi(p,2r) < 0,$$
$$C_3 := \frac{1}{2}\sum_{p\in\mathbb{P}_{2K}^{-}}\sum_{r=1}^{K}\phi(p,2r-1) < -\epsilon$$

and take

$$0 < \epsilon' < \frac{\epsilon}{4K(C_1 - C_3)}.$$

Then, as in the proof of Lemma 2.28, it follows from Proposition 2.30, Proposition 2.31 and Proposition 2.32 that there exists $T_0 \in \mathbb{R}$ such that

$$|p^{\mathrm{i}2T_0} + 1| < \epsilon',\ \ p \in \mathbb{P}_{2K}^{-}, \quad |q^{\mathrm{i}2T_0} - 1| < \epsilon',\ \ q \in \mathbb{P}_{2K}^{+}.$$

Respectively, by using three factorizations;

$$\begin{aligned} x^r - 1 &= (x-1)(x^{r-1} + \cdots + 1),\\ x^{2r} - 1 &= (x+1)(x-1)(x^{2r-2} + x^{2r-4} + \cdots + 1),\\ x^{2r-1} + 1 &= (x+1)(x^{2r-2} - x^{2r-3} + \cdots + 1), \end{aligned}$$

we have the following inequalities;

$$\begin{aligned} |p^{\mathrm{i}rT_0} - 1| &< r\epsilon',\ 1 \le r \le 2K,\ p \in \mathbb{P}_{2K}^{+},\\ |p^{\mathrm{i}2rT_0} - 1| &< 2r\epsilon',\ 1 \le r \le K,\ p \in \mathbb{P}_{2K}^{-},\\ |p^{\mathrm{i}(2r-1)T_0} + 1| &< (2r-1)\epsilon',\ 1 \le r \le K,\ p \in \mathbb{P}_{2K}^{-}. \end{aligned}$$

These inequalities imply that

$$\begin{aligned} -4K\epsilon' < p^{\mathrm{i}rT_0} + p^{-\mathrm{i}rT_0} - 2 &\le 0,\ 1 \le r \le 2K,\ p \in \mathbb{P}_{2K}^{+},\\ -4K\epsilon' < p^{\mathrm{i}2rT_0} + p^{-\mathrm{i}2rT_0} - 2 &\le 0,\ 1 \le r \le K,\ p \in \mathbb{P}_{2K}^{-},\\ -4 \le p^{\mathrm{i}(2r-1)T_0} + p^{-\mathrm{i}(2r-1)T_0} - 2 &< -4 + 4K\epsilon',\ 1 \le r \le K,\ p \in \mathbb{P}_{2K}^{-}. \end{aligned}$$

Thus, by (3.5) and taking $\vec{t_0} \in \mathbb{R}^d$ such that $T_0 = \langle \vec{c}, \vec{t_0}\rangle$, we have

$$\begin{aligned} D(\vec{t_0}) > -\epsilon + \frac{1}{2}\Big\{ &\sum_{p\in\mathbb{P}_{2K}^{+}}\sum_{r=1}^{2K}\phi(p,r)\big(p^{\mathrm{i}rT_0} + p^{-\mathrm{i}rT_0} - 2\big)\\ &+ \sum_{p\in\mathbb{P}_{2K}^{-}}\sum_{r=1}^{K}\phi(p,2r)\big(p^{\mathrm{i}2rT_0} + p^{-\mathrm{i}2rT_0} - 2\big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{p\in\mathbb{P}_{2K}^-}\sum_{r=1}^{K}\phi(p,2r-1)\big(p^{\mathrm{i}(2r-1)T_0}+p^{-\mathrm{i}(2r-1)T_0}-2\big)\Big\}\\
&>-\epsilon-4KC_1\epsilon'+0\cdot C_2+(4K\epsilon'-4)C_3\\
&=-\epsilon-4C_3-4K(C_1-C_3)\epsilon'\\
&>-\epsilon+4\epsilon-\epsilon=2\epsilon>0.
\end{aligned}
$$

Hence we have $|f_{\vec{\sigma},\vec{\sigma}'}(\vec{t_0})|>1$. This completes the proof. □

Next, we consider another product of multidimensional Shintani zeta functions. Let $m=1$, $r\in\mathbb{N}$, and $\lambda_{1k}=1$, $k=1,\dots,r$. We take $u_1,\dots,u_r\in\mathbb{R}$ and $\vec{c},\vec{\sigma}\in\mathbb{R}^d$ such that $\sum_{j=1}^r u_j=1$ and $\langle\vec{c},\vec{\sigma}\rangle>r$. For each function ψ from $\mathbb{N}$ to $\mathbb{R}$ satisfying $|\psi(n)|=O(n^\epsilon)$ for any $\epsilon>0$, we define a character θ_ψ by

$$\theta_\psi(n_1,\dots,n_r)=\psi(n_1+\cdots+n_r+1),\ \ (n_1,\dots,n_r)\in\mathbb{Z}_{\geq 0}^r,$$

and for $\vec{s}=\vec{\sigma}+\mathrm{i}\vec{t}\in\mathbb{C}$, $\vec{t}\in\mathbb{R}^d$, we also define a multidimensional Shintani zeta function $Z_L(\vec{s},\psi)$ by

$$Z_L(\vec{s},\psi)=\sum_{n_1,\dots,n_r=0}^{\infty}\frac{\theta_\psi(n_1,\dots,n_r)}{\prod_{l=1}^m(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r))^{\langle\vec{c}_l,\vec{s}\rangle}}.$$

Then, we have

$$
\begin{aligned}
Z_L(\vec{s},\psi)&=\sum_{n_1,\dots,n_r=0}^{\infty}\frac{\theta_\psi(n_1,\dots,n_r)}{\prod_{l=1}^m(\lambda_{l1}(n_1+u_1)+\cdots+\lambda_{lr}(n_r+u_r))^{\langle\vec{c}_l,\vec{s}\rangle}}\\
&=\sum_{n_1,\dots,n_r=0}^{\infty}\frac{\psi(n_1+\cdots+n_r+1)}{(n_1+\cdots+n_r+1)^{\langle\vec{c},\vec{\sigma}\rangle}}\\
&=\sum_{n=0}^{\infty}A(r,n)\frac{\psi(n+1)}{(n+1)^{\langle\vec{c},\vec{\sigma}\rangle}},
\end{aligned}
$$

where $A(r,n):=\sharp\{(n_1,\dots,n_r)\in\mathbb{Z}_{\geq 0}^r\mid n_1+\cdots+n_r=N\}$. Here we denote by $\sharp S$ the number of elements in a set S.

We have the following.

Lemma 3.8. *Fix $r\in\mathbb{N}$ and $N\in\mathbb{Z}_{\geq 0}$. Then, we have*

(i) $A(r+1,N)=\sum_{l=0}^{N}A(r,N-l)$,

(ii) $A(r+1, N+1) = A(r, N+1) + A(r+1, N)$,

(iii) $(1-x)^{-r} = \sum_{n=0}^{\infty} A(r,n)\, x^n$ *for* $|x| < 1$,

(iv) $A(l, k) = A(k+1, l-1)$ *if* $k \geq l \geq 1$,

(v) *there exist* $P(r+1, 1), \ldots, P(r+1, r) > 0$ *such that, for all* $n \in \mathbb{Z}_{\geq 0}$

$$P(r+1,1) + \cdots + P(r+1,r) = 1, \quad A(r+1,n) = \sum_{j=1}^{r} P(r+1,j)\,(n+1)^j. \tag{3.6}$$

Proof. We have (i), since

$$\begin{aligned}
A(r+1, N) &= \sharp\{(n_1, \ldots, n_r, n_{r+1}) \in \mathbb{Z}_{\geq 0}^r \mid n_1 + \cdots + n_r + n_{r+1} = N\} \\
&= \sharp\left\{\bigcup_{l=0}^{N} \{(n_1, \ldots, n_r, l) \in \mathbb{Z}_{\geq 0}^r \mid n_1 + \cdots + n_r + l = N\}\right\} \\
&= \sum_{l=0}^{N} \sharp\{(n_1, \ldots, n_r, l) \in \mathbb{Z}_{\geq 0}^r \mid n_1 + \cdots + n_r + l = N\} \\
&= \sum_{l=0}^{N} A(r, N-l).
\end{aligned}$$

We also have (ii), since

$$\begin{aligned}
A(r+1, N+1) &= \sum_{l=0}^{N+1} A(r, N+1-l) \\
&= A(r, N+1) + \sum_{l=0}^{N} A(r, N-l) \\
&= A(r, N+1) + A(r+1, N)
\end{aligned}$$

by (i).

Now we show (iii). Since the function $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ converges absolutely in the region $|x| < 1$, we have

$$(1-x)^{-r} = \Big(\sum_{n=0}^{\infty} x^n\Big)^r = \sum_{n_1, \ldots, n_r = 0}^{\infty} x^{n_1 + \cdots + n_r} = \sum_{n=0}^{\infty} A(r,n)\, x^n.$$

By differentiating both side of the equation of (iii) j times, we have

$$r(r+1)\cdots(r+k-1)(1-x)^{-r-j} = \sum_{n=j}^{\infty} n(n-1)\cdots(n-j+1)A(r,n)x^{n-j}.$$

Since $(1-x)^{-r-j} = \sum_{n=0}^{\infty} A(r+j,n)\,x^n$, it follows from comparing the coefficients of the variable x in the above equation that

$$\frac{(r+j-1)\,!}{(r-1)\,!}A(r+j,n) = \frac{(n+j)\,!}{n\,!}A(r,n+j), \quad n \in \mathbb{Z}_{\geq 0}. \tag{3.7}$$

Take $n = l-1$, $j = k-l+1$, $r = l$, then the equation of (iv) holds.

Finally, we show (v). Obviously, we have $A(2,n) = n+1$ for $n \in \mathbb{Z}_{\geq 0}$. So (3.6) holds when $P(2,1) = 1$. Assume that $r \geq 2$ and there exist $P(r,1), \ldots, P(r,r-1) > 0$ such that

$$P(r,1) + \cdots + P(r,r-1) = 1, \quad A(r,n) = \sum_{j=1}^{r-1} P(r,j)\,(n+1)^j, \ n \in \mathbb{Z}_{\geq 0}.$$

By (3.7) and the assumption, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} A(r+1,n)\,x^n &= \frac{1}{r}\sum_{n=0}^{\infty}(n+1)A(r,n+1)\,x^n \\
&= \frac{1}{r}\sum_{n=0}^{\infty}\sum_{j=1}^{r-1} P(r,j)(n+2)^j(n+1)\,x^n \\
&= \frac{1}{r}\sum_{n=0}^{\infty}\sum_{j=1}^{r-1} P(r,j)\sum_{l=0}^{j}\binom{j}{l}\,(n+1)^{l+1}\,x^n.
\end{aligned}$$

By comparing the coefficients of the variable x in the above equation for each $n \in \mathbb{Z}_{\geq 0}$, we obtain

$$\begin{aligned}
A(r+1,n) &= \frac{1}{r}\sum_{j=1}^{r-1} P(r,j)\sum_{l=1}^{j+1}\binom{j}{l-1}\,(n+1)^l \\
&= \frac{1}{r}\sum_{j=1}^{r}(n+1)^j\sum_{l=j-1}^{r-1}\binom{l}{j-1}\,P(r,l).
\end{aligned}$$

Put

$$P(r+1,j) := \frac{1}{r}\sum_{l=j-1}^{r-1}\binom{l}{j-1}\,P(r,l), \quad 1 \leq j \leq r.$$

Then, we have

$$A(r+1,n) = \sum_{j=1}^{r} P(r+1,j)\,(n+1)^j, \quad n \in \mathbb{Z}_{\geq 0},$$

where

$$P(r+1,j) > 0, \ 1 \leq j \leq r, \quad P(r+1,1) + \cdots + P(r+1,r) = 1,$$

since $P(r,j) > 0$, $1 \leq j \leq r-1$, and $A(r+1,0) = 1$. Inductively, we have (v). $\square$

Now we focus on Lemma 3.8 (v), which causes a certain compound distribution when a product of a multidimensional Shintani zeta function and $Z_L(\vec{s},\psi)$ can be a characteristic function.

Example 3.9. Let $r \geq 2$. Put $\psi = \chi$, where χ be a real valued Dirichlet character. We consider a normalized multidimensional Shintani zeta function $\widetilde{f}_{\vec{\sigma},\vec{\sigma}'}$ given by

$$\widetilde{f}_{\vec{\sigma},\vec{\sigma}'}(\vec{t}) := \frac{Z_L(\vec{\sigma}+\mathrm{i}\vec{t},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}'+\mathrm{i}\vec{t})}{Z_L(\vec{\sigma},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}')}, \quad \vec{t} \in \mathbb{R}^d.$$

For $\vec{s} = \vec{\sigma} + \mathrm{i}\vec{t} \in \mathbb{C}$, $\vec{t} \in \mathbb{R}^d$, and $\vec{\sigma}_j \in \mathbb{R}^d$ with $\langle \vec{c}, \vec{\sigma}_j \rangle = \langle \vec{c}, \vec{\sigma} \rangle - j$, we have

$$Z_L(\vec{s},\chi) = \sum_{n=0}^{\infty} A(r,n) \frac{\chi(n+1)}{(n+1)^{\langle \vec{c},\vec{\sigma} \rangle}} = \sum_{j=1}^{r-1} P(r,j)\, L_{(\vec{c})}(\vec{\sigma}_j + \mathrm{i}\vec{t},\chi)$$

by Lemma 3.8 (v). Thus we obtain

$$\begin{aligned}
\widetilde{f}_{\vec{\sigma},\vec{\sigma}'}(\vec{t}) &= \frac{Z_L(\vec{\sigma}+\mathrm{i}\vec{t},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}'+\mathrm{i}\vec{t})}{Z_L(\vec{\sigma},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}')} \\
&= \sum_{j=1}^{r-1} \frac{P(r,j)}{Z_L(\vec{\sigma},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}')} L(\vec{\sigma}_j+\mathrm{i}\vec{t},\chi)\,\zeta_{(\vec{c})}(\vec{\sigma}'+\mathrm{i}\vec{t}) \\
&= \sum_{j=1}^{r-1} \widetilde{P}(r,j)\, f_{\vec{\sigma}_j,\vec{\sigma}'}(\vec{t}),
\end{aligned}$$

where $\widetilde{P}(r,j) := P(r,j)\,L(\vec{\sigma}_j,\chi)/Z_L(\vec{\sigma},\chi) \geq 0$. Note that $\sum_{j=1}^{r-1} \widetilde{P}(r,j) = 1$.

By Theorem 3.7, for each $1 \leq j \leq r-1$, $f_{\vec{\sigma}_j,\vec{\sigma}'}$ is a characteristic function if and only if $\langle \vec{c}, \vec{\sigma}_j - \vec{\sigma}' \rangle \geq 0$. If $\langle \vec{c}, \vec{\sigma} - \vec{\sigma}' \rangle \geq r-1$, then $f_{\vec{\sigma}_j,\vec{\sigma}'}$ is a characteristic function for all $1 \leq j \leq r-1$. Therefore, we can see that $\widetilde{f}_{\vec{\sigma},\vec{\sigma}'}$ is a characteristic function of a compound distribution.

Chapter 4

An approximation scheme for diffusion processes based on an anti-symmetric calculus on Wiener space

In this chapter, we show that every anti-symmetric multiple stochastic (Ito's) integral has a polynomial form of single and double ones. As an application, a new approximating scheme for the solution to a stochastic differential equation is proposed.

4.1 Backgrounds

Let X be a diffusion process in $\mathbb{R}^d$, $d \geq 1$, given by a solution to a stochastic differential equation, which is written in the Stratonovich form as

$$X_t = x + \sum_{j=0}^{d} \int_0^t V_j(X_s) \circ dW_s^j, \tag{4.1}$$

where $x \in \mathbb{R}^d$, $W = (W^1, \dots, W^d)$ is a d-dimensional Wiener process, dt is denoted by dW_t^0 by a convention, and V_j, $j = 0, 1, \dots, d$ are in $C_b^\infty(\mathbb{R}^d \to \mathbb{R}^d)$, smooth functions whose derivatives of any order are bounded.

For the purpose of approximating the law of X, we use the so-called stochastic Taylor expansion (or Itô-Taylor expansion);

$$\begin{aligned} &X_t - x \\ &\sim \sum_{n=1}^{N} \sum_{j_1,\dots,j_n=0}^{d} V_{j_1} V_{j_2} \cdots V_{j_n}(x) \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \circ dW_{s_n}^{j_n} \circ dW_{s_{n-1}}^{j_{n-1}} \cdots \circ dW_{s_1}^{j_1} \end{aligned} \tag{4.2}$$

which is often a key ingredient (see e.g. Kloeden and Platen [17]). Here, as is common in the literature, we identify the functions $V_{j_i}, \dots, V_{j_{n-1}}$ with the vector field $\sum_i V^i_{j_1}\partial_i$, and so on ($V_{j_n}$ is still a function). The expansion of $N = 1$, with $x \to X_{u_i}$ and $X_t \to X_{u_{i+1}}$, and by the repetition in i, implies the Euler-Maruyama approximation. The expansions of $N > 1$ also imply a higher-order method, but the method becomes by far problematic since when $d \geq 2$, no explicit form of the joint law of multiple stochastic integrals are available. Many new schemes on this higher-order higher-dimensional approximation problem have been proposed in recent years driven by growing needs from financial practices. Among these, so-called Kusuoka's scheme (Kusuoka [18], see also Kusuoka and Ninomiya [19]), including Ninomiya-Victoir's [24] and Ninomiya-Ninomiya's [23], and the cuvature method by T. Lyons and N. Victoir [20] are well appreciated.

In the present chapter, we propose a new framework in which we rely on, instead of generic multiple Wiener integrals, symmetric and anti-symmetric ones. It is well-known that the former integrals are actually expressed by an Hermite polynomial of first order integrals. In this chapter we point out that

- the latter are also expressed as a polynomial of first- and the second order anti-symmetric integrals (stochastic areas),
- for which semi-explicit forms (the Fourier transform) of the joint distribution are available.

These are the contributions of the present paper to the literature.

Here we state briefly another background of our study. Akahori et al. [2] constructed an isomorphism between $L^2(\mathcal{W}^d, u^d)$ and the anti-symmetric Fock space over $L^2([0,1] \to \mathbb{R}^d)$;

$$L^2(\mathcal{W}^d, u^d) \;\backsimeq\; \bigoplus_{n=0}^{\infty} \bigwedge_n L^2([0,1] \to \mathbb{R}^d).$$

With this, all L^2-martingales can be identified with an infinite series expansion of multiple anti-symmetric stochastic integrals. In the present paper, we show that all such integrals have polynomial forms of single and double ones. This will lead to a new (anti-symmetric) calculus though this paper will not study further this subject.

This chapter is organized as follows. First, in section 4.2, a representation of a Clifford algebra on $L^2(\mathcal{W}^d, u^d)$ is constructed as in the same way as in Akahori et al.

[2]. Secondly, in section 4.3 given are the main result of this chapter and its proof. Then in section 4.4, the new scheme is introduced, with some discussions.

4.2 Construction of a Clifford algebra on $L^2(\mathcal{W}^d, u^d)$

Let $\mathbf{H}$ be a real Hilbert space with an orthonormal basis $\{e_n \mid n \in \mathbb{Z} + \frac{1}{2}\}$. First, we will construct a Clifford algebra acting on the Fock space over $\mathbf{H}$.

We consider its n-th antisymmetric power $\mathbf{H}_n := \bigwedge_n \mathbf{H}$, and define the creation operator φ_h^+ indexed by a vector $h \in \mathbf{H}$: φ_h^+ maps $\mathbf{H}_n$ into $\mathbf{H}_{n+1}$ as follows: For $u_j \in \mathbf{H}$ $(j = 1, \cdots, n)$,

$$\varphi_h^+(u_1 \wedge \cdots \wedge u_n) = h \wedge u_1 \wedge \cdots \wedge u_n$$

where $\wedge$ is the exterior product. The annihilation operator $\varphi_{h'}^-$ is indexed by an element h' of the dual space $\mathbf{H}^*$ of $\mathbf{H}$, mapping $\mathbf{H_{n+1}}$ into $\mathbf{H_n}$ as follows: For $u_j \in \mathbf{H}^*$ $(j = 1, \cdots, n+1)$

$$\varphi_{h'}^-(u_1 \wedge \cdots \wedge u_{n+1}) = \sum_{i=1}^{n+1} (-1)^i h'(u_i) u_1 \wedge \cdots \widehat{u_i} \cdots \wedge u_{n+1},$$

where a hat on a vector means that it is omitted. In particular, for the vacuum vector $|\mathbf{1}\rangle (= 1 \in \mathbb{R})$, $\varphi_h^+|\mathbf{1}\rangle = h$, $\varphi_{h'}^-|\mathbf{1}\rangle = 0$.

Here, we note that the creation and annihilation operators have the linearity and the boundedness, so that they are extended to the whole space.

Next we denote by e_n' the dual element of e_n, and define

$$\varphi_n := \varphi_{e_n}^+, \qquad \varphi_n^* := \varphi_{e_n'}^-.$$

Then, some calculations lead to the relations

$$[\varphi_n, \varphi_m]_+ = [\varphi_n^*, \varphi_m^*]_+ = 0, \qquad [\varphi_n, \varphi_m^*]_+ = \delta_{n+m,0},$$

where $[A, B]_+ = AB + BA$ is the anti-commutator and $\delta_{n,m} = 1$ if $n = m$, $\delta_{n,m} = 0$ if $n \neq m$, n, $m \in \mathbb{N}$. Therefore, the creation and annihilation operators $\{\varphi_n, \varphi_m^* \mid n, m \in \mathbb{Z} + \frac{1}{2}\}$ generate a Clifford algebra $\mathcal{A}$, which acts on the Fock space $\bigoplus_{n=0}^{\infty} \bigwedge_n \mathbf{H}$ over $\mathbf{H}$.

Secondly, let $\mathbf{H} = L^2([0, 1] \to \mathbb{R}^d)$ and we will identify the Fock space over $\mathbf{H}$ with L^2 space of Wiener functionals in the following way. Let $(w_k)_{k=1}^d$ be the canonical basis of $\mathbb{R}^d$ and $\otimes$ be its tensor product. For each $n \in \mathbb{N}$ we write $\Delta^n := \{(s_1, \ldots, s_n) \in [0,1]^n \mid s_1 < s_2 < \cdots < s_n\}$, and for each $g^n \in L^2(\Delta^n \to (\mathbb{R}^d)^{\otimes n})$,

$$g^n(s_1, \ldots, s_n) = \sum_{i_1, \cdots, i_n=1}^{d} g^n_{i_1,\ldots,i_n}(s_1, \ldots, s_n)\, w_{i_1} \otimes \cdots \otimes w_{i_n}.$$

Then, we can define its multiple stochastic integral:

$$I_n(g^n) := \sum_{i_1,\ldots,i_n=1}^{d} \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} g^n_{i_1,\ldots,i_n}(s_1, \ldots, s_n) dW^{i_n}_{s_n} \cdots dW^{i_1}_{s_1},$$

where $W = (W^k)_{k=1}^d$ is the Brownian motion on $\mathcal{W}^d$.

It is well known in the standard Malliavin calculus that there exists an isomorphism

$$\bigoplus_{n=0}^{\infty} L^2(\Delta^n \to (\mathbb{R}^d)^{\otimes n}) \to L^2(\mathcal{W}^d, u^d), \quad g^n \mapsto I_n(g^n).$$

We have also another isomorphism

$$\bigoplus_{n=0}^{\infty} \bigwedge_n \mathbf{H} \to \bigoplus_{n=0}^{\infty} L^2(\Delta^n \to (\mathbb{R}^d)^{\otimes n}), \quad \widetilde{g^n} \mapsto g^n$$

by $g^n(s_1, \ldots, s_n) = n\,!\,\widetilde{g^n}(s_1, \ldots s_n)$ for $(s_1, \ldots, s_n) \in \Delta^n$. (Here we note that a tensor product $\otimes'$ in $\mathbf{H}$ is identified as follows:

$$(u_1 \otimes' \cdots \otimes' u_n)(s_1, \ldots, s_n) = \sum_{i_1,\ldots,i_n=1}^{d} u_1^{i_1}(s_1) \cdots u_n^{i_n}(s_n)\, w_{i_1} \otimes \cdots \otimes w_{i_n},$$

for $u_j = (u_j^k)_{k=1}^d \in \mathbf{H}$. Then, $u_1 \otimes' \cdots \otimes' u_n \in L^2([0,1]^n \to (\mathbb{R}^d)^{\otimes n})$, and the exterior product $\wedge$ in $\mathbf{H}$ is naturally defined through the tensor product $\otimes'$).

Hence, the composition of the two isomorphisms brings about the isomorphism

$$\bigoplus_{n=0}^{\infty} \bigwedge_n \mathbf{H} \;\backsimeq\; L^2(\mathcal{W}^d, u^d).$$

which induces an action of a Clifford algebra on $L^2(\mathcal{W}^d, u^d)$ by the one on $\bigoplus_{n=0}^{\infty} \bigwedge_n \mathbf{H}$.

In the following, we consider the elements of the form $\varphi_1 \varphi_2 \cdots \varphi_n |\mathbf{1}\rangle$ in $L^2(\mathcal{W}^d, u^d)$.

For simplicity, we will see the case of $d = 2$ where a basis of $L^2([0,1] \to \mathbb{R}^2)$ is given as follows:

$$e_n(s) = \begin{pmatrix} h_n(s) \\ 0 \end{pmatrix}, \quad e_{-n}(s) = \begin{pmatrix} 0 \\ h_n(s) \end{pmatrix}, \;\; n \in \mathbb{Z}_{\geq 0} + \frac{1}{2},$$

where $\{h_n \mid n \in \mathbb{Z} + \frac{1}{2}\}$ is a basis of $L^2([0,1] \to \mathbb{R})$.

It is easy to see that for $n, m \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$,

$$\varphi_n|\mathbf{1}\rangle = \int_0^1 h_n(s) dW_s^1, \quad \varphi_{-m}|\mathbf{1}\rangle = \int_0^1 h_m(s) dW_s^2$$

$$\varphi_n \varphi_{-m}|\mathbf{1}\rangle = \frac{1}{2}\Big(\int_0^1 h_n(t) \int_0^t h_m(s) dW_s^2 dW_t^1 - \int_0^1 h_m(t) \int_0^t h_n(s) dW_s^1 dW_t^2\Big),$$

$\cdots$ etc.

In the next section, we shall see that the elements of the form $\varphi_1 \varphi_2 \cdots \varphi_n|\mathbf{1}\rangle$ in $L^2(\mathcal{W}^d, u^d)$, are actually polynomials of $\varphi_i|\mathbf{1}\rangle, \varphi_i\varphi_j|\mathbf{1}\rangle$, $(i, j = 1, \ldots, n)$.

4.3 Anti-symmetric expansion

Let $\{(e_i^k)_{i=1}^d \mid k \geq 1\}$ be a family of functions in $L^2([0,1] \to \mathbb{R}^d)$. Define

$$f^{n_k}(s) := \sum_{i=1}^d \int_0^s e_i^{n_k}(u) dW_u^i, \ \ s \in [0,\, 1],$$

and

$$f^{n_1,n_2,\ldots,n_k}(s) := \sum_{i=1}^d \int_0^s e_i^{n_k}(u) f^{n_1,n_2,\ldots,n_{k-1}}(u) dW_u^i, \ \ s \in [0,\, 1],$$

where $W = (W^i)$ is a d-dimensional Wiener process on some probability space and $\int H dW$ denotes the Itô integral of H with respect to W. Then, the integration-by-parts formula tells us that for $p, q \geq 1$ and $s \in [0,\, 1]$, we have

$$\begin{aligned}
&f^{n_1,n_2,\ldots,n_p}(s)\, f^{m_1,m_2,\ldots,m_q}(s) \\
&= \sum_{i=1}^d \int_0^s e_i^{n_p}(u) f^{n_1,n_2,\ldots,n_{p-1}}(u) f^{m_1,m_2,\ldots,m_q}(u) dW_u^i \\
&\quad + \sum_{i=1}^d \int_0^s e_i^{m_q}(u) f^{m_1,m_2,\ldots,m_{q-1}}(u) f^{n_1,n_2,\ldots,n_p}(u) dW_u^i \\
&\quad + \sum_{i=1}^d \int_0^s e_i^{n_p}(u) e_i^{m_q}(u) f^{n_1,n_2,\ldots,n_{p-1}}(u) f^{m_1,m_2,\ldots,m_{q-1}}(u) du.
\end{aligned}$$

Note that $f^{n_1,n_2,\ldots,n_{p-1}}(u) := 1$ if $p = 1$.

Put

$$(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_W := \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) f^{n_{\sigma(1)}, n_{\sigma(2)}, \ldots, n_{\sigma(k)}}(1),$$

where $\mathfrak{S}_k$ is the k-th symmetric group. Then we have the following.

Theorem 4.1. *For $n_1, n_2, \ldots, n_k \in \mathbb{N}$, there is a polynomial F such that*

$$\begin{aligned}(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_W \\ = F\big((e_{n_1})_W, \ldots, (e_{n_k})_W, (e_{n_1} \wedge e_{n_2})_W, \\ \ldots, (e_{n_1} \wedge e_{n_k})_W, (e_{n_2} \wedge e_{n_3})_W, \ldots, (e_{n_{k-1}} \wedge e_{n_k})_W\big).\end{aligned}$$

In particular, when k is even, we have

$$(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_W = \operatorname{Pf}[((e_{n_i} \wedge e_{n_j})_W)_{1 \le i,j \le k}],$$

where $\operatorname{Pf}[A]$ is the Pfaffian of an antisymmetric matrix $A := (A_{ij})_{1 \le i,j \le 2n}$ defined by

$$\operatorname{Pf}(A) := \frac{1}{n\,!\,2^n} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1)\,\sigma(2i)}.$$

When k is odd, we have

$$(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_W = \sum_{l=1}^{k} c_l \, (e_{n_l})_W \operatorname{Pf}[((e_{n_i} \wedge e_{n_j})_W)_{1 \le i,j \le k,\, i,j \ne l}],$$

for some $c_1, \ldots, c_k \in \mathbb{R}$.

Remark 4.2. If $\{(e_i^k)_{i=1}^d \mid k \ge 1\}$ is a basis for $L^2([0,1] \to \mathbb{R}^d)$, then we have

$$\varphi_{n_k} \varphi_{n_2} \cdots \varphi_{n_1} |\mathbf{1}\rangle = \frac{1}{k!} (e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_W,$$

where $\varphi_{n_1} \varphi_{n_2} \cdots \varphi_{n_k}$ are creation operators in section 4.2.

We shall use the following lemmas to prove the theorem.

Lemma 4.3. *For each $p, q \in \mathbb{N}$,*

$$\begin{aligned}&\sum_{\sigma \in \mathfrak{S}_{p+q}} \operatorname{sgn}(\sigma) f^{\sigma(1), \ldots, \sigma(p)}(1) f^{\sigma(p+1), \ldots, \sigma(p+q)}(1) \\ &= \sum_{\lambda \in \Lambda_{p,q}} \operatorname{sgn}(\sigma^\lambda) (e_{k_1} \wedge \cdots \wedge e_{k_p})_W (e_{l_1} \wedge \cdots \wedge e_{l_p})_W,\end{aligned}$$

where the sum in the right-hand-side is taken over the set defined by $\{\{k_i\}_{i=1}^p, \{l_j\}_{j=1}^q\} \in \Lambda_{p,q}$ *if and only if* (i) $k_i, l_j \in \mathbb{N}$ *and* $k_i \neq l_j$ *for* $1 \leq i \leq p,\ 1 \leq j \leq q$, (ii) $1 \leq k_1 < k_2 < \cdots < k_p \leq p+q,\ 1 \leq l_1 < l_2 < \cdots < l_q \leq p+q$, *and for each* $\lambda := \{\{k_i\}_{i=1}^p, \{l_j\}_{j=1}^q\} \in \Lambda_{p,q}$, $\sigma^\lambda := (k_1, \ldots, k_p, l_1, \ldots, l_q)$ *is a permutation such that* $\sigma^\lambda(i) = k_i,\ \sigma^\lambda(p+j) = l_j\ \ (1 \leq i \leq p,\ 1 \leq j \leq q)$.

Proof. It suffices to rearrange the elements of the symmetric group. □

Lemma 4.4. *For each* $s \in [0,\ 1]$ *and* $k \in \mathbb{N}$,

$$\sum_{\sigma \in \mathfrak{S}_{2k}} \mathrm{sgn}(\sigma) f^{n_{\sigma(1)}}(s) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2k)}}(s) = 0.$$

Proof. It is clear that

$$\sum_{\sigma \in \mathfrak{S}_{2}} \mathrm{sgn}(\sigma) f^{n_{\sigma(1)}}(s) f^{n_{\sigma(2)}}(s) = 0.$$

For $k \geq 2$ and $l = 1, 2, \ldots, k$, we assume that

$$\sum_{\sigma \in \mathfrak{S}_{2l}} \mathrm{sgn}(\sigma) f^{n_{\sigma(1)}}(s) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2l)}}(s) = 0.$$

Then, we have

$$\begin{aligned}
&\sum_{\sigma \in \mathfrak{S}_{2k+2}} \mathrm{sgn}(\sigma) f^{n_{\sigma(1)}}(s) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2k+2)}}(s) \\
&= \sum_{\sigma \in \mathfrak{S}_{2k+2}} \mathrm{sgn}(\sigma) f^{n_{\sigma(1)}}(s) \Big\{ f^{n_{\sigma(2k+2)}}(s) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2k+1)}}(s) \\
&\qquad - \sum_{i=1}^d \int_0^s e_i^{n_{\sigma(2k+1)}}(u) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2k)}}(u) f^{n_{\sigma(2k+2)}}(u) dW_u^i \\
&\qquad - \sum_{i=1}^d \int_0^s e_i^{n_{\sigma(2k+2)}}(u) e_i^{n_{\sigma(2k+1)}}(u) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2k)}}(u) du \Big\}.
\end{aligned}$$

On the other hand, we obtain

$$\sum_{\sigma \in \mathfrak{S}_{2k+2}} \mathrm{sgn}(\sigma) f^{n_{\sigma(1)}}(s) f^{n_{\sigma(2k+2)}}(s) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2k+1)}}(s) = 0,$$

and for $i = 1, 2, \ldots, d$,

$$\sum_{\sigma \in \mathfrak{S}_{2k+2}} \mathrm{sgn}(\sigma) f^{n_{\sigma(1)}}(s) \int_0^s e_i^{n_{\sigma(2k+2)}}(u) e_i^{n_{\sigma(2k+1)}}(u) f^{n_{\sigma(2)}, \ldots, n_{\sigma(2k)}}(u) du = 0.$$

Then by the assumption, we have

$$\sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) f^{n_{\sigma(1)}}(s) \int_0^s e_i^{n_{\sigma(2k+1)}}(u) f^{n_{\sigma(2)},\dots,n_{\sigma(2k)}}(u) f^{n_{\sigma(2k+2)}}(u) dW_u^i = 0.$$

Hence, the lemma follows from induction since

$$\sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) f^{n_{\sigma(1)}}(s) f^{n_{\sigma(2)},\dots,n_{\sigma(2k+2)}}(s) = 0.$$

□

Lemma 4.5. *For $k\in\mathbb{N}$ and $p=0,1,\dots,k-1$,*

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) f^{\sigma(1),\sigma(2)}(s) f^{\sigma(3),\dots,\sigma(2k+2)}(s) \\
&= \sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \Big\{ \sum_{l=0}^{p} f^{\sigma(3),\dots,\sigma(2k-2l+2),\sigma(1),\sigma(2),\sigma(2k-2l+3),\dots,\sigma(2k+2)}(s) \\
&+ \sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_{2p+4}=1}^{d} \int_0^s e_{i_1}^{\sigma(2k+2)}(s_1) \int_0^{s_1} e_{i_2}^{\sigma(2k+1)}(s_2) \int_0^{s_2}\cdots\int_0^{s_{2p+3}} e_{i_{2p+4}}^{\sigma(2k-2p+1)}(u) \\
&\quad \cdot f^{\sigma(3),\dots,\sigma(2k-2p)}(u) f^{\sigma(1),\sigma(2)}(u) dW_u^{i_{2p+4}} dW_{s_{2p+3}}^{i_{2p+3}} \cdots dW_{s_1}^{i_1} \Big\},
\end{aligned}$$

where $f^{\sigma(3),\dots,\sigma(2k-2p)}(u) := 1$ if $p=k-1$.

Proof. Observe that

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) f^{\sigma(1),\sigma(2)}(s) f^{\sigma(3),\dots,\sigma(2k+2)}(s) \\
&= \sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \sum_{i=1}^{d} \Big\{ \int_0^s e_i^{\sigma(2)}(u) f^{\sigma(1)}(u) f^{\sigma(3),\dots,\sigma(2k+2)}(u) dW_u^i \\
&\quad + \int_0^s e_i^{\sigma(2k+2)}(u) f^{\sigma(3),\dots,\sigma(2k+1)}(u) f^{\sigma(1),\sigma(2)}(u) dW_u^i \\
&\quad + \int_0^s e_i^{\sigma(2)}(u) e_i^{\sigma(2k+2)}(u) f^{\sigma(1)}(u) f^{\sigma(3),\dots,\sigma(2k+1)}(u) du \Big\} \\
&= \sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \sum_{i=1}^{d} \Big\{ \int_0^s e_i^{\sigma(2)}(u) f^{\sigma(1)}(u) f^{\sigma(3),\dots,\sigma(2k+2)}(u) dW_u^i \\
&\quad + \int_0^s e_i^{\sigma(2k+2)}(u) f^{\sigma(3),\dots,\sigma(2k+1)}(u) f^{\sigma(1),\sigma(2)}(u) dW_u^i \Big\}
\end{aligned}$$

$$= \sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \sum_{i=1}^{d}\sum_{j=1}^{d} \Big\{ \int_0^s e_i^{\sigma(2)}(u) \int_0^u e_j^{\sigma(1)}(v) f^{\sigma(3),\ldots,\sigma(2k+2)}(v) dW_v^j dW_u^i$$
$$+ \int_0^s e_i^{\sigma(2)}(u) \int_0^u e_j^{\sigma(2k+2)}(v) f^{\sigma(3),\ldots,\sigma(2k+1)}(v) f^{\sigma(1)}(v) dW_v^j dW_u^i$$
$$+ \int_0^s e_i^{\sigma(2)}(u) \int_0^u e_j^{\sigma(1)}(v) e_j^{\sigma(2k+2)}(v) f^{\sigma(3),\ldots,\sigma(2k+1)}(v) dv dW_u^i$$
$$+ \int_0^s e_i^{\sigma(2k+2)}(u) \int_0^u e_j^{\sigma(2)}(v) f^{\sigma(3),\ldots,\sigma(2k+1)}(v) f^{\sigma(1)}(v) dW_v^j dW_u^i$$
$$+ \int_0^s e_i^{\sigma(2k+2)}(u) \int_0^u e_j^{\sigma(2k+1)}(v) f^{\sigma(3),\ldots,\sigma(2k)}(v) f^{\sigma(1),\sigma(2)}(v) dW_v^j dW_u^i$$
$$+ \int_0^s e_i^{\sigma(2k+2)}(u) \int_0^u e_j^{\sigma(2)}(v) e_j^{\sigma(2k+1)}(v) f^{\sigma(1)}(v) f^{\sigma(3),\ldots,\sigma(2k)}(v) dv dW_u^i \Big\}.$$

It is clear that

$$\sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \int_0^s e_i^{\sigma(2)}(u) \int_0^u e_j^{\sigma(1)}(v) e_j^{\sigma(2k+2)}(v) f^{\sigma(3),\ldots,\sigma(2k+1)}(v) dv dW_u^i$$
$$= \sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \int_0^s e_i^{\sigma(2k+2)}(u) \int_0^u e_j^{\sigma(2)}(v) e_j^{\sigma(2k+1)}(v) f^{\sigma(1)}(v) f^{\sigma(3),\ldots,\sigma(2k)}(v) dv dW_u^i$$
$$= 0.$$

From Lemma 4.4, we have

$$\sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \int_0^s e_i^{\sigma(2)}(u) \int_0^u e_j^{\sigma(2k+2)}(v) f^{\sigma(3),\ldots,\sigma(2k+1)}(v) f^{\sigma(1)}(v) dW_v^j dW_u^i$$
$$= \sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \int_0^s e_i^{\sigma(2k+2)}(u) \int_0^u e_j^{\sigma(2)}(v) f^{\sigma(3),\ldots,\sigma(2k+1)}(v) f^{\sigma(1)}(v) dW_v^j dW_u^i$$
$$= 0.$$

Hence we have

$$\sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) f^{\sigma(1),\sigma(2)}(s) f^{\sigma(3),\ldots,\sigma(2k+2)}(s)$$
$$= \sum_{\sigma\in\mathfrak{S}_{2k+2}} \operatorname{sgn}(\sigma) \Big\{ f^{\sigma(3),\ldots,\sigma(2k+2),\sigma(1),\sigma(2)}(s)$$
$$+ \sum_{i=1}^{d}\sum_{j=1}^{d} \int_0^s e_i^{\sigma(2k+2)}(u) \int_0^u e_j^{\sigma(2k+1)}(v) f^{\sigma(3),\ldots,\sigma(2k)}(v) f^{\sigma(1),\sigma(2)}(v) dW_v^j dW_u^i \Big\}.$$

By calculating $f^{\sigma(3),\ldots,\sigma(2k)}(v) f^{\sigma(1),\sigma(2)}(v)$ similarly and repeating the procedure to $f^{\sigma(3),\ldots,\sigma(2k-2)}(w)\, f^{\sigma(1),\sigma(2)}(w), \ldots, f^{\sigma(3),\ldots,\sigma(2k-2p+2)}(r) f^{\sigma(1),\sigma(2)}(r)$, we see that this lemma now follows. □

Proof of Theorem 4.1. Fix $k \in \mathbb{N}$. If $n = 2k+1$, then we have

$$
\begin{aligned}
&(e_1 \wedge \cdots \wedge e_n)_W \\
&= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) f^{\sigma(1),\ldots,\sigma(n)}(1) \\
&= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \Big\{ f^{\sigma(n)}(1) f^{\sigma(1),\ldots,\sigma(n-1)}(1) \\
&\qquad - \sum_{i=1}^{d} \int_0^1 e_i^{\sigma(n-1)}(u) f^{\sigma(1),\ldots,\sigma(n-2)}(u) f^{\sigma(n)}(u) dW_u^i \\
&\qquad - \sum_{i=1}^{d} \int_0^1 e_i^{\sigma(n-1)}(u) e_i^{\sigma(n)}(u) f^{\sigma(1),\ldots,\sigma(n-2)}(u) du \Big\} \\
&= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \Big\{ f^{\sigma(n)}(1) f^{\sigma(1),\ldots,\sigma(n-1)}(1) \\
&\qquad - \sum_{i=1}^{d} \int_0^1 e_i^{\sigma(n-1)}(u) f^{\sigma(1),\ldots,\sigma(n-2)}(u) f^{\sigma(n)}(u) dW_u^i \Big\} \\
&= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) f^{\sigma(n)}(1) f^{\sigma(1),\ldots,\sigma(n-1)}(1) \qquad (\because \mathit{Lemma} 4.4) \\
&= \sum_{\lambda \in \Lambda_{n-1,1}} \operatorname{sgn}(\sigma^\lambda) (e_{k_1} \wedge \cdots \wedge e_{k_{n-1}})_W (e_{l_1})_W. \qquad (\because \mathit{Lemma} 4.3)
\end{aligned}
$$

If $n = 2k+2$, then Lemma 4.5 shows that

$$
\begin{aligned}
&\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) f^{\sigma(1),\sigma(2)}(1) f^{\sigma(3),\ldots,\sigma(n)}(1) \\
&= (p+1)(e_1 \wedge \cdots \wedge e_n)_W \\
&\qquad + \sum_{i_1=1}^{d} \cdots \sum_{i_{2p+4}=1}^{d} \int_0^1 e_{i_1}^{\sigma(2k+2)}(s_1) \int_0^{s_1} \cdots \int_0^{s_{2p+3}} e_{i_{2p+4}}^{\sigma(2k-2p+1)}(u) \\
&\qquad\qquad \cdot f^{\sigma(3),\ldots,\sigma(2k-2p)}(u) f^{\sigma(1),\sigma(2)}(u) dW_u^{i_{2p+4}} dW_{s_{2p+3}}^{i_{2p+3}} \cdots dW_{s_1}^{i_1}.
\end{aligned}
$$

Taking $p = k-1$, then above equation yields that

$$
\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) f^{\sigma(1),\sigma(2)}(1) f^{\sigma(3),\ldots,\sigma(n)}(1) = (k+1)(e_1 \wedge \cdots \wedge e_n)_W,
$$

so that

$$
\begin{aligned}
&(e_1 \wedge \cdots \wedge e_n)_W \\
&= \frac{2}{n} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) f^{\sigma(1),\sigma(2)}(1) f^{\sigma(3),\ldots,\sigma(n)}(1)
\end{aligned}
$$

$$= \frac{2}{n} \sum_{\lambda \in \Lambda_{2,n-2}} \operatorname{sgn}(\sigma^{\lambda}) (e_{k_1} \wedge e_{k_2})_W (e_{l_1} \wedge \cdots \wedge e_{l_{n-2}})_W. \qquad (\because \textit{Lemma}4.3)$$

Now, suppose that

$$(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_{2k}})_W = \mathrm{Pf}[((e_{n_i} \wedge e_{n_j})_W)_{1 \le i,j \le 2k}].$$

Then, we have

$$\begin{aligned}(e_1 \wedge \cdots \wedge e_{2k+2})_W &= \frac{1}{k+1} \sum_{\lambda \in \Lambda_{2,2k}} \operatorname{sgn}(\sigma^{\lambda}) (e_{k_1} \wedge e_{k_2})_W (e_{l_1} \wedge \cdots \wedge e_{l_{2k}})_W \\ &= \frac{1}{(k+1)!\,2^{k+1}} \sum_{\lambda \in \Lambda_{2,2k}, \tau \in \mathfrak{S}_{2k}} \operatorname{sgn}(\sigma^{\lambda}\tau) \Big((e_{k_1} \wedge e_{k_2})_W \\ &\qquad - (e_{k_2} \wedge e_{k_1})_W \Big) \prod_{i=1}^{k} (e_{l_{\tau(2i-1)}} \wedge e_{l_{\tau(2i)}})_W \\ &= \frac{1}{(k+1)!\,2^{k+1}} \sum_{\sigma \in \mathfrak{S}_{2k+1}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k+1} (e_{\sigma(2i-1)} \wedge e_{\sigma(2i)})_W.\end{aligned}$$

By induction, we have completed the proof. □

Corollary 4.6. *Let $\{(e_i^k)_{i=1}^d \mid k \ge 1\}$ be a family of continuous functions from $[0,1]$ to $\mathbb{R}^d$ with finite variation. Put*

$$\widetilde{f}^{n_k}(s) := \sum_{i=1}^{d} \int_0^s e_i^{n_k}(u) \circ dW_u^i, \quad s \in [0,\,1],$$

and

$$\widetilde{f}^{n_1,n_2,\ldots,n_k}(s) := \sum_{i=1}^{d} \int_0^s e_i^{n_k}(u) f^{n_1,n_2,\ldots,n_{k-1}}(u) \circ dW_u^i \quad s \in [0,\,1],$$

where $\int H \circ dW$ denotes the Stratonovich integral of H with respect to W. Define

$$(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_{\widetilde{W}} := \sum_{\sigma \in \mathfrak{S}_k} \widetilde{f}^{n_{\sigma(1)},n_{\sigma(2)},\ldots,n_{\sigma(k)}}.$$

Then, we have

$$(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_W = (e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_{\widetilde{W}}$$

Proof. By Theorem 4.1, there exits a polynomial F such that

$$\begin{aligned}&(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_W \\ &\qquad = F\big((e_{n_1})_W, \ldots, (e_{n_k})_W, (e_{n_1} \wedge e_{n_2})_W, \\ &\qquad\qquad \ldots, (e_{n_1} \wedge e_{n_k})_W, (e_{n_2} \wedge e_{n_3})_W, \ldots, (e_{n_{k-1}} \wedge e_{n_k})_W\big).\end{aligned}$$

and similarly

$$\begin{aligned}&(e_{n_1} \wedge e_{n_2} \wedge \cdots \wedge e_{n_k})_{\widetilde{W}} \\ &\qquad = F\big((e_{n_1})_{\widetilde{W}}, \ldots, (e_{n_k})_{\widetilde{W}}, (e_{n_1} \wedge e_{n_2})_{\widetilde{W}}, \\ &\qquad\qquad \ldots, (e_{n_1} \wedge e_{n_k})_{\widetilde{W}}, (e_{n_2} \wedge e_{n_3})_{\widetilde{W}}, \ldots, (e_{n_{k-1}} \wedge e_{n_k})_{\widetilde{W}}\big).\end{aligned}$$

Therefore, it is enough to prove that $(e_{n_1})_W = (e_{n_1})_{\widetilde{W}}$ and $(e_{n_1} \wedge e_{n_2})_W = (e_{n_1} \wedge e_{n_2})_{\widetilde{W}}$. However, it is clear that

$$(e_{n_1})_{\widetilde{W}} = \sum_{i=1}^{d} \int_0^1 e_i^{n_k}(u) \circ dW_u^i = \sum_{i=1}^{d} \int_0^1 e_i^{n_k}(u)\, dW_u^i = (e_{n_1})_W$$

and

$$\begin{aligned}(e_{n_1} \wedge e_{n_2})_{\widetilde{W}} &= \sum_{i=1}^{d} \sum_{j=1}^{d} \Big\{ \int_0^1 e_i^{n_2}(s) \int_0^s e_j^{n_1}(u) \circ dW_u^j \circ dW_s^i \\ &\qquad - \int_0^1 e_i^{n_1}(s) \int_0^s e_j^{n_2}(u) \circ dW_u^j \circ dW_s^i \Big\} \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} \Big\{ \int_0^1 e_i^{n_2}(s) \int_0^s e_j^{n_1}(u)\; dW_u^j dW_s^i + \frac{\delta_{ij}}{2} \int_0^1 e_i^{n_2}(s) e_j^{n_1}(s)\, ds \\ &\qquad - \int_0^1 e_i^{n_1}(s) \int_0^s e_j^{n_2}(u) dW_u^j dW_s^i - \frac{\delta_{ij}}{2} \int_0^1 e_i^{n_1}(s) e_j^{n_2}(s)\, ds \Big\} \\ &= (e_{n_1} \wedge e_{n_2})_W.\end{aligned}$$

□

Remark 4.7. Let $X = (X^i)_{i=1}^d$ be a continuous semimartingale and the the quadratic covariation of the distinct X^i and X^j be zero. Then, even though we replace the Wiener process W with X and the family of functions $\{(e_i^k)_{i=1}^d \mid k \geq 1\}$ with one which enables us to define the stochastic integral by X, the claim is also correct.

4.4 Applications

In this section, we introduce a new approximating scheme for the solution of a stochastic differential equation, which could be an application of Theorem 4.1.

The basic idea is as follows: If the joint distribution of $(e_{n_1})_W, \dots, (e_{n_k})_W$, $(e_{n_1} \wedge e_{n_2})_W$, $(e_{n_1} \wedge e_{n_3})_W, \dots, (e_{n_{k-1}} \wedge e_{n_k})_W$, $k = 1, 2, \dots$ are available, then we can explicitly calculate the expectation of $G(\{(e_{n_1} \wedge \cdots \wedge e_{n_k})_W : n_1, n_2, \dots, n_k \in \mathbb{Z} + \frac{1}{2}\}_{k=1,2,\dots})$ for a measurable function G. This can be an anti-symmetric counterpart of the standard Gaussian based approximations. In our new framework, the (joint) distribution of the stochastic area(s) plays a central role.

Example 4.8. Here is an example. Let $d = 2$ and put

$$e^n_k(s) := \frac{1}{\sqrt{n}} \begin{pmatrix} I[\frac{k}{n}, \frac{k+1}{n}](s) \\ 0 \end{pmatrix}, \quad e^n_{-k}(s) := \frac{1}{\sqrt{n}} \begin{pmatrix} 0 \\ I[\frac{k}{n}, \frac{k+1}{n}](s) \end{pmatrix}, \ \ 0 \le k \le n-1,$$

where $I[\frac{k}{n}, \frac{k+1}{n}](s)$ is the indicator function of the interval $[\frac{k}{n}, \frac{k+1}{n}]$. Then, for each $0 \le k < l \le n-1$, we have the following equations.

$$\begin{aligned}
(e^n_k)_W = \frac{1}{\sqrt{n}}\big(W^1_{\frac{k+1}{n}} - W^1_{\frac{k}{n}}\big), &\quad (e^n_{-k})_W = \frac{1}{\sqrt{n}}\big(W^2_{\frac{k+1}{n}} - W^2_{\frac{k}{n}}\big), \\
(e^n_k \wedge e^n_l)_W &= \frac{1}{n}\big((W^1_{\frac{l+1}{n}} - W^1_{\frac{l}{n}})(W^1_{\frac{k+1}{n}} - W^1_{\frac{k}{n}})\big), \\
(e^n_{-k} \wedge e^n_{-l})_W &= \frac{1}{n}\big((W^2_{\frac{l+1}{n}} - W^2_{\frac{l}{n}})(W^2_{\frac{k+1}{n}} - W^2_{\frac{k}{n}})\big), \\
(e^n_k \wedge e^n_{-l})_W &= \frac{1}{n}\big((W^2_{\frac{l+1}{n}} - W^2_{\frac{l}{n}})(W^1_{\frac{k+1}{n}} - W^1_{\frac{k}{n}})\big), \\
(e^n_{-k} \wedge e^n_l)_W &= \frac{1}{n}\big((W^2_{\frac{k+1}{n}} - W^2_{\frac{k}{n}})(W^1_{\frac{l+1}{n}} - W^1_{\frac{l}{n}})\big), \\
(e^n_{-k} \wedge \ e^n_k)_W &= \frac{1}{n}\bigg(\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\frac{k}{n}}^{t} dW^2_s dW^1_t - \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\frac{k}{n}}^{t} dW^1_s dW^2_t\bigg).
\end{aligned}$$

The Fourier transform of the joint distribution have been obtained by Aihara et al. [1].

When we approximate the solution to a SDE, we rely on the stochastic Taylor expansion (4.2), as is remarked in the introduction. The expansion gives, however, a linear combination of multiple Wiener integrals, which is neither necessarily symmetric nor anti-symmetric. So we need to work on the following class:

Definition 4.9. For the stochastic differential equation (4.1), or equivalently, its associated vector fields $V_0, V_1, \ldots, V_d$, we say that it has a k-th order reduction if for each $n \leq k$, the linear combination of stochastic integrals

$$\sum_{j_1,\ldots,j_n=0}^{d} V_{j_1}V_{j_2}\cdots V_{j_n}(x)\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-1}} \circ dW_{s_n}^{j_n}\circ dW_{s_{n-1}}^{j_{n-1}}\cdots\circ dW_{s_1}^{j_1} \tag{4.3}$$

is represented as a **finite** sum of polynomials of (e_l) and $(e_l \wedge e_m)_W$ for a finite set of functions $e_1, \ldots, e_M$ in $L^2([0,T] \to \mathbb{R}^d)$.

We note that every SDE has a 2nd order reduction since

$$\begin{aligned}&V_{i,j}\int\int dW^i dW^j\\ &= \frac{V_{i,j}}{2}\left(\int\int dW^i dW^j + \int\int dW^j dW^i\right) + \frac{V_{i,j}}{2}\left(\int\int dW^i dW^j - \int\int dW^j dW^i\right).\end{aligned}$$

Since every symmetric multiple integral is represented by an Hermite polynomial, an SDE has a k-th order reduction if (4.3) is decomposed into a sum of symmetric and anti-symmetric integrals. The following lemma gives a necessary and sufficient condition for that:

Proposition 4.10. *We keep the notations in the section 4.3. A linear combination of multiple Wiener integrals*

$$X = \sum_k \sum_{1\leq i_1\leq\cdots\leq i_k\leq d}\sum_{\sigma\in\mathfrak{S}_k} a_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} f^{i_{\sigma(1)},\ldots,\sigma(k)}(1), \tag{4.4}$$

becomes a sum of symmetric and anti-symmetric integrals if and only if the following condition satisfied: For each $k > 2$, $1 \leq i_1 \leq \cdots \leq i_k \leq d$, there exist $A_{i_1,\ldots,i_k}$ and $B_{i_1,\ldots,i_k}$ such that for all $\operatorname{sgn}(\sigma) = 1$ *and* $\operatorname{sgn}(\tau) = -1$ $(\sigma, \tau \in \mathfrak{S}_k)$,

$$A_{i_1,\ldots,i_k} = a_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} + a_{i_{\tau(1)},\ldots,i_{\tau(k)}}, \quad B_{i_1,\ldots,i_k} = a_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} - a_{i_{\tau(1)},\ldots,i_{\tau(k)}}. \tag{4.5}$$

Proof. Suppose that X is a linear combination of multiple stochastic integrals given by (4.4) and there exist $A_{i_1,\ldots,i_k}$ and $B_{i_1,\ldots,i_k}$ satisfying (4.5) for all $\sigma \in \mathfrak{S}_k^+ := \{\sigma \in \mathfrak{S}_k \mid \operatorname{sgn}(\sigma) = 1\}$ and $\tau \in \mathfrak{S}_k^- := \{\sigma \in \mathfrak{S}_k \mid \operatorname{sgn}(\sigma) = -1\}$. Then, we have

$$X = \sum_{k=1}^{\infty}\sum_{1\leq i_1\leq\cdots\leq i_k}\Bigg\{\sum_{\sigma\in\mathfrak{S}_k^+} a_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} + \sum_{\sigma\in\mathfrak{S}_k^-} a_{i_{\sigma(1)},\ldots,i_{\sigma(k)}}\Bigg\} f^{i_{\sigma(1)},\ldots,i_{\sigma(k)}}(1)$$

$$= \sum_{k=1}^{\infty} \sum_{1 \le i_1 \le \cdots \le i_k} \Big\{ \sum_{\sigma \in \mathfrak{S}_k^+} \frac{A_{i_1,\dots,i_k} + B_{i_1,\dots,i_k}}{2} + \sum_{\sigma \in \mathfrak{S}_k^-} \frac{A_{i_1,\dots,i_k} - B_{i_1,\dots,i_k}}{2} \Big\} f^{i_{\sigma(1)},\dots,i_{\sigma(k)}}(1)$$

$$= \sum_{k=1}^{\infty} \sum_{1 \le i_1 \le \cdots \le i_k} \Big\{ A_{i_1,\dots,i_k} \sum_{\sigma \in \mathfrak{S}_k} f^{i_{\sigma(1)},\dots,i_{\sigma(k)}}(1) + \frac{B_{i_1,\dots,i_k}}{2} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) f^{i_{\sigma(1)},\dots,i_{\sigma(k)}}(1) \Big\}$$

$$= \sum_{k=1}^{\infty} \sum_{1 \le i_1 \le \cdots \le i_k} \Big\{ A_{i_1,\dots,i_k} (e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_k})_W + \frac{B_{i_1,\dots,i_k}}{2} (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})_W \Big\},$$

where $(e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_k})_W$ is a symmetric integral defined by

$$(e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_k})_W := \sum_{\sigma \in \mathfrak{S}_k} f^{i_{\sigma(1)},\dots,i_{\sigma(k)}}(1).$$

Therefore X is a sum of symmetric and anti-symmetric integrals.

Conversely, if X is a sum of symmetric and anti-symmetric integrals given by

$$X = \sum_{k=1}^{\infty} \sum_{1 \le i_1 \le \cdots \le i_k} \Big\{ C_{i_1,\dots,i_k} (e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_k})_W + D_{i_1,\dots,i_k} (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})_W \Big\},$$

then X is a linear combination of multiple stochastic integrals whose coefficients $a_{i_{\sigma(1)},\dots,i_{\sigma(k)}}$ and $a_{i_{\tau(1)},\dots,i_{\tau(k)}}$ in (4.4) are equal to $(C_{i_1,\dots,i_k} + 2D_{i_1,\dots,i_k})/2$ and $(C_{i_1,\dots,i_k} - 2D_{i_1,\dots,i_k})/2$ for each $\sigma \in \mathfrak{S}_k^+$ and $\tau \in \mathfrak{S}_k^-$, respectively. □

We give an example without 3-th order reduction, which is a 3rd-order scheme in dimension 2.

Example 4.11. We use the Itô expression;

$$X_t = X_0 + \int_0^t a(X_s)\, ds + \sum_{j=1}^{2} \int_0^t b_j(X_s)\, dW_s^j,$$

where $a = (a^1, a^2)$, $b_1 = (b_1^1, b_1^2)$, $b_2 = (b_2^1, b_2^2) \in C^4(\mathbb{R}^2 \to \mathbb{R}^2)$. For $f = (f^1, f^2) \in C^2(\mathbb{R}^2 \to \mathbb{R}^2)$, Itô formula implies that

$$f(X_t) = f(X_0) + \int_0^t \Big((f'a)(X_s) + \frac{1}{2} \sum_{j=1}^{2} (b_j^* f'' b_j)(X_s) \Big)\, ds + \sum_{j=1}^{2} \int_0^t (f' b_j)(X_s) dW_s^j,$$

where $f'a$ and $b_j^* f'' b_j$ are functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$(f'a)^i(x) = \sum_{k=1}^{2} \frac{\partial f^i}{\partial x_k}(x) a^k(x), \quad (b_j^* f'' b_j)^i(x) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial^2 f^i}{\partial x_k \partial x_l}(x) b_j^k(x) b_j^l(x), \ \ x \in \mathbb{R}^2.$$

By apply Itô formula to a and b respectively, we have

$$\begin{aligned}X_t - X_s =& a(X_s)(t-s) + \sum_{j=1}^{2} b_j(X_s)(W_t^j - W_s^j) \\ &+ \int_s^t \int_s^u \Big(a'a + \frac{1}{2}\sum_{j=1}^{2} b_j^* a'' b_j\Big)(X_v)\, dv\, du + \sum_{j=1}^{2}\int_s^t \int_s^u a' b_j(X_v) dW_v^j du \\ &+ \sum_{j=1}^{2}\int_s^t \int_s^u \Big(b_j' a + \frac{1}{2}\sum_{k=1}^{2} b_k^* b_j'' b_k\Big)(X_v)\, dv\, dW_u^j \\ &+ \sum_{j=1}^{2}\sum_{k=1}^{2}\int_s^t \int_s^u b_j' b_k(X_v)\, dW_v^k\, dW_u^j\end{aligned}$$

For each $j, k = 1, 2$, put

$$c_{00} := a'a + \frac{1}{2}\sum_{j=1}^{2} b_j^* a'' b_j, \quad c_{j0} := a' b_j, \quad c_{0j} := b_j' a + \frac{1}{2}\sum_{k=1}^{2} b_k^* b_j'' b_k \quad c_{kj} := b_j' b_k$$

and we also apply Itô formula to them, then we have

$$\begin{aligned}X_t - X_s =& a(X_s)(t-s) + \sum_{j=1}^{2} b_j(X_s)(W_t^j - W_s^j) + \sum_{j=0}^{2}\sum_{k=0}^{2} c_{kj}(X_s)\int_s^t \int_s^u dW_v^k dW_u^j \\ &+ \sum_{j=0}^{2}\sum_{k=0}^{2}\int_s^t \int_s^u \int_s^v \Big((c_{kj}' a)(X_r) + \frac{1}{2}\sum_{l=1}^{2}(b_l^* c_{kj}'' b_l)(X_r)\Big)\, dW_r^0\, dW_v^k\, dW_u^j \\ &+ \sum_{j=0}^{2}\sum_{k=0}^{2}\sum_{l=1}^{2}\int_s^t \int_s^u \int_s^v (c_{kj}' b_j)(X_s)\, dW_r^l\, dW_v^k\, dW_u^j.\end{aligned}$$

Note that $W_t^0 = t$ in a convention.

Next, we consider a discretization of X_T. For each $N \in \mathbb{N}$ and $n = 0, 1, \ldots, N$, let $\Delta = T/N$ and $t_n = n \times \Delta$. We take $\{e_m \mid m \in Z_{\geq 0}\}$ as an orthonormal basis for $L^2([0,1] \to \mathbb{R})$ with $e_0 = 1$, and observe the Fourier expansion of triple integral :

$$\begin{aligned}&\int_{t_n}^{t_{n+1}} \int_{t_n}^u \int_{t_n}^v dW_r^l\, dW_v^k\, dW_u^j \\ &\overset{d}{=} \Delta^{o(l,k,j)} \int_0^1 \int_0^u \int_0^v dW_r^l\, dW_v^k\, dW_u^j \\ &= \Delta^{o(l,k,j)} \sum_{m=0}^{\infty}\Big(\int_0^1 e_m(r) W_r^l\, dr\Big)\int_0^1 \int_0^u e_m(v)\, dW_v^k\, dW_u^j\end{aligned}$$

$$\overset{d}{=} \sum_{m=0}^{\infty} \Big(\int_{t_n}^{t_{n+1}} e_m^N(u) \int_{t_n}^{u} dW_r^l \, du \Big) \int_{t_n}^{t_{n+1}} \int_{t_n}^{u} e_m^N(v) \, dW_v^k \, dW_u^j,$$

where $o(l,k,j) := (3 + \delta_{0,l} + \delta_{0,k} + \delta_{0,j})/2$ and

$$e_m^N(u) := \frac{1}{\sqrt{\Delta}} e_m((u - t_n)/\Delta), \;\; t_n < u \leq t_{n+1}.$$

Therefore, we discretize X_T by replacing the triple integrals such as

$$\int_s^t \int_s^u \int_s^v c(X_r) \, dr \, dv \, du$$

and

$$\int_s^t \int_s^u \int_s^v c(X_r) \, dW_r^l \, dW_v^k \, dW_u^j, \;\; (l,k,j) \neq (0,0,0)$$

with

$$c(X_r) \int_s^t \int_s^u \int_s^v \, dr \, dv \, du = c(X_r)(t-u)^3/6$$

and

$$c(X_r) \sum_{m=0}^{N} \Big(\int_s^t e_m^N(u) \int_s^u dW_r^l \, dr \Big) \int_s^t \int_s^u e_m^N(v) \, dW_v^k \, dW_u^j, \;\; (l,k,j) \neq (0,0,0).$$

The N-th approximation of $X^{(N)}$ of X is given by the following.

$$\begin{aligned}
X_0^{(N)} =& X_0, \\
X_{t_{n+1}}^{(N)} =& X_{t_n}^{(N)} + a(X_{t_n}^{(N)})\Delta + \sum_{j=1}^{2} b_j(X_{t_n}^{(N)})\Delta W_n^j + \sum_{j=0}^{2}\sum_{k=0}^{2} c_{kj}(X_{t_n}^{(N)})\Delta A_{n,0}^{k,j} \\
&+ \Big((c_{00}' a)(X_{t_n}^{(N)}) + \frac{1}{2}\sum_{l=1}^{2} (b_l^* c_{00}'' b_l)(X_{t_n}^{(N)}) \Big) \Delta^3/6 \\
&+ \sum_{j,k=0,(j,k)\neq(0,0)}^{2} \Big((c_{kj}' a)(X_{t_n}^{(N)}) + \frac{1}{2}\sum_{l=1}^{2} (b_l^* c_{kj}'' b_l)(X_{t_n}^{(N)}) \Big) \sum_{m=0}^{N} \Delta \bar{A}_{n,m}^{0,0} \Delta A_{n,m}^{k,j} \\
&+ \sum_{j=0}^{2}\sum_{k=0}^{2}\sum_{l=1}^{2} (c_{kj}' b_j)(X_{t_n}^{(N)}) \sum_{m=0}^{N} \Delta \bar{A}_{n,m}^{l,0} \Delta A_{n,m}^{k,j},
\end{aligned}$$

where $\Delta W_n^j = W_{t_n}^j - W_{t_{n-1}}^j$,

$$\Delta A_{n,m}^{k,j} := \int_{s_n}^{t_n} \int_{s_n}^{u} e_m^n(v) \, dW_v^k \, dW_u^j, \quad \Delta \bar{A}_{n,m}^{l,0} := \int_{s_n}^{t_n} e_m^n(r) \int_{s_n}^{u} dW_r^l \, du.$$

This corresponds to $2(+\alpha)$ weak Taylor scheme, which needs to simulate double stochastic integral. We can obtain an explicit form of the Fourier transform of their joint distribution when we take a good orthonormal basis $\{e_m \mid m \in \mathbb{Z}_{\geq 0}\}$ such as Walsh functions. In fact, we see that

$$\begin{aligned}
\Delta A^{0,j}_{n,m} &\overset{d}{=} \frac{1}{\sqrt{n}} \int_0^1 e_0(u) \int_0^u e_m(v)\, dv\, dW^j_u \\
&= \frac{1}{2\sqrt{n}} \Big(\delta_{j,1}(\bar{e}_m)_W + \delta_{j,2}(\underline{e}_m)_W - (\bar{e}_0 \wedge \underline{e}_m)_j \Big), \\
\Delta A^{j,0}_{n,m} &\overset{d}{=} \frac{1}{\sqrt{n}} \int_0^1 e_0(u) \int_0^u e_m(v)\, dW^j_v\, du \\
&= \frac{1}{2\sqrt{n}} \Big(\delta_{j,1}(\bar{e}_m)_W + \delta_{j,2}(\underline{e}_m)_W + (\bar{e}_m \wedge \underline{e}_0)_j \Big), \\
\Delta A^{1,1}_{n,m} &\overset{d}{=} \frac{1}{n} \int_0^1 e_0(u) \int_0^u e_m(v)\, dW^1_v\, dW^1_u = \frac{1}{2n} \Big(((\bar{e}_m)_W)^2 - (\bar{e}_0 \wedge \bar{e}_m)_W \Big), \\
\Delta A^{2,2}_{n,m} &\overset{d}{=} \frac{1}{n} \int_0^1 e_0(u) \int_0^u e_m(v)\, dW^2_v\, dW^2_u = \frac{1}{2n} \Big(((\underline{e}_m)_W)^2 - (\underline{e}_0 \wedge \underline{e}_m)_W \Big), \\
\Delta A^{1,2}_{n,m} &\overset{d}{=} \frac{1}{n} \int_0^1 e_0(u) \int_0^u e_m(v)\, dW^1_v\, dW^2_u = \frac{1}{2n} \Big((\bar{e}_m)_W (\underline{e}_0)_W - (\underline{e}_0 \wedge \bar{e}_m)_W \Big), \\
\Delta A^{2,1}_{n,m} &\overset{d}{=} \frac{1}{n} \int_0^1 e_0(u) \int_0^u e_m(v)\, dW^2_v\, dW^1_u = \frac{1}{2n} \Big((\underline{e}_m)_W (\bar{e}_0)_W + (\underline{e}_m \wedge \bar{e}_0)_W \Big), \\
\Delta \bar{A}^{j,0}_{n,m} &\overset{d}{=} \frac{1}{\sqrt{n}} \int_0^1 e_m(u) \int_0^u e_0(v)\, dW^j_v\, du \\
&= \frac{1}{2\sqrt{n}} \bigg(\Big(\int_0^1 e_m(s)\, ds \Big) (\delta_{j,1}(\bar{e}_0)_W + \delta_{j,2}(\underline{e}_0)_W) + (\bar{e}_0 \wedge \underline{e}_m)_j \bigg) \\
&= \frac{1}{2\sqrt{n}} (\bar{e}_0 \wedge \underline{e}_m)_j.
\end{aligned}$$

Here $j = 1, 2$, $\delta_{jj} = 1$, $\delta_{jk} = 0$ $(j \neq k)$ and

$$\bar{e}_m(s) := \begin{pmatrix} e_m(s) \\ 0 \end{pmatrix}, \quad \underline{e}_m(s) := \begin{pmatrix} 0 \\ e_m(s) \end{pmatrix},$$

$$(\bar{e}_l \wedge \underline{e}_m)_j := \int_0^1 e_m(u) \int_0^u e_l(v)\, dW^j_v\, du - \int_0^1 e_l(u) \int_0^u e_m(v)\, dv\, dW^j_u.$$

In above equations, it follows from the integration by parts formula:

$$\begin{aligned}
&\Big(\int_0^1 e_n(s)\, dW^k \Big) \Big(\int_0^1 e_m(s)\, dW^l \Big) \\
&= \int_0^1 e_n(s) \int_0^s e_m(u) dW^l_u dW^k_s + \int_0^1 e_m(s) \int_0^s e_n(u) dW^k_u dW^l_s, \quad k, l = 0, 1, 2.
\end{aligned}$$

To obtain the Fourier transform of their joint distribution, we need that of W_1^1, W_1^2, A^{ij}, U_1 and U_2, where

$$A^{ij} = \int_0^1 W_s^j \, dW_s^i - \int_0^1 W_s^i \, dW_s^j, \quad U^i = \int_0^1 W_s^i \, ds - \int_0^1 s \, dW_s^i,$$

which can be obtained by Proposition 4.12 below.

The following proposition is supplementary to a result by Helmes and Schwane [13].

Proposition 4.12. *For each $\lambda_{ij} \in \mathbb{R}$ satisfying*

$$v^2 := \sum_{k=1}^d (\lambda_{1k} - \lambda_{k1})^2 = \cdots = \sum_{k=1}^d (\lambda_{dk} - \lambda_{kd})^2,$$

let Λ, S and V^λ be matrices given by $\Lambda_{ij} := \lambda_{ij} - \lambda_{ji}$ and $S = \Lambda\Lambda^*$, respectively. Here Λ^* denotes the transposed matrix of Λ. Denote unite matrix by I_d, and put

$$G(\lambda) := \prod_{i=1}^d \Big(\frac{\sqrt{v_i^\lambda}}{\sinh \sqrt{v_i^\lambda}} \Big)^{\frac{1}{2}}, \quad F(\lambda) = \operatorname{diag}\bigg(\frac{\sqrt{v_i^\lambda} \coth \sqrt{v_i^\lambda} - 1}{2 v_i^\lambda} \bigg)$$

and $H(\lambda) := O(\lambda)^* F(\lambda) O(\lambda)$, where $O(\lambda)$ is an orthogonal matrix and a diagonal matrix $V^\lambda = \operatorname{diag}(v_i^\lambda)$ such that $S = O(\lambda)^* V^\lambda O(\lambda)$. For each η, $\zeta \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\Big\{ \sqrt{-1} \sum_{i=1}^d \sum_{j=1}^d \lambda_{ij} A^{ij} + \sqrt{-1} \sum_{j=1}^d (\eta_j U^j + \zeta_j W_1^j) \Big\} \bigg] \\
&= \frac{G(\lambda)}{\sqrt{\det(I_d + 2\Lambda H(\lambda)\Lambda^*)}} \exp\Big\{ 2\langle (I_d + 2\Lambda H(\lambda)\Lambda^*)^{-1} \Lambda H(\lambda)\eta, \Lambda H(\lambda)\eta \rangle - \langle H(\lambda)\eta, \eta \rangle \\
&\quad - 2\sqrt{-1} \langle (I_d + 2\Lambda H(\lambda)\Lambda^*)^{-1} \Lambda H(\lambda)\eta, \zeta \rangle - \frac{1}{2} \langle ((I_d + 2\Lambda H(\lambda)\Lambda^*)^{-1} \zeta, \zeta \rangle \Big\}.
\end{aligned}$$

In particular, when $d = 2$, for each λ, η_1, η_2, ζ_1 and $\zeta_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\Big\{ \sqrt{-1}(\lambda A + \eta_1 U_1 + \eta_2 U_2 + \zeta_1 W_1^1 + \zeta_2 W_1^2) \Big\} \bigg] \\
&= \frac{1}{\cosh \lambda} \bigg(\frac{\lambda \coth \lambda - 1}{2\lambda^3 \coth \lambda} \big(2\lambda\sqrt{-1}(\eta_1 \zeta_2 - \eta_2 \zeta_1) - \eta_1^2 - \eta_2^2 \big) - \frac{\zeta_1^2 + \zeta_2^2}{2\lambda \coth \lambda} \bigg).
\end{aligned}$$

Proof. For each $n \in \mathbb{N}$, put

$$e_0(\theta) := 1,$$

$$e_n(\theta) := \sqrt{2} \sin 2n\pi\theta, \quad e_n^*(\theta) := \sqrt{2} \cos 2n\pi\theta$$

and

$$\begin{aligned}
a_0(s) &:= \int_0^s e_0(\theta)\, d\theta = s, \\
a_n(s) &:= \int_0^s e_n(\theta)\, d\theta = \frac{1}{2} c_n \, (\sqrt{2} e_0(s) - e_n^*(s)), \\
a_n^*(s) &:= \int_0^s e_n^*(\theta)\, d\theta = \frac{1}{2} c_n \, e_n(s),
\end{aligned}$$

where $c_n = 1/n\pi$. Let $\langle f, g \rangle_{L^2}$ be the inner product in $L^2([0,1]) := L^2([0,1] \to \mathbb{R})$ given by

$$\langle f, g \rangle_{L^2} = \int_0^1 f(s)\, g(s)\, ds, \;\; f, g \in L^2([0,1]).$$

Then we have the following equations.

$$\begin{aligned}
&\langle a_0, e_0 \rangle_{L^2} = \frac{1}{2}, \quad \langle a_n, e_0 \rangle_{L^2} = \frac{\sqrt{2}}{2}\, c_n, \quad \langle a_n^*, e_0 \rangle_{L^2} = 0, \\
&\langle a_0, e_n \rangle_{L^2} = -\frac{\sqrt{2}}{2}\, c_n, \quad \langle a_0, e_n^* \rangle_{L^2} = 0, \\
&\langle a_n, e_m \rangle_{L^2} = \langle a_n^*, e_m^* \rangle_{L^2} = 0, \\
&\langle a_n, e_m^* \rangle_{L^2} = -\frac{1}{2}\, c_n \delta_{n,m}, \quad \langle a_n^*, e_m \rangle_{L^2} = \frac{1}{2}\, c_n \delta_{n,m}.
\end{aligned}$$

We also put

$$\xi_{i,0} := \int_0^1 e_0(s)\, dW_s^i, \quad \xi_{i,n} := \int_0^1 e_n(s)\, dW_s^i, \quad \xi_{i,n}^* := \int_0^1 e_n^*(s)\, dW_s^i.$$

Then, $\{\xi_{i,0},\ \xi_{i,n},\ \xi_{i,n}^* \mid i = 1, \ldots, d, n \in \mathbb{N}\}$ is a collection of independent Gaussian random variables with mean 0 and variance 1, since $\{e_0, e_n, e_m^* \mid n, m \in \mathbb{N}\}$ is an orthonormal basis of $L^2([0,1])$. By the Fourier expansion to the Brownian motion, we have

$$W_t^i = \langle W^i, e_0 \rangle_{L^2} + \sum_{n=1}^{\infty} \big(\langle W^i, e_n \rangle_{L^2} e_n(t) + \langle W^i, e_n^* \rangle_{L^2} e_n^*(t) \big),$$

so that

$$\int_0^i W_s^j\, dW_s^i = \langle W^j, e_0 \rangle_{L^2} \xi_{i,0} + \sum_{n=1}^{\infty} \big(\langle W^j, e_n \rangle_{L^2} \xi_{i,n} + \langle W^j, e_n^* \rangle_{L^2} \xi_{i,n}^* \big).$$

By the integral-by-parts formula, we also have

$$\begin{aligned}
\langle W^j, e_n\rangle_{L^2} &= \int_0^1 W_s^j\, e_n(s)\, ds \\
&= W_1^j a_n(1) - \int_0^1 a_n(s)\, dW_s^j \\
&= -\int_0^1 \langle a_n, e_0\rangle_{L^2} e_0(s)\, dW_s^j \\
&\qquad - \sum_{m=1}^{\infty} \Big\{ \int_0^1 \langle a_n, e_m\rangle_{L^2} e_m(s)\, dW_s^j + \int_0^1 \langle a_n, e_m^*\rangle_{L^2} e_m^*(s)\, dW_s^j \Big\} \\
&= \frac{1}{2}\, c_n\, (\xi_{j,n}^* - \sqrt{2}\xi_{j,0}). \\
\langle W^j, e_n^*\rangle_{L^2} &= \int_0^1 W_s^j\, e_n^*(s)\, ds \\
&= W_1^j a_n^*(1) - \int_0^1 a_n^*(s)\, dW_s^j \\
&= -\int_0^1 \langle a_n^*, e_0\rangle_{L^2} e_0(s)\, dW_s^j \\
&\qquad - \sum_{m=1}^{\infty} \Big\{ \int_0^1 \langle a_n^*, e_m\rangle_{L^2} e_m(s) dW_s^j + \int_0^1 \langle a_n^*, e_m^*\rangle_{L^2} e_m^*(s)\, dW_s^j \Big\} \\
&= -\frac{1}{2}\, c_n\, \xi_{j,n}. \\
\langle W^j, e_0\rangle_{L^2} &= \int_0^1 W_s^j\, e_0(s)\, ds \\
&= W_1^j - \int_0^1 s\, dW_s^j \\
&= W_1^j - \int_0^1 \langle a_0, e_0\rangle_{L^2} e_0(s) dW_s^j \\
&\qquad - \sum_{m=1}^{\infty} \Big\{ \int_0^1 \langle a_0, e_m\rangle_{L^2} e_m\, dW_s^j + \int_0^1 \langle a_0, e_m^*\rangle_{L^2} e_m^*\, dW_s^j \Big\} \\
&= \frac{1}{2} \left(\xi_{j,0} + \sqrt{2} \sum_{n=1}^{\infty} c_n \xi_{j,n} \right).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
A^{ij} &= \int_0^1 W_s^j\, dW_s^i - \int_0^1 W_s^i\, dW_s^j \\
&= \sum_{n=1}^{\infty} \Big\{ \sqrt{2} c_n (\xi_{i,0}\xi_{j,n} - \xi_{i,n}\xi_{j,0}) + c_n (\xi_{i,n}\xi_{j,n}^* - \xi_{i,n}^*\xi_{j,n}) \Big\}.
\end{aligned}$$

and

$$
\begin{aligned}
U^i &:= \int_0^1 W_s^i\,ds - \int_0^1 s\,dW_s^i \\
&= \int_0^1 (1-2s)\,dW_s^i \\
&= \xi_{i,0} - 2\int_0^1 a_0(s)\,dW_s^i \\
&= \xi_{i,0} - 2\int_0^1 \langle a_0, e_0\rangle_{L^2} e_0(s)\,dW_s^i \\
&\qquad - 2\sum_{m=1}^{\infty}\Big\{\int_0^1 \langle a_0, e_m\rangle_{L^2} e_m(s)\,dW_s^i + \int_0^1 \langle a_0, e_m^*\rangle_{L^2} e_m^*(s)\,dW_s^i\Big\} \\
&= \sqrt{2}\sum_{n=1}^{\infty} c_n\,\xi_{i,n}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&L_n(\lambda,\eta:z) \\
&= \mathbb{E}\Big[\exp\Big\{\sqrt{-1}\sum_{i=1}^d\sum_{j=1}^d \lambda_{ij}\big\{\sqrt{2}c_n(\xi_{i,0}\xi_{j,n}-\xi_{i,n}\xi_{j,0}) + c_n(\xi_{i,n}\xi_{j,n}^* - \xi_{i,n}^*\xi_{j,n})\big\} \\
&\qquad + \sqrt{-1}\sum_{j=1}^d \eta_j(\sqrt{2}c_n\xi_{j,n})\Big\}\Big|\xi_{1,0}=z_1,\ldots,\xi_{d,0}=z_d\Big] \\
&= \Big(\frac{1}{\sqrt{2\pi}}\Big)^{2d}\int_{\mathbb{R}^{2d}} \exp\Big\{\sqrt{-1}\sum_{i=1}^d\sum_{j=1}^d \lambda_{ij}c_n\big\{\sqrt{2}(z_ix_j - x_iz_j) + (x_iy_j - y_ix_j)\big\} \\
&\qquad + \sqrt{-1}\sum_{j=1}^d \sqrt{2}c_n\eta_jx_j - \frac{1}{2}\sum_{j=1}^d (x_j^2+y_j^2)\Big\}\,dx_1\cdots dx_d\,dy_1\cdots dy_d \\
&= \Big(\frac{1}{\sqrt{2\pi}}\Big)^{d}\int_{\mathbb{R}^{d}} \exp\Big\{\sqrt{-1}\sum_{i=1}^d\sum_{j=1}^d \sqrt{2}c_n(\lambda_{ij}-\lambda_{ji})z_ix_j \\
&\qquad - \frac{c_n^2}{2}\sum_{j=1}^d\Big(\sum_{i=1}^d(\lambda_{ij}-\lambda_{ji})x_i\Big)^2 + \sqrt{-1}\sum_{j=1}^d\sqrt{2}c_n\eta_jx_j - \frac{1}{2}\sum_{j=1}^d x_j^2\Big\}\,dx_1\cdots dx_d \\
&= \Big(\frac{1}{\sqrt{2\pi}}\Big)^{d}\int_{\mathbb{R}^{d}} \exp\Big\{\sqrt{-1}\sum_{j=1}^d \sqrt{2}c_n\Big(\eta_i + \sum_{i=1}^d(\lambda_{ij}-\lambda_{ji})z_i\Big)x_j \\
&\qquad - \frac{1}{2}\sum_{i=1}^d\Big(1 + c_n^2\sum_{j=1}^d(\lambda_{ij}-\lambda_{ji})^2\Big)x_i^2
\end{aligned}
$$

$$-\frac{c_n^2}{2}\sum_{k=1}^{d}\sum_{i,j=1,i\neq j}^{d}(\lambda_{ik}-\lambda_{ki})(\lambda_{jk}-\lambda_{kj})x_ix_j\Big\}\,dx_1\cdots dx_d.$$

Now, for each $i,j=1,\dots,d$, we put

$$\widetilde{\lambda}_{ij}:=\lambda_{ij}-\lambda_{ji},\quad \sigma_{jn}:=\left(1+c_n^2\sum_{i=1}^{d}\widetilde{\lambda}_{ij}^2\right)^{-\frac{1}{2}}=(1+c_n^2v^2)^{-\frac{1}{2}}.$$

Let D_n be a matrix defined by $D_n:=\mathrm{diag}(\sigma_{in}^1(\lambda))$ and $Z_1,\dots,Z_d$ be independent random variables whose distributions are standard normal. Then, we have that S is symmetric, positive definite, and therefore, there exist an orthogonal matrix O and a diagonal matrix $V^\lambda:=\mathrm{diag}(v_i^\lambda)$ such that $S=O^*V^\lambda O$. Since

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{\sqrt{-1}\langle\vec{t},D_nZ\rangle-\frac{c_n^2}{2}\langle SD_nZ,D_nZ\rangle+\frac{v^2c_n^2}{2}\langle D_nZ,D_nZ\rangle\Big\}\Big]\\
&=\mathbb{E}\Big[\exp\Big\{\sqrt{-1}\langle O\vec{t},OD_nZ\rangle-\frac{c_n^2}{2}\langle V^\lambda OD_nZ,OD_nZ\rangle+\frac{v^2c_n^2}{2}\langle OD_nZ,OD_nZ\rangle\Big\}\Big]\\
&=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\exp\Big\{(1+c_n^2v^2)^{-\frac{1}{2}}\sqrt{-1}\langle O\vec{t},x\rangle-\frac{1}{2}\langle M_nx,x\rangle\Big\}\,dx\\
&=(\det M_n)^{-\frac{1}{2}}\exp\Big\{-\frac{1}{2}(1+c_n^2v^2)^{-1}\langle M_n^{-1}O\vec{t},O\vec{t}\rangle\Big\},
\end{aligned}$$

where $M_n:=(1+c_n^2v^2)^{-1}I_d+c_n^2(1+c_n^2v^2)^{-1}V^\lambda$, we obtain

$$\begin{aligned}
L_n(\lambda,\eta:z)&=(1+c_n^2v^2)^{-\frac{d}{2}}\mathbb{E}\Big[\exp\Big\{\sqrt{-1}\langle\sqrt{2}c_n(\eta+\Lambda^*z),D_nZ\rangle\\
&\qquad-\frac{c_n^2}{2}\langle SD_nZ,D_nZ\rangle+\frac{v^2c_n^2}{2}\langle D_nZ,D_nZ\rangle\Big\}\Big]\\
&=(1+c_n^2v^2)^{-\frac{d}{2}}(\det M_n)^{-\frac{1}{2}}\\
&\qquad\cdot\exp\Big\{-c_n^2(1+c_n^2v^2)^{-1}\langle M_n^{-1}O(\eta+\Lambda^*z),O(\eta+\Lambda^*z)\rangle\Big\}
\end{aligned}$$

By well-known formulas;

$$\begin{aligned}
&\prod_{n=1}^{\infty}(1+v^2c_n^2)^{-\frac{d}{2}}(\det M_n)^{-\frac{1}{2}}=\prod_{i=1}^{d}\prod_{n=1}^{\infty}\Big(\frac{1}{1+v_i^\lambda c_n^2}\Big)^{\frac{1}{2}}=\prod_{i=1}^{d}\Big(\frac{\sqrt{v_i^\lambda}}{\sinh\sqrt{v_i^\lambda}}\Big)^{\frac{1}{2}}=G(\lambda),\\
&\sum_{n=1}^{\infty}\frac{c_n^2}{1+c_n^2v^2}M_n^{-1}=\mathrm{diag}\left(\sum_{n=1}^{\infty}\frac{1}{\pi^2n^2+v_i^\lambda}\right)=\mathrm{diag}\left(\frac{\sqrt{v_i^\lambda}\coth\sqrt{v_i^\lambda}-1}{2v_i^\lambda}\right)=F(\lambda),
\end{aligned}$$

we have the following.

$$\mathbb{E}\Big[\exp\Big\{\sqrt{-1}\sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_{ij}A^{ij}+\sqrt{-1}\sum_{j=1}^{d}(\eta_jU^j+\zeta_jW_1^j)\Big\}\Big]$$

$$
\begin{aligned}
&= (\sqrt{2\pi})^{-d} \int_{\mathbb{R}^d} \exp\Big\{\sqrt{-1}\langle \zeta, z\rangle - \frac{1}{2}\langle z, z\rangle\Big\} \prod_{n=1}^{\infty} L_n(\lambda, \eta : z)\, dz \\
&= (\sqrt{2\pi})^{-d} G(\lambda) \exp\Big\{ - \langle H(\lambda)\eta, \eta\rangle\Big\} \\
&\qquad \cdot \int_{\mathbb{R}^d} \exp\Big\{\sqrt{-1}\langle \zeta, z\rangle - \frac{1}{2}\langle (I_d + 2\Lambda H(\lambda)\Lambda^*)z, z\rangle - 2\langle \Lambda H(\lambda)\eta, z\rangle\Big\}\, dz \\
&= \frac{G(\lambda)}{\sqrt{\det(I_d + 2\Lambda H(\lambda)\Lambda^*)}} \exp\Big\{2\langle (I_d + 2\Lambda H(\lambda)\Lambda^*)^{-1}\Lambda H(\lambda)\eta, \Lambda H(\lambda)\eta\rangle - \langle H(\lambda)\eta, \eta\rangle \\
&\qquad - 2\sqrt{-1}\langle (I_d + 2\Lambda H(\lambda)\Lambda^*)^{-1}\Lambda H(\lambda)\eta, \zeta\rangle - \frac{1}{2}\langle((I_d + 2\Lambda H(\lambda)\Lambda^*)^{-1}\zeta, \zeta\rangle\Big\}.
\end{aligned}
$$

□

Bibliography

[1] Aihara, H., Akahori, J., Fujii, H. and Nitta, Y. : "Tau functions of KP solitons realized in Wiener space", Bull. London Math. Soc. 45 (2013), no. 6, 1301–1309.

[2] Akahori, J., Matsushita, T. and Nitta, Y.: "A realization of Sato's Grassmanian in Wiener space", preprint, Ritsumeikan University, 2010.

[3] Aoyama, T. and Nakamura, T.: "Behaviors of multivariable finite Euler products in probabilistic view", Math. Nachr. 286 (2013), no. 17-18, 1691–1700.

[4] Aoyama, T. and Nakamura, T.: "Multidimensional polynomial Euler products and infinitely divisible distributions on $\mathbb{R}^d$", submitted, http://arxiv.org/abs/1204.4041, 2013.

[5] Aoyama, T. and Nakamura, T.: "Multidimensional Shintani zeta functions and zeta distributions on $\mathbb{R}^d$", Tokyo J. Math. 36 (2013), 521–538.

[6] Aoyama, T. and Yoshikawa, K.: "Some examples of multidimensional Shintani zeta distributions", JSIAM Letters, vol. 6 (2014), 41–44.

[7] Aoyama, T. and Yoshikawa, K.: "Multinomial distributions in Shintani zeta class", to appeared in Japan Journal of Industrial and Applied Mathematics, 2014.

[8] Aoyama, T. and Yoshikawa, K.: "Characteristic functions of discrete distributions on $\mathbb{R}^d$", preprint, 2014.

[9] Apostol, T. M.: *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[10] Apostol, T. M.: *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Mathematics 41, Springer-Verlag, New York, 1990.

[11] Baker, A.: *Transcendental Number Theory*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1975.

[12] Gnedenko, B.V. and Kolmogorov, A.N.: *Limit Distributions for Sums of Independent Random Variables* (translated from the Russian by K. L. Chung), Addison-Wesley, 1968.

[13] Helmes, K. and Schwane, A.: "Lévy's stochastic area formula in higher dimensions", J. Funct. Anal. 54 (1983), 177–192.

[14] Hu, C.-Y., Iksanov, A.M., Lin, G.D. and Zakusylo, O.K.: "The Hurwitz zeta distribution", Aust. N. Z. J. Stat. 48 (2006), 1–6.

[15] Jessen, B. and Wintner, A.: "Distribution functions and the Riemann zeta function", Trans. Amer. Math. Soc. 38 (1935), 48–88.

[16] Khinchine, A.Ya.: *Limit Theorems for Sums of Independent Random Variables* (in Russian), Moscow and Leningrad, 1938.

[17] Kloeden, P.E. and Platen, E.: *Numerical Solution of Stochastic Differential Equations*, Application of Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[18] Kusuoka, S.: "Approximation of expectation of diffusion processes based on Lie algebra and Malliavin calculus", Advances in mathematical economics, vol. 6 (2004), 69–83.

[19] Kusuoka, S. and Ninomiya, S.: "A new simulation method of diffusion processes applied to finance", Stochastic processes and applications to mathematical finance (2004), 233–253, World Sci. Publ., River Edge, NJ.

[20] Lyons, T., and Victoir, N.: "Cubature on Wiener space", Stochastic analysis with applications to mathematical finance. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 460 (2004), no. 2041, 169–198.

[21] Meyer, P.A.: *Quantum Probability for Probabilists*, 2nd edition, Lecture Notes in Mathematics, vol. 1538, Springer-Verlag, Berlin, Heidelberg, New York, 1995.

[22] Nakamura, T.: "The joint universality and the generalized strong recurrence for Dirichlet L-functions", Acta Arith. 138 (2009), no. 4, 357–362.

[23] Ninomiya, M. and Ninomiya, S.: "A new higher-order weak approximation scheme for stochastic differential equations and the Runge-Kutta method", Finance Stoch. 13 (2009), no. 3, 415–443.

[24] Ninomiya, S. and Victoir, N.: "Weak approximation of stochastic differential equations and application to derivative pricing", Appl. Math. Finance 15 (2008), no. 1-2, 107–121.

[25] Sato, K.: *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.

[26] Yoshikawa, K.: "An approximation scheme for diffusion processes based on an antisymmetric calculus on Wiener space", Asia-Pacific Financial Markets, published online, http://link.springer.com/article/10.1007/s10690-014-9199-2, 2014.