2013 (Heisei 25)
Doctoral Thesis

Analysis of Wiener- and Poisson- space using representations of Lie algebras

Doctoral Program in Integrated Science and Engineering
Graduate School of Science and Engineering
Ritsumeikan University

Takafumi Amaba

Preface.

In probability theory, Brownian noises and Poisson noises play fundamental roles when we consider classical noises (for example, Lévy noises or, more generally, noises generated by Markov-type stochastic differential equations which admit strong solutions). One of the reason is the fact that with such noises, several important quantities can be computed explicitly. Moreover, any Brownian noise and any Poisson noise (with non-random intensity) are automatically independent, and hence we may think that the underlying probability space splits into the direct product of a probability space supporting Brownian noises and a space supporting Poisson noises. On each of these spaces, Brownian motions or stationary Poisson point processes can be regarded as a system of infinite dimensional "coordinates". In fact, such circumstance seems to affect the Itô-Lévy decomposition theorem and (also/hence) the framework of the Malliavin calculus for Lévy processes. Thus, for the study of classical noises, it is enough to investigate its Brownian component and Poissonian component separately.

The utilities of the two noises are more than that. They satisfy the "consistency", by which we mean that a Brownian motion or a stationary Poisson point process (more strictly, their laws) can be viewed as a sort of "inverse limits" of a "projective systems" (with respect to a class of conditional expectations), which also appears as an aspect of "infinite divisibility", and is stronger than Kolmogorov's consistency condition for construction of Markov processes. If we speak only on Brownian motion $B = (B_t)_{0\le t\le T}$, it can be understood that the "consistency" implies that any finite dimensional Euclidean space $\mathbb{R}^n$ is "embedded" into the probability space (Wiener space) by folding, i.e., we map $(x^1, \cdots, x^n) \mapsto (x^1 + \cdots + x^k)_{k=1}^n$. Conversely, by "spreading" $B = (B_t)_{0\le t\le T}$ out finitely, we obtain a system $(\Delta B_1, \cdots, \Delta B_n)$ of a part of orthogonally stacked "coordinates". As far as the case of continuous motion, a noise with these properties is essentially unique (except for trivial noise), and is the Brownian noise, which has been stated as in the Itô-Lévy decomposition theorem. In the case of Poisson noises, the corresponding "coordinates" takes a bit different form: They will take its values in a space of measures.

It is known that there are (essentially equivalent) representations of the Heisenberg algebra on Brownian noises and Poisson noises. In particular, the action of Heisenberg algebra is inherited, because of the "consistency", even when we are in the space charted by the system of orthogonally stacked "coordinates", and equivalently, even when we discretize (in time) the framework of Malliavin calculus. We will employ this property in our framework. Although it appears that our framework depends strongly on these nature and thus is restrictive, but it covers several important objects such as the Euler-Maruyama scheme for stochastic differential equations, and ultimately by taking the limiting, everything described by Brownian noises and Poisson noises.

In this thesis, we give applications of representations of the Heisenberg algebra. The study is divided into two parts. In Part 1, we study the change of variable formula on the classical Wiener space, which is called the Ramer-Kusuoka formula. We will see that the Ramer-Kusuoka formula can be described as a formula in the ring of formal power series with the coefficients in a (generalized) Heisenberg algebra. Although the arguments are limited only on the classical Wiener space, the formula would describe also the Girsanov formula on the Poisson space. In that sense, our formula has to unify both the change of variable formulae on the Wiener and Poisson spaces. Part 2 is devoted to study a discrete version of Clark-Ocone formulae. The Clark-Ocone formula is a stochastic version of the fundamental theorem of calculus, which is also an explicit expression of the martingale representation theorem. It is an important problem to ask whether or not a given noise has martingale representation property, that is, whether it has a finite number of martingale basis. The Brownian noises and Poissonian noises have the martingale representation property, however, when we discretize the noises, this property fails. This is the starting point of our study. Because we are always in separable Hilbert spaces, so we have countably many martingale basis, and in fact, our discrete Clark-Ocone formula will use these countable basis. After we establish the discrete Clark-Ocone formula, we will see how the superfluous bases tend to vanish, when we take infinitesimally small partitions of the time interval. Such studies will be designed as the error analysis for martingale representation error.

Finally, I want to mention further research directions, in the case of continuous models. The framework presented here, and even that of Malliavin calculus does not cover analyses for stochastic differential equations which doesn't admit any strong solutions since a solution to such equation is not a function of only the driving Brownian motion in general. Such solutions might be described completely by the driving Brownian motion and some additional noises suitably correlated with the driving noise, and thus it seems to be impossible to apply, in principle, the Malliavin calculus via methods which are already established. I believe, at least in the case where the stochastic differential equation has symmetries, that there are frameworks, broader than that of Malliavin calculus, in which we can deal with stochastic differential equations with non-strong solutions as mentioned above, and there are discretization techniques which keep the structure of symmetries or "Galois group" of the stochastic differential equation.

Acknowledgments

I am deeply grateful to Prof. Jirô Akahori, my Ph.D. supervisor, for his guidances and insightful comments, Prof. Kazufumi Nakajima, my undergraduate supervisor who have taught me many fundamentals of mathematics, which, even now are still essential parts of my research, and also Prof. Kazuhiro Kuwae and Assoc. Prof. Kazumasa Kuwada. I am also in debt to Assis. Prof. Takahiro Aoyama and Libo Li who gave me many useful advices, in particular, the conventions and rules of the academic society, and English writting. Furthermore, I am obliged to Shigeki Yanase, whose sudden passing has gravely sadden us all, for his profitable seminar on Galois theory, at which I spent my first time as a tutor. Since that time, I began to consider the possibility of constructing Galois theory for stochastic differential equations. However, most of all, I must thank specially Takehisa Hara, S. Matsuse (he finally couldn't tell me his last name) and Shinji Nakazato, all of who have taught me elementary mathematics throughout my youth and triggered my interest to study mathematics.

December 10, 2013

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Part 1

Change of Variable Formula on the Wiener Space

For each bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$ and smooth transformation $z : \mathbb{R}^n \to \mathbb{R}^n$, it is elementary to deduce the change of variable formula

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\, \frac{e^{-\frac{|x|^2}{2\Delta t}}}{(2\pi\Delta t)^{n/2}}\, \mathrm{d}x \\ = \int_{\mathbb{R}^n} f(x - \Delta t\, z(x)) \left|\det(1 - J_{\Delta t\, z}(x))\right| \\ \times \exp\left\{\langle z(x), x\rangle - \frac{|z(x)|^2}{2}\Delta t\right\} \frac{e^{-\frac{|x|^2}{2\Delta t}}}{(2\pi\Delta t)^{n/2}}\, \mathrm{d}x,\end{aligned} \tag{0.1}$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are the canonical inner product on $\mathbb{R}^n$ and the associated norm respectively, $J_{\Delta t\, z}$ is the Jacobian matrix of $\Delta t\, z$ given by

$$J_{\Delta t\, z}(x) = \begin{pmatrix} \dfrac{\partial z^1}{\partial x^1} & \cdots & \dfrac{\partial z^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial z^n}{\partial x^1} & \cdots & \dfrac{\partial z^n}{\partial x^n} \end{pmatrix} \Delta t,$$

$z = (z^1, \cdots, z^n)$ and Δt is an arbitrary positive constant.

On the other hand, the Wiener process $W = (W_t)_{0 \le t \le 1}$ is defined by

$$(W_t)(w) = w_t, \quad 0 \le t \le 1, \quad w \in \mathscr{W} = C([0,1] \to \mathbb{R}).$$

Any equidistant partition $\Delta : 0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval $[0,1]$ induces a mapping

$$(\Delta W_1, \cdots, \Delta W_n) : \mathscr{W} \to \mathbb{R}^n \tag{0.2}$$

where $\Delta W_l = W_{t_l} - W_{t_{l-1}}$.

The change of variable formula (0.1) with $\Delta t = 1/n$ can be pulled-back onto the Wiener space $\mathscr{W}$ by the mapping (0.2). Furthermore, one can take the limit $n \to +\infty$ in the pulled-back formula, and the resulting formula gives a change of variable formula on the Wiener space.

Although formula (0.1) is a step before taking the limit, it indicates several aspects of the change of variable formula on the Wiener space. From the definition of the map (0.2), it seems natural to regard $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$ as the process $X = (X_l)_{l=0}^n$ defined by $X_0 = 0$ and $X_l = x^1 + \cdots + x^l$ for $l = 1, 2, \cdots, n$. The filtration generated by the process X coincides with the coordinate filtration $\mathcal{F} = (\mathcal{F}_l)_{l=0}^n$ defined by $\mathcal{F}_0 = \{\emptyset, \mathbb{R}^n\}$ and $\mathcal{F}_l = \sigma(x^1, \cdots, x^l)$ for $l = 1, 2, \cdots, n$. Then the transformation $z\Delta t$ can be naturally identified with a random process

$Z = (Z_l)_{l=0}^n$ given by

$$Z_l = \int_0^{t_l} \sum_{k=1}^n 1_{\{t_{k-1} \le s < t_k\}} z^k \, \mathrm{d}s = \sum_{k=1}^l \dot{Z}_k \Delta t$$

where $\dot{Z}_l := z^l$. Under these notations, roughly speaking, the change of variable formula on the Wiener space is called

– *Cameron-Martin formula*: when z is a constant map, i.e., non-random.

– *Cameron-Martin-Maruyama-Girsanov formula*: when Z is a $\mathcal{F}$-predictable, or equivalently,

$$\frac{\partial z^l}{\partial x^k} = 0 \quad \text{if } l \le k.$$

If this is the case, the Jacobian matrix $J_{\Delta t\, z}$ is nilpotent, so that $\det(1 - J_{\Delta t\, z}) \equiv 1$. Moreover, under the identification $x = (\Delta W_1, \cdots, \Delta W_n)$,

$$\langle z(x), x\rangle = \sum_{l=1}^n \dot{Z}_l \, \Delta W_l = \int_0^1 \dot{Z}_s \, \mathrm{d}W_s,$$

$$|z(x)|^2 \Delta t = \sum_{l=1}^n \dot{Z}_l^2 \, \Delta t = \int_0^1 \dot{Z}_s^2 \, \mathrm{d}s$$

where, in the last equalities of each above line, we identify discrete-time processes with continuous-time processes which are piecewise constant.

– *Ramer-Kusuoka formula*: when z is generic. This being the case, $\langle z(x), x\rangle$ is understood using the notions of the Skorohod integral $\int \dot{Z}\, \delta W$ or the Ogawa integral $\int \dot{Z} \, {*\mathrm{d}W}$ as

$$\langle z(x), x\rangle = \int_0^1 \dot{Z}_s \, \delta W_s + \mathrm{tr}(J_Z) = \int_0^1 \dot{Z}_s \, {*\mathrm{d}W_s},$$

where J_Z is the Jacobian matrix of $(Z_1, \cdots, Z_n) : \mathbb{R}^n \to \mathbb{R}^n$.

After the limiting procedure, one has that, for a differentiable random process $Z = (Z_t)_{0 \le t \le 1}$ (which need not to be adapted to the natural filtration of W) and bounded measurable $F : \mathscr{W} \to \mathbb{R}$,

$$\begin{aligned} &\mathbf{E}[F(W)] \\ &= \mathbf{E}\Big[F(W - Z)\,|\det(1 - DZ)|\,\mathrm{e}^{\mathrm{tr}\, DZ} \exp\Big\{\int_0^1 \dot{Z}_s \, \delta W_s - \frac{1}{2}\int_0^1 \dot{Z}_s^2 \, \mathrm{d}s\Big\}\Big], \end{aligned} \tag{0.3}$$

where $\mathbf{E}$ means the expectation with respect to the Wiener measure $\mathbf{P}$. This is a general form of the change of variable formula on the Wiener space, and is called the *Ramer-Kusuoka* formula.

Originally, such a change of variable formula (0.3) is studied by Cameron and Martin in [**13**] when $Z \equiv \theta \in H$ is a non-random path, where H is the subspace of $\mathscr{W}$ consisting of all paths h with square integrable derivative and $h_0 = 0$. The space H is now called the *Cameron-Martin subspace*. Their work was extended by Gross [**20**] and Kuo [**26**] in the framework of more general abstract Wiener spaces.

For another generalization, Girsanov [**18**], Maruyama [**33**], [**34**] and Motoo [**37**] studied the case where Z is an adapted process and $Z \in H$ a.s. from a viewpoint of stochastic differential equation and showed that the Itô integral appeared in the density function. In this case, the formula (0.3) is simplified to the Cameron-Martin-Maruyama-Girsanov formula

$$\mathbf{E}[\,F(W)\,] = \mathbf{E}\Big[\,F(\,W - Z\,)\,\exp\Big\{\int_0^1 \dot{Z}_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^1 \dot{Z}_s^2\,\mathrm{d}s\Big\}\Big] \tag{0.4}$$

as explained before.

Ramer [**47**] studied the case where Z is a non-adapted random process and deduced the formula (0.3). He introduced an abstract version of the Itô integral which is called the *Itô-Ramer integral* in [**27**] and he showed that the density factorizes into two factors. One is the Carleman-Fredholm determinant $\det(\,1 - DZ\,)\mathrm{e}^{\mathrm{tr}\,DZ}$ of the operator $1 - DZ$ (1 denotes the identity map) and the other is the Girsanov type density in which, because of non-adaptedness, the Itô integral is replaced by the Itô-Ramer integral or the *Skorohod integral* from a point of view of the Malliavin calculus. For an extension of applicable class, this result is generalized by Kusuoka [**27**].[1]

Zakai [**61**] characterized the class of Z for which the Carleman-Fredholm determinant is equal to one by using quasi-nilpotency and explained how the Ramer-Kusuoka formula (0.3) is reduced to the Maruyama-Girsanov formula (0.4).

As a particular case, Buckdahn and Föllmer [**6**] studied the law of the solution of *anticipative* stochastic differential equation of the form $\mathrm{d}\xi_t = \mathrm{d}W_t + k_t(\xi, W)\mathrm{d}t$ where the drift $k_t(\xi, \omega)$ depends on the past behavior of ξ and the future behavior of the Brownian motion W. Yano [**60**] studied the composition of functional on an abstract Wiener space taking its value in a finite dimensional vector space and the Ramer type translation on an extended abstract Wiener space.

[1] In [**47**] and [**27**], the authors worked on abstract Wiener spaces. If we want to write the first factor as just a Fredholm determinant rather than the Carleman-Fredholm determinant, one will get an expression with using the *Ogawa integral* under some integrability condition.

CHAPTER 1

Cameron-Martin-Maruyama-Girsanov Formula via an Action of Heisenberg Algebra

This part is based on the joint work [**4**].

1. Introduction

Let $(\mathscr{W}, \mathcal{B}(\mathscr{W}), \mathbf{P})$ be the Wiener space on the interval $[0,1]$, that is, $\mathscr{W}$ is the set of all continuous paths in $\mathbb{R}$ defined on $[0,1]$ which starts from zero, $\mathcal{B}(\mathscr{W})$ is the σ-field generated by the topology of uniform convergence. and $\mathbf{P}$ is the Wiener measure on the measurable space $(\mathscr{W}, \mathcal{B}(\mathscr{W}))$. Then the canonical Wiener process $(W(t))_{t\geq 0}$ is defined by $W(t,w) = w(t)$ for $0 \leq t \leq 1$ and $w \in \mathscr{W}$.

Let H denote the Cameron-Martin subspace of $\mathscr{W}$, i.e., $h \in \mathscr{W}$ belongs to H if and only if $h(t)$ is absolutely continuous in t and the derivative $\dot{h}(t)$ is square-integrable. Note that H is a Hilbert space under the inner product

$$\langle h_1, h_2 \rangle_H = \int_0^1 \dot{h}_1(t)\dot{h}_2(t)\,\mathrm{d}t, \qquad h_1, h_2 \in H.$$

It is a fundamental fact in stochastic calculus that the Cameron-Martin (henceforth CM) formula (see, e.g. [**32**], pp 25) in the following form holds:

$$\int_{\mathscr{W}} F(w+\theta)\,\mathbf{P}(\mathrm{d}w) = \int_{\mathscr{W}} F(w) \exp\Big\{ \int_0^1 \dot{\theta}(t)\mathrm{d}w(t) - \frac{1}{2}\int_0^1 \dot{\theta}(t)^2 \mathrm{d}t \Big\} \mathbf{P}(\mathrm{d}w) \tag{1.1}$$

where F is a bounded measurable function on $\mathscr{W}$ and $\theta \in H$.

The motivation of the present study comes from the following observation(s). In the above CM formula (1.1), the integrand of the left-hand-side can be seen as an action of a translation operator, which is an exponentiation of a differentiation D_θ:

$$\int_{\mathscr{W}} F(w+\theta)\,\mathbf{P}(\mathrm{d}w) \ \text{“}=\text{”}\ \mathbf{E}[e^{D_\theta} F]. \tag{1.2}$$

On the other hand, the right-hand-side can be seen as a "coupling" of the exponential martingale and F:

$$\int_{\mathscr{W}} F(w) \exp\Big\{ \int_0^1 \dot\theta(t) \mathrm{d}w(t) - \frac{1}{2} \int_0^1 \dot\theta(t)^2 \mathrm{d}t \Big\} \mathbf{P}(\mathrm{d}w) = \Big\langle F, \exp\Big\{ \int_0^1 \dot\theta(t) \mathrm{d}W(t) - \frac{1}{2} \int_0^1 \dot\theta(t)^2 \mathrm{d}t \Big\} \Big\rangle.$$

Since we can read the right-hand-side of (1.2) as

$$\mathbf{E}[\mathrm{e}^{D_\theta} F] \text{ “=” } \Big\langle 1, \mathrm{e}^{D_\theta} F \Big\rangle,$$

the Cameron-Martin formula

$$\Big\langle 1, \mathrm{e}^{D_\theta} F \Big\rangle \text{ “=” } \Big\langle F, \exp\Big\{ \int_0^1 \dot\theta(t) \mathrm{d}W(t) - \frac{1}{2} \int_0^1 \dot\theta(t)^2 \mathrm{d}t \Big\} \Big\rangle$$

leads to the following interpretation:

$$\exp\Big\{ \int_0^1 \dot\theta(t) \mathrm{d}W(t) - \frac{1}{2} \int_0^1 \dot\theta(t)^2 \mathrm{d}t \Big\} \text{ “=” } \mathrm{e}^{D_\theta^*}(1),$$

where D_θ^* is an "adjoint operator" of D_θ.

The observation, conversely, suggests that the CM formula could be proved directly by the duality relation between e^{D_θ} and $\mathrm{e}^{D_\theta^*}$, without resorting to the stochastic calculus. The program is successfully carried out in section 2. We may say this program runs by the calculus of functionals of Wiener integrals.

Along the line, we also give an algebraic proof of the Maruyama-Girsanov (henceforth MG) formula (see e.g. [**50**, IV.38, Theorem (38.5)]), an extension of the CM formula. Note that MG formula cannot be written in the quasi-invariant form as (1.1), but in the following way:

$$\begin{aligned} &\int_{\mathscr{W}} F(w) \mathbf{P}(\mathrm{d}w) \\ &= \int_{\mathscr{W}} F(w - Z(w)) \exp\Big\{ \int_0^1 \dot Z(t,w) \mathrm{d}w(t) - \frac{1}{2} \int_0^1 \dot Z(t,w)^2 \mathrm{d}t \Big\} \mathbf{P}(\mathrm{d}w). \end{aligned} \tag{1.3}$$

Here $Z : \mathscr{W} \to H$ is a "predictable" map such that

$$\int_{\mathscr{W}} \exp\Big\{ \int_0^1 \dot Z(t,w) \mathrm{d}w(t) - \frac{1}{2} \int_0^1 \dot Z(t,w)^2 \mathrm{d}t \Big\} \mathbf{P}(\mathrm{d}w) = 1.$$

In this non-linear situation, infinite dimensional vector fields like $X_Z \equiv Z^i D_{e_i}$[1], where $\{e_i\}$ is a basis of H and $Z^i = \langle Z, e_i \rangle_H$, may play a role of D_θ in the linear case,

[1] Here we use Einstein's convention.

but its exponentiation e^{X_Z} does not make sense anymore. Instead, we need to consider "tensor fields"

$$D_Z^{\otimes n} = Z^{i_1} \cdots Z^{i_n} D_{e_{i_1}} \cdots D_{e_{i_n}}$$

and its formal series

$$\sum_{n=0}^{\infty} \frac{1}{n!} D_Z^{\otimes n} =: \widetilde{\mathrm{e}}^{D_Z}.$$

We will show in Proposition 3.1.2 that the operator $\widetilde{\mathrm{e}}^{D_Z}$ is the translation by Z; $\widetilde{\mathrm{e}}^{D_Z}(f(W)) = f(W+Z)$. To understand MG formula (1.3) in terms of the translation operator $\widetilde{\mathrm{e}}^{D_Z}$, we additionally introduce another sequence $\{L_n\}$ of tensor fields (see subsection 3.2 for the definition), which has the property (Lemma 3.3.1) of

$$\sum_{n=1}^{\infty} \frac{1}{n!} L_n = \exp\Big\{ \int_0^1 \dot{Z}(t) \mathrm{d}w(t) - \frac{1}{2} \int_0^1 \dot{Z}^2(t) \mathrm{d}t \Big\} (\widetilde{\mathrm{e}}^{D_Z} - 1).$$

Then, as a corollary to the adjoint formula for L_n (Theorem 3.2.1), MG formula can be obtained (Corollary 3.3.2).

The proof of key theorem (Theorem 3.2.1), however, is not "algebraic" since it involves the use of Itô's formula. This means, we feel, a considerable part of the "algebraic structure" of MG formula is still unrevealed. We then try to give a purely algebraic proof (=without resorting the results from the stochastic calculus) to MG formula in section 4 at the cost that we only consider the case where $\dot{Z}$ is a simple predictable process such as

$$\dot{Z} = \sum_{i=1}^{N} z_i 1_{(t_i, t_{i+1}]}(t).$$

We will consider a family of vector fields like $z_i D_i$, where D_i is the differentiation in the direction of $\int 1_{(t_i,t_{i+1}]}(t)\,\mathrm{d}t$. A key ingredient in our (second) algebraic proof of MG formula is the following semi-commutativity:

$$z_i D_j = D_j z_i \quad \text{if } j \geq i, \tag{1.4}$$

which may be understood as "causality".

Actually, the relation (1.4) implies that the Jacobian matrix $DZ = (D_{e_i} Z_j)_{ij}$, if it is defined, is upper triangular. In a coordinate-free language, it is nilpotent. Equivalently, $\mathrm{Tr}(DZ)^n = 0$ for every n, or $\mathrm{Tr} \wedge^n DZ = 0$ for every n. Since the statements are coordinate-free(=independent of the choice of $\{e_i\}$), they can be a characterization of the causality (=predictability) in the infinite dimensional setting as well. This observation retrieves the result in **[61]** that Ramer-Kusuoka formula (**[47]**,**[27]**) is reduced to MG formula when DZ is nilpotent in this sense. The observation also implies that Ramer-Kusuoka formula itself can be approached in our algebraic way. This program has been successfully carried out in [**3**].

Throughout this chapter, the domains of the operators are basically restricted to "polynomials" (precise definition of which will be given soon) in order to concentrate on algebraic structures. We leave in section 5 a lemma and its proof to ensure the continuity of the operators and hence to have a standard version of CM-MG formula.

To the best of our knowledge, an algebraic proof like ours for CMMG formula have never been proposed. Although we only treat a simplest one-dimensional Brownian case, our method can be applied to more general cases if only they have a proper action of the infinite dimensional Heisenberg algebra. The present study is largely motivated by P. Malliavin's way to look at stochastic calculus, which for example appears in [**32**] and [**31**], and also by some operator calculus often found in the quantum fields theory (see e.g. [**36**]).

2. An Algebraic Proof of the Cameron-Martin Formula

2.1. Preliminaries. For any $h \in H$, we set

$$[h](w) := \int_0^1 \dot{h}(t) \mathrm{d}w(t), \quad w \in \mathscr{W}.$$

A function $F : \mathscr{W} \to \mathbb{R}$ is called a *polynomial functional* if there exist an $n \in \mathbb{N}$, $h_1, h_2, \cdots, h_n \in H$ and a polynomial $p(x_1, x_2, \cdots, x_n)$ of n-variables such that

$$F(w) = p\Big([h_1](w), [h_2](w), \cdots, [h_n](w) \Big), \quad w \in \mathscr{W}.$$

The set of all polynomial functionals is denoted by $\mathcal{P}$. This is an algebra over $\mathbb{R}$ included densely in $L^p(\mathscr{W})$ for any $1 \le p < \infty$ (see e.g. [**24**], pp 353, Remark 8.2).

Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of H. If we set

$$\xi_i(w) := [e_i](w) = \int_0^1 \dot{e}_i(t) \mathrm{d}w(t), \qquad i = 1, 2, \cdots$$

then $\xi_1, \xi_2, \cdots$ are mutually independent standard Gaussian random variables. Let $H_n[\xi]$, $n = 1, 2, \cdots$ be the n-th Hermite polynomial in ξ defined by the generating function identity

$$\exp\Big(\lambda\xi - \frac{\lambda^2}{2}\Big) = \sum_{n=0}^\infty \frac{\lambda^n}{n!} H_n[\xi], \quad \lambda \in \mathbb{R},$$

and put

$$\mathbf{\Lambda} := \left\{ \mathbf{a} = (a_i)_{i=1}^\infty : \begin{array}{l} a_i \in \mathbb{Z}^+, \\ a_i = 0 \text{ except for a finite number of } i\text{'s} \end{array} \right\}.$$

We write $\mathbf{a}! := \prod_{i=1}^{\infty} a_i!$ for $\mathbf{a} = (a_i)_{i=1}^{\infty} \in \mathbf{\Lambda}$. We define $H_{\mathbf{a}}(w) \in \mathbf{P}$, $\mathbf{a} \in \mathbf{\Lambda}$ by

$$H_{\mathbf{a}}(w) := \prod_{i=1}^{\infty} H_{a_i}[\xi_i(w)], \quad w \in \mathscr{W}$$

and then $\{\frac{1}{\sqrt{\mathbf{a}!}} H_{\mathbf{a}} : \mathbf{a} \in \mathbf{\Lambda}\}$ forms an orthonormal basis of $L^2(\mathscr{W})$ (see e.g. **[24]**).

For a differentiable function f on $\mathbb{R}$ measured by $N_1(\mathrm{d}\xi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\xi^2/2} \mathrm{d}\xi$, if we define ∂ and ∂^* as

$$\partial f(\xi) = f'(\xi) \text{ and } \partial^* f(\xi) = -\partial f(\xi) + \xi f(\xi), \quad \xi \in \mathbb{R}$$

then ∂^* is adjoint to ∂ on the differentiable class in $L^2(\mathbb{R}, N_1)$. We note that the n-th Hermite polynomial H_n can be given by $H_n[\xi] = (\partial^{*n} 1)(\xi)$.

2.2. Directional Differentiations and its Exponentials. For a function F on $\mathscr{W}$ and $\theta \in H$, the differentiation of F in the direction θ, $D_\theta F$ is defined by

$$D_\theta F(w) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big\{ F(w + \varepsilon\theta) - F(w) \Big\}, \quad w \in \mathscr{W}$$

if it exists(see e.g. **[24]**). Note that $D_\theta F(w)$ is linear in θ and F and satisfies the Leibniz' formula $D_\theta(FG)(w) = D_\theta F(w) \cdot G(w) + F(w) D_\theta G(w)$ for functions F and G on $\mathscr{W}$ such that $D_\theta F(w)$ and $D_\theta G(w)$ exist. If $F(w)$ is of the form $F(w) = f([h](w))$ where f is a differentiable function on $\mathbb{R}$ and $h \in H$, then we have

$$D_\theta F(w) = \langle \theta, h \rangle_H f'([h](w)). \tag{1.5}$$

For $\theta \in H$, we define the *exponential* of D_θ by

$$\mathrm{e}^{D_\theta} F(w) := \sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n F(w), \quad F \in \mathcal{P} \text{ and } w \in \mathscr{W}$$

which is actually a finite sum by (1.5).

Lemma 2.2.1. *For $F, G \in \mathcal{P}$, we have*

$$\mathrm{e}^{D_\theta}(FG) = \mathrm{e}^{D_\theta}(F) \cdot \mathrm{e}^{D_\theta}(G). \tag{1.6}$$

Proof. is a straightforward computation:

$$\begin{aligned}
\mathrm{e}^{D_\theta}(F)\cdot\mathrm{e}^{D_\theta}(G) &= \Big(\sum_{n=0}^{\infty}\frac{1}{n!}D_\theta^n F\Big)\cdot\Big(\sum_{n=0}^{\infty}\frac{1}{n!}D_\theta^n G\Big)\\
&= \Big(F + D_\theta F + \frac{1}{2!}D_\theta^2 F + \frac{1}{3!}D_\theta^3 F + \cdots\Big)\\
&\qquad\cdot\Big(G + D_\theta G + \frac{1}{2!}D_\theta^2 G + \frac{1}{3!}D_\theta^3 G + \cdots\Big)\\
&= FG + \Big\{D_\theta F\cdot G + F D_\theta G\Big\}\\
&\qquad + \Big\{\frac{1}{2!}D_\theta^2 F\cdot G + D_\theta F\cdot D_\theta G + F\cdot\frac{1}{2!}D_\theta^2 G\Big\}\\
&\quad + \Big\{\frac{1}{3!}D_\theta^3 F\cdot G + \frac{1}{2!}D_\theta^2 F\cdot D_\theta G + D_\theta F\cdot\frac{1}{2!}D_\theta^2 G + F\cdot\frac{1}{3!}D_\theta^3 G\Big\}\\
&\qquad + \cdots\\
&= FG + D_\theta(FG) + \frac{1}{2!}D_\theta^2(FG) + \frac{1}{3!}D_\theta^3(FG) + \cdots = \mathrm{e}^{D_\theta}(FG).
\end{aligned}$$

□

Proposition 2.2.2. *For every $F \in \mathcal{P}$, we have*

$$\mathrm{e}^{D_\theta}F(w) = F(w+\theta), \quad w \in \mathscr{W}. \tag{1.7}$$

Proof. By Lemma 2.2.1, it suffices to show (1.7) for the functional $F \in \mathcal{P}$ of the form $F(w) = f([h](w))$ where $f(x)$ is a polynomial in one-variable and $h \in H$. Then using (1.5), we obtain

$$\begin{aligned}
\mathrm{e}^{D_\theta}F(w) &= \sum_{n=0}^{\infty}\frac{1}{n!}D_\theta^n f(\,[h](w)\,) = \sum_{n=0}^{\infty}\frac{1}{n!}\langle\theta,h\rangle_H^n f^{(n)}(\,[h](w)\,)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(\,[h](w)\,)\Big\{\Big([h](w)+\langle\theta,h\rangle_H\Big) - [h](w)\Big\}^n\\
&= f\Big(\,[h](w)+\langle\theta,h\rangle_H\Big) = F(w+\theta),
\end{aligned}$$

where $f^{(n)}(x)$ denotes the n-th derivative of $f(x)$. □

2.3. Formal Adjoint Operator and its Exponential. In the analogy of ∂ and ∂^* in the previous section, we define D_θ^*, $\theta \in H$ by

$$D_\theta^* F(w) := -D_\theta F(w) + \int_0^1 \dot\theta(t)\mathrm{d}w(t)\cdot F(w), \quad F \in \mathcal{P},\ w \in \mathscr{W}.$$

Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of H and put $\xi_i(w) := [e_i](w)$ for $i = 1, 2, \cdots$. Then we have

LEMMA 2.3.1. *It holds that*

$$\mathbf{E}\Big[D_\theta H_n[\xi_k]\cdot H_m[\xi_l]\Big]=\mathbf{E}\Big[H_n[\xi_k]D^*_\theta H_m[\xi_l]\Big]$$

for any $k,l,m,n=1,2,\cdots$.

PROOF. Since $t\mapsto H_n[\int_0^t e_k(s)\mathrm{d}w(s)]$ $(n\geq 1)$ is a martingale with initial value zero, if $k\neq l$ the independence of ξ_k and ξ_l and the formula (1.5) imply that both sides become zero when $n,m\geq 1$. If $n=m=0$, it is clear that the left-hand side is zero. Then the right-hand side equals to

$$\mathbf{E}[\,D^*_\theta 1\,]=\mathbf{E}[\,-D_\theta 1+\int_0^1\dot\theta(t)\mathrm{d}w(t)\,]=\mathbf{E}[\,\int_0^1\dot\theta(t)\mathrm{d}w(t)\,]=0.$$

Hence the case $k=l$ suffices. Noting that ξ_k is a normal Gaussian random variable, we have

$$\begin{aligned}
\mathbf{E}\Big[D_\theta H_n[\xi_k]\cdot H_m[\xi_k]\Big]&=\langle\theta,e_k\rangle_H\mathbf{E}\Big[H_n'[\xi_k]H_m[\xi_k]\Big]\\
&=\langle\theta,e_k\rangle_H\int_{-\infty}^{\infty}\partial H_n[\xi]\cdot H_m[\xi]\,\gamma_1(\mathrm{d}\xi)\\
&=\langle\theta,e_k\rangle_H\int_{-\infty}^{\infty}H_n[\xi]\partial^*H_m[\xi]\,\gamma_1(\mathrm{d}\xi)\\
&=\langle\theta,e_k\rangle_H\int_{-\infty}^{\infty}H_n[\xi]\Big\{-H_m'[\xi]+\xi H_m[\xi]\Big\}\gamma_1(\mathrm{d}\xi)\\
&=\langle\theta,e_k\rangle_H\mathbf{E}\Big[H_n[\xi_k]\Big\{-H_m'[\xi_k]+\xi_kH_m[\xi_k]\Big\}\Big]\\
&=\mathbf{E}\Big[H_n[\xi_k]\Big\{-D_\theta H_m[\xi_k]+\langle\theta,e_k\rangle_H\xi_kH_m[\xi_k]\Big\}\Big].
\end{aligned}$$

Since θ can be written as $\theta=\sum_{k=1}^\infty\langle\theta,e_k\rangle_He_k$, $\int_0^1\dot\theta(t)\mathrm{d}w(t)$ admits the L^2-expansion

$$\int_0^1\dot\theta(t)\mathrm{d}w(t)=\sum_{k=1}^\infty\langle\theta,e_k\rangle_H\xi_k.$$

Now the independence of $\{\xi_i\}_{i=1}^\infty$ shows that

$$\mathbf{E}\Big[H_n[\xi_k]\int_0^1\dot\theta(t)\mathrm{d}w(t)\,H_m[\xi_k]\Big]=\mathbf{E}\Big[H_n[\xi_k]\,\langle\theta,e_k\rangle_H\xi_kH_m[\xi_k]\Big].$$

□

PROPOSITION 2.3.2. *For every* $F,G\in\mathcal{P}$, *it holds that*

$$\mathbf{E}[\,D_\theta F\cdot G\,]=\mathbf{E}[\,F D^*_\theta G\,].\tag{1.8}$$

Proof. For fixed $F, G \in \mathcal{P}$, there exist a positive integer $n \in \mathbb{N}$ and an orthonormal system $\{e_1, e_2, \cdots, e_n\}$ in H and polynomials $f(x_1, x_2, \cdots, x_n)$ and $g(x_1, x_2, \cdots, x_n)$ of n-variables such that

$$F(w) = f\Big([e_1](w), [e_2](w), \cdots, [e_n](w)\Big) \quad \text{and}$$
$$G(w) = g\Big([e_1](w), [e_2](w), \cdots, [e_n](w)\Big).$$

Extend $\{e_1, e_2, \cdots, e_n\}$ to an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of H. Since the degree of the n-th Hermite polynomial is exactly n, f and g can be written as linear combinations of finite products of Hermite polynomials. From this fact and by the linearity of D_θ and D_θ^* and the independence, F and G may be assumed without loss of generality to be of the form

$$F(w) = \prod_{i=0}^{p} H_{n_i}[\xi_{k_i}(w)] \quad \text{and} \quad G(w) = \prod_{i=0}^{p} H_{m_i}[\xi_{k_i}(w)].$$

where $\xi_k(w) = [e_k](w)$ and $k_1, k_2, \cdots, k_p$ are mutually distinct. Then, using the Leibniz' rule, Lemma 2.3.1 and the independence of $\xi_1, \xi_2, \cdots$, we have

$$\begin{aligned}
\mathbf{E}[D_\theta F \cdot G] &= \mathbf{E}\Big[D_\theta \prod_{i=1}^{p} H_{n_i}[\xi_{k_i}] \cdot \prod_{i=1}^{p} H_{m_i}[\xi_{k_i}]\Big] \\
&= \sum_{i=1}^{p} \mathbf{E}\Big[D_\theta H_{n_i}[\xi_{k_i}] \cdot \prod_{j\neq i} H_{n_j}[\xi_{k_j}] \cdot \prod_{i=1}^{p} H_{m_i}[\xi_{k_i}]\Big] \\
&= \sum_{i=1}^{p} \mathbf{E}\Big[D_\theta H_{n_i}[\xi_{k_i}] \cdot H_{m_i}[\xi_{k_i}]\Big] \mathbf{E}\Big[\prod_{j\neq i} H_{n_j}[\xi_{k_j}] H_{m_j}[\xi_{k_j}]\Big] \\
&= \sum_{i=1}^{p} \mathbf{E}\Big[H_{n_i}[\xi_{k_i}]\Big\{-D_\theta H_{m_i}[\xi_{k_i}] + \langle e_{k_i}, \theta\rangle_H \xi_{k_i} H_{m_i}[\xi_{k_i}]\Big\}\Big] \\
&\qquad\qquad \times \mathbf{E}\Big[\prod_{j\neq i} H_{n_j}[\xi_{k_j}] H_{m_j}[\xi_{k_j}]\Big] \\
&= \sum_{i=1}^{p} \mathbf{E}\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}]\Big\{-D_\theta H_{m_i}[\xi_{k_i}] + \langle e_{k_i}, \theta\rangle_H \xi_{k_i} H_{m_i}[\xi_{k_i}]\Big\} \prod_{j\neq i} H_{m_j}[\xi_{k_j}]\Big] \\
&= \sum_{i=1}^{p} \mathbf{E}\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}]\Big(-D_\theta H_{m_i}[\xi_{k_i}]\Big)\Big] \\
&\qquad\qquad + \mathbf{E}\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}]\Big\{\sum_{i=1}^{p} \langle e_{k_i}, \theta\rangle_H \xi_{k_i}\Big\} \prod_{j=1}^{p} H_{m_j}[\xi_{k_j}]\Big].
\end{aligned}$$

By the orthogonality of $\xi_1, \xi_2, \cdots$, the last term is equal to

$$\mathbf{E}\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}] \cdot \int_0^1 \dot{\theta}(t)\mathrm{d}w(t) \prod_{j=1}^{p} H_{m_j}[\xi_{k_j}]\Big],$$

which completes the proof. □

Remark 2.1. Note that $\{D_\theta : \theta \in H\}$ determines a linear operator $D : \mathcal{P} \to \mathcal{P} \otimes H$ such that $\langle DF, \theta\rangle_H = D_\theta F$ for each $F \in \mathcal{P}$ and $\theta \in H$. The operator can be extended to an operator $D : \mathcal{P} \otimes H \to \mathcal{P} \otimes H \otimes H$ by $D(F \otimes \theta) = DF \otimes \theta$. This operator is commonly used in Malliavin calculus (see e.g. [**24**]). Its "adjoint" $D^* : \mathcal{P} \otimes H \to \mathcal{P}$ is defined by $D^*F(w) = -\mathrm{tr}\, DF(w) + [F](w)$, $F \in \mathcal{P} \otimes H$. Then the "integration by parts formula";

$$\int_{\mathscr{W}} \langle DF(w), G(w)\rangle_H \gamma(\mathrm{d}w) = \int_{\mathscr{W}} F(w) D^* G(w)\gamma(\mathrm{d}w)$$

holds for all $F \in \mathcal{P}$ and $G \in \mathcal{P} \otimes H$ (see e.g. [**24**], pp 361). Under these notations, $D^*_\theta F = D^*(F \otimes \theta)$ for each $F \in \mathcal{P}$ and hence the above adjointness follows immediately from our result and vice versa.

Next we define the *exponential* $\mathrm{e}^{D^*_\theta}$ of D^*_θ by the formal series

$$\mathrm{e}^{D^*_\theta} := \sum_{n=0}^{\infty} \frac{1}{n!} D^{*n}_\theta.$$

Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis of H as above.

Theorem 2.3.3. *For every $\theta \in H$ such that $|\theta|_H = 1$, it holds that*

$$D^{*n}_\theta 1(w) = H_n\Big[\int_0^1 \dot{\theta}(t)\mathrm{d}w(t)\Big] \in \mathcal{P}, \quad n = 0, 1, 2, \cdots \tag{1.9}$$

and hence $\mathrm{e}^{D^*_\theta}1$ can be defined. In fact, it is the exponential martingale (evaluated at time 1)

$$\mathrm{e}^{D^*_\theta}1(w) = \exp\Big\{\int_0^1 \dot{\theta}(t)\mathrm{d}w(t) - \frac{1}{2}\Big\}, \quad w \in \mathscr{W}. \tag{1.10}$$

Furthermore, it holds that

$$\mathbf{E}[\mathrm{e}^{D_\theta} F] = \mathbf{E}[F \cdot \mathrm{e}^{D^*_\theta} 1], \quad F \in \mathcal{P}. \tag{1.11}$$

Proof. We use the induction on n to prove (1.9). It is clear that

$$D^*_\theta 1(w) = \int_0^1 \dot{\theta}(t)\mathrm{d}w(t) = H_1\Big[\int_0^1 \dot{\theta}(t)\mathrm{d}w(t)\Big].$$

Suppose that (1.9) holds for n. We recall that the Hermite polynomials satisfy the identity

$$H_{n+1}[x] = xH_n[x] - nH_{n-1}[x]. \tag{1.12}$$

Put $\Theta(w) := \int_0^1 \dot{\theta}(t) \mathrm{d}w(t)$. Then, noting that $\langle \theta, \theta \rangle_H = 1$ and using (1.5),

$$\begin{aligned} D_\theta^{*(n+1)} 1 &= D_\theta^* H_n[\Theta] = -D_\theta H_n[\Theta] + \Theta H_n[\Theta] \\ &= \Theta H_n[\Theta] - nH_{n-1}[\Theta] = H_{n+1}[\Theta]. \end{aligned}$$

Hence (1.9) holds for every $n = 0, 1, 2, \cdots$. Then (1.10) follows immediately from (1.9).

Finally we shall prove (1.11). By using Proposition 2.3.2, for $F \in \mathcal{P}$ we have

$$\mathbf{E}[\, \mathrm{e}^{D_\theta} F\,] = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{E}[\, D_\theta^n F\,] = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{E}[\, F \cdot D_\theta^{*n} 1\,] = \mathbf{E}[\, F \cdot \mathrm{e}^{D_\theta^*} 1\,].$$

□

Corollary 2.3.4. *For every $\theta \in H$, it holds that*

$$\mathrm{e}^{D_\theta^*} 1(w) = \exp\Big\{ \int_0^1 \dot{\theta}(t) \mathrm{d}w(t) - \frac{1}{2} \int_0^1 \dot{\theta}(t)^2 \mathrm{d}t \Big\}, \quad w \in \mathscr{W}. \tag{1.13}$$

Furthermore, it holds that

$$\mathbf{E}[\, \mathrm{e}^{D_\theta} F\,] = \mathbf{E}[\, F \cdot \mathrm{e}^{D_\theta^*} 1\,], \quad F \in \mathcal{P}. \tag{1.14}$$

Proof. Let $\eta = \theta / |\theta|_H$ and then it follows that

$$D_\theta^{*n} 1(w) = |\theta|_H^n D_\eta^{*n} 1(w) = |\theta|_H^n H_n[\, \int_0^1 \dot{\eta}(t) \mathrm{d}w(t)\,]$$

for $n = 0, 1, 2, \cdots$ and $w \in \mathscr{W}$ by Theorem 2.3.3. Hence we have

$$\mathrm{e}^{D_\theta^*} 1(w) = \sum_{n=0}^{\infty} \frac{|\theta|_H^n}{n!} H_n[\, \int_0^1 \dot{\eta}(t) \mathrm{d}w(t)\,] = \exp\Big\{ |\theta|_H \int_0^1 \dot{\eta}(t) \mathrm{d}w(t) - \frac{|\theta|_H^2}{2} \Big\}.$$

The identity (1.14) can be shown by the same argument as Theorem 2.3.3. □

Now, we have the Cameron-Martin formula in this polynomial framework.

Corollary 2.3.5. *For every $\theta \in H$ and $F \in \mathcal{P}$, it holds that*

$$\begin{aligned} &\int_{\mathscr{W}} F(w + \theta) \gamma(\mathrm{d}w) = \mathbf{E}[\, \mathrm{e}^{D_\theta} F\,] = \mathbf{E}[\, F \cdot \mathrm{e}^{D_\theta^*} 1\,] \\ &= \int_{\mathscr{W}} F(w) \exp\Big\{ \int_0^1 \dot{\theta}(t) \mathrm{d}w(t) - \frac{1}{2} \int_0^1 \dot{\theta}(t)^2 \mathrm{d}t \Big\} \gamma(\mathrm{d}w). \end{aligned} \tag{1.15}$$

3. An Algebraic Proof of MG Formula

In this section, we will give an algebraic proof of the MG formula using an adjoint relation similar to (1.11). As we have announced in the introduction, for the proof of the adjoint relation we will rely on the standard stochastic calculus.

Let $Z : \mathscr{W} \to H$ be a predictable map; i.e. $\dot{Z}(t)$, $0 \le t \le 1$ is a predictable process such that

$$\|Z\|_H^2 = \int_0^1 \dot{Z}(s)^2 \mathrm{d}s < +\infty \quad \text{a.s.}$$

Suppose $\mathcal{E}(\int \dot{Z}\mathrm{d}W)$ is a true martingale where for a martingale $M = (M(t))_{0\le t\le 1}$ the process $\mathcal{E}(M)$ is defined by

$$\mathcal{E}(M)_t = \exp\left\{M(t) - \frac{1}{2}\langle M\rangle(t)\right\}.$$

3.1. Infinite Dimensional Tensor Fields. We fix a c.o.n.s. $\{e_i : i \in \mathbf{N}\}$ of H and will write simply D_i for D_{e_i} for each $i \in \mathbf{N}$. We define a differentiation along Z. For $\phi \in \mathcal{P}$, we define D_Z in the following way:

$$D_Z\phi(W) := \sum_{i=1}^{\infty} \langle Z, e_i\rangle(W)\, D_i\phi(W),$$

where $\langle\cdot,\cdot\rangle$ is the inner product of H. Moreover, we define the n-th D_Z, which we write as $D_Z^{\otimes n}$ by the following:

$$\begin{aligned} D_Z^{\otimes n} &:= \underbrace{D_Z \otimes D_Z \otimes \cdots \otimes D_Z}_{n\text{-times}} \\ &:= \sum_{i,j,k,\cdots} \underbrace{\langle Z, e_i\rangle\langle Z, e_j\rangle\langle Z, e_k\rangle\cdots}_{n\text{-members}} \underbrace{D_iD_jD_k\cdots}_{n\text{-members}}. \end{aligned}$$

Next we define the exponential of D_Z by the formal series of

$$\begin{aligned} \widetilde{\mathrm{e}}^{D_Z} &:= 1 + D_Z + \frac{1}{2!}D_Z^{\otimes 2} + \frac{1}{3!}D_Z^{\otimes 3} + \cdots \\ &= 1 + \sum_i \langle Z, e_i\rangle D_i + \frac{1}{2!}\sum_{i,j}\langle Z, e_i\rangle\langle Z, e_j\rangle D_iD_j \\ &\qquad + \frac{1}{3!}\sum_{i,j,k}\langle Z, e_i\rangle\langle Z, e_j\rangle\langle Z, e_k\rangle D_iD_jD_k + \cdots . \end{aligned}$$

We denote $\langle Z, e_i\rangle$ by Z_i, so we may write $\langle Z, e_i\rangle\langle Z, e_j\rangle D_iD_j$ as $Z_iZ_jD_iD_j$ and furthermore $D_Z^{\otimes 2} = \sum_{i,j} Z_iZ_jD_iD_j$ as $\langle Z\otimes Z, \nabla\otimes\nabla\rangle, \cdots, D_Z^{\otimes n} = \langle Z^{\otimes n}, \nabla^{\otimes n}\rangle$, and so on.

Lemma 3.1.1. *For any $k \in \mathbb{N}$, we have*

$$\widetilde{\mathrm{e}}^{D_Z}\Big(H_{n_1}(\int_0^1 \dot{e}_{m_1}\mathrm{d}W)\cdots H_{n_k}(\int_0^1 \dot{e}_{m_k}\mathrm{d}W)\Big) = \widetilde{\mathrm{e}}^{D_Z}\Big(H_{n_1}(\int_0^1 \dot{e}_{m_1}\mathrm{d}W)\Big)\cdots \widetilde{\mathrm{e}}^{D_Z}\Big(H_{n_k}(\int_0^1 \dot{e}_{m_k}\mathrm{d}W)\Big). \tag{1.16}$$

Proof. First note that the equation (1.16) is equivalent to

$$\sum_{l=0}^{n_1+\cdots+n_k}\frac{1}{l!}\langle Z^{\otimes l},\nabla^{\otimes l}\rangle\Big(H_{n_1}(\int_0^1 \dot{e}_{m_1}\mathrm{d}W)\cdots H_{n_k}(\int_0^1 \dot{e}_{m_k}\mathrm{d}W)\Big) = \sum_{l_1=0}^{n_1}\frac{1}{l_1!}\langle Z^{\otimes l_1},\nabla^{\otimes l_1}\rangle H_{n_1}(\int_0^1 \dot{e}_{m_1}\mathrm{d}W)\cdots\sum_{l_k=0}^{n_k}\frac{1}{l_k!}\langle Z^{\otimes l_k},\nabla^{\otimes l_k}\rangle H_{n_k}(\int_0^1 \dot{e}_{m_k}\mathrm{d}W). \tag{1.17}$$

Fixing $l_1,\cdots l_k$ such that $l_1 \le n_1,\cdots,l_k \le n_k$, it suffices to prove that the coefficients of

$$\nabla^{\otimes l_1}H_{n_1}\nabla^{\otimes l_2}H_{n_2}\cdots\nabla^{\otimes l_k}H_{n_k}$$

of the left-hand after applying Leibniz rule correspond to those of right-hand. The coefficients of the left-hand are the following.

$$\frac{1}{(l_1+l_2+\cdots+l_k)!}\binom{l_1+l_2+\cdots+l_k}{l_1}\binom{l_2+\cdots+l_k}{l_2}\cdots\binom{l_k}{l_k}.$$

This is equal to $\frac{1}{l_1!l_2!\cdots l_k!}$, so we get (1.17). □

Proposition 3.1.2. *For $f \in \mathcal{P}$, we have*

$$\widetilde{\mathrm{e}}^{D_Z}(f(W)) = f(W+Z). \tag{1.18}$$

Proof. Since $\widetilde{\mathrm{e}}^{D_Z}$ is linear and by Lemma 3.1.1, we only prove in the case of $f(W) = H_n(\int_0^1 \dot{e}_i(s)\mathrm{d}W_s)$, that is, it suffices to show

$$\widetilde{\mathrm{e}}^{D_Z}\Big(H_n(\int_0^1 \dot{e}_i(s)\mathrm{d}W_s)\Big) = H_n\Big(\int_0^1 \dot{e}_i(s)\mathrm{d}W_s + \langle Z, e_i\rangle\Big).$$

By the definition, we have

$$\widetilde{\mathrm{e}}^{D_Z}\Big(H_n(\int_0^1 \dot{e}_i(s)\mathrm{d}W_s)\Big) = \sum_{k=0}^{n}\binom{n}{k}\langle Z, e_i\rangle^k H_{n-k}(\int_0^1 \dot{e}_i(s)\mathrm{d}W_s).$$

For this, apply $H_n(x+y) = \sum_{k=0}^n \binom{n}{k}H_{n-k}(x)y^k$, then we get (1.18). □

3.2. The Operator L_n^Z. To prove Maruyama-Girsanov formula, we additionally introduce a sequence $\{L_n^Z\}$ of new operators associated with Z as follows. For any $n \in \mathbb{N}$, L_n^Z is defined by $L_0^Z = \mathrm{id}$ and

$$L_n^Z = -\sum_{k=1}^{n} \binom{n}{k} \widehat{H}_{n-k}\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s, \|Z\|_H^2\Big) D_{-Z}^{\otimes k}, \quad n \in \mathbf{N} \tag{1.19}$$

where the polynomials $\widehat{H}_n(x,y)$, $n = 1, 2, \cdots$, are defined by means of the formula

$$\mathrm{e}^{\lambda x - \frac{\lambda^2}{2}y^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \widehat{H}_n(x,y).$$

With this notation, the Hermite polynomials we have used so far are can be written as

$$H_n[x] = \widehat{H}_n(x, 1).$$

Theorem 3.2.1. *For any $F \in \mathcal{P}$, we have*

$$\mathbf{E}\Big[\sum_{n=0}^{\infty} \frac{1}{n!} L_n^Z F\Big] = \mathbf{E}[\mathcal{E}\Big(\int_0^{\cdot} \dot{Z}(s)\mathrm{d}W_s\Big)_1 F]. \tag{1.20}$$

Proof. It suffices to show

$$\mathbf{E}[L_n^Z F] = \mathbf{E}[\widehat{H}_n\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s, \|Z\|_H^2\Big) F] \tag{1.21}$$

for each $n \in \mathbb{N}$ and $F \in \mathcal{P}$. If we can prove that

$$\mathbf{E}[L_n^Z\Big(\mathcal{E}(\int \dot{f}\,\mathrm{d}W)_1\Big)] = \mathbf{E}\Big[\widehat{H}_n\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s, \|Z\|_H^2\Big)\mathcal{E}(\int \dot{f}\,\mathrm{d}W)_1\Big] \tag{1.22}$$

for arbitrary $f \in H$, then (1.21) is deduced. In fact, for a finite orthonormal system $\{e_1, \cdots, e_m\}$, take $f := \lambda_1 e_1 + \cdots \lambda_m e_m$ for $\lambda_1, \cdots, \lambda_m \in \mathbf{R}$. Then,

$$\begin{aligned}\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1 &= \prod_{i=1}^{m} \mathcal{E}\Big(\lambda_i \int \dot{e}_i\,\mathrm{d}W\Big)_1 \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{n_1+\cdots+n_m=N} \frac{N!}{n_1!\cdots n_m!} \prod_{i=1}^{m} \lambda_i^{n_i} H_{n_i}\Big(\int_0^1 \dot{e}_i(s)\mathrm{d}W_s\Big),\end{aligned}$$

and we notice that $\sum_{N=0}^{\infty} a_N$ where

$$a_N = \mathbf{E}\Big[\sum_{n_1+\cdots+n_m=N} \frac{N!}{n_1!\cdots n_m!} \prod_{i=1}^{m} \lambda_i^{n_i} H_{n_i}\Big(\int_0^1 \dot{e}_i(s)\mathrm{d}W_s\Big)\Big] = \begin{cases} 1 & \text{if } N = 0, \\ 0 & \text{otherwise} \end{cases}$$

is absolutely convergent. This means that (1.21) is valid for arbitrary monomials and hence for all polynomials.

So, let us prove (1.22). First we note that

$$\begin{aligned}
&\mathbf{E}[L_n^Z\Big(\mathcal{E}(\int \dot{f}\,\mathrm{d}W)_1\Big)] \\
&= \mathbf{E}\Big[\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\widehat{H}_{n-k}\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s, \|Z\|_H^2\Big)D_Z^{\otimes k}\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1\Big],
\end{aligned}$$

where $\widehat{H}_n(s)$ denotes $\widehat{H}_n(\int_0^s \dot{Z}(u)\mathrm{d}W_u, \int_0^s \dot{Z}(u)^2\mathrm{d}u)$ and $\widehat{H}_n := \widehat{H}_n(1)$. Since $D_i\mathcal{E}(\int \dot{f}\mathrm{d}W)_1 = \langle f, e_i\rangle\mathcal{E}(\int \dot{f}\mathrm{d}W)_1$, we have

$$\begin{aligned}
&\mathbf{E}[L_n^Z\Big(\mathcal{E}(\int \dot{f}\,\mathrm{d}W)_1\Big)] \\
&= \mathbf{E}\Big[\mathcal{E}\Big(\int \dot{f}\mathrm{d}W\Big)_1\Big\{\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\widehat{H}_{n-k}\sum_{i_1,\cdots,i_k} Z_{i_1}\cdots Z_{i_k}\langle f, e_{i_1}\rangle\cdots\langle f, e_{i_k}\rangle\Big\}\Big] \\
&= \mathbf{E}\Big[\mathcal{E}\Big(\int \dot{f}\mathrm{d}W\Big)_1\Big\{\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\widehat{H}_{n-k}\langle Z, f\rangle^k\Big\}\Big].
\end{aligned}$$

We will use the following formulas to obtain (1.22) which will complete the proof;

$$\widehat{H}_n(t) = n\int_0^t \widehat{H}_{n-1}(s)\dot{Z}(s)\,\mathrm{d}W_s,$$

$$\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_t = 1 + \int_0^t \mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_s \dot{f}(s)\,\mathrm{d}W_s,$$

and

$$\mathrm{d}\Big\langle \widehat{H}_n, \mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)\Big\rangle_s = n\widehat{H}_{n-1}(s)\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_s \dot{f}(s)\dot{Z}(s)\mathrm{d}s. \tag{1.23}$$

As a first step we have

$$\begin{aligned}
&\mathbf{E}\Big[\widehat{H}_n\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s, \int_0^1 \dot{Z}(s)^2\mathrm{d}s\Big)\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1\Big] \\
&= \mathbf{E}\Big[n\int_0^1 \widehat{H}_{n-1}(s)\dot{Z}(s)\mathrm{d}W_s\Big] \\
&\qquad + \mathbf{E}\Big[n\int_0^1 \widehat{H}_{n-1}(s)\dot{Z}(s)\mathrm{d}W_s \int_0^1 \mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_s \dot{f}(s)\,\mathrm{d}W_s\Big] \\
&= \mathbf{E}\Big[n\int_0^1 \widehat{H}_{n-1}(s)\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_s \dot{f}(s)\dot{Z}(s)\,\mathrm{d}s\Big] =: I.
\end{aligned}$$

By Ito's formula, we have

$$
\begin{aligned}
&\widehat{H}_{n-1}(1)\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1 \int_0^1 \dot{f}(s)\dot{Z}(s)\,\mathrm{d}s \\
&= \int_0^1 \widehat{H}_{n-1}(s)\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_s \dot{f}(s)\dot{Z}(s)\,\mathrm{d}s + \int_0^1 \int_0^s \dot{f}(u)\dot{Z}(u)\mathrm{d}u\,\mathrm{d}\Big\langle \widehat{H}_{n-1}, \mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)\Big\rangle_s \\
&\qquad + \text{a martingale.}
\end{aligned}
$$

Then by using (1.23), we have

$$
\begin{aligned}
I &= \mathbf{E}\Big[n\,\widehat{H}_{n-1}\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1 \int_0^1 \dot{f}(s)\dot{Z}(s)\,\mathrm{d}s \Big] \\
&\qquad - \mathbf{E}\Big[n(n-1) \int_0^1 \dot{f}(s)\dot{Z}(s) \int_0^s \dot{f}(u)\dot{Z}(u)\,\mathrm{d}u\,\widehat{H}_{n-2}(s)\,\mathcal{E}\Big(\int \dot{f}\mathrm{d}W\Big)_s \mathrm{d}s \Big] \\
&=: \mathbf{E}\Big[n\,\widehat{H}_{n-1}\,\mathcal{E}\Big(\int \dot{f}\mathrm{d}W\Big)_1 \langle f, Z\rangle \Big] - I.
\end{aligned}
$$

Again we apply Ito's formula to get

$$
\begin{aligned}
&\widehat{H}_{n-2}(1)\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1 \langle f, Z\rangle^2 \\
&= 2\int_0^1 \widehat{H}_{n-2}(s)\,\mathcal{E}\Big(\int \dot{f}\mathrm{d}W\Big)_s \int_0^s \dot{f}(u)\dot{Z}(u)\,\mathrm{d}u\, f(s)Z(s)\,\mathrm{d}s \\
&\qquad + \int_0^1 \Big\{ \int_0^s \dot{f}(u)\dot{Z}(u)\,\mathrm{d}u \Big\}^2 \mathrm{d}\Big\langle \widehat{H}_{n-2}, \mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)\Big\rangle_s + \text{a martingale}
\end{aligned}
$$

and by using (1.23) again, we obtain

$$
\begin{aligned}
I &= \mathbf{E}\Big[\frac{n(n-1)}{2}\,\widehat{H}_{n-2}\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1 \langle f, Z\rangle^2 \Big] \\
&\quad - \mathbf{E}\Big[\frac{n(n-1)(n-2)}{2} \int_0^1 \widehat{H}_{n-3}(s)\,\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_s \dot{f}(s)\dot{Z}(s) \Big\{ \int_0^s \dot{f}(u)\dot{Z}(u)\,\mathrm{d}u \Big\}^2 \mathrm{d}s \Big].
\end{aligned}
$$

Hence we have

$$\begin{aligned}
&\mathbf{E}\Big[\widehat{H}_n\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s, \int_0^1 \dot{Z}(s)^2\mathrm{d}s\Big)\cdot\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1\Big] = I\\
&= \mathbf{E}\Big[n\widehat{H}_{n-1}\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1\langle f, Z\rangle\Big]\\
&\quad - \mathbf{E}\Big[\frac{n(n-1)}{2}\widehat{H}_{n-2}\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1\langle f, Z\rangle^2\Big]\\
&\quad + \mathbf{E}\Big[\frac{n(n-1)(n-2)}{2}\int_0^1 \dot{f}(s)\dot{Z}(s)\Big\{\int_0^s \dot{f}(u)\dot{Z}(u)\,\mathrm{d}u\Big\}^2\widehat{H}_{n-3}(s)\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_s\mathrm{d}s\Big].
\end{aligned}$$

By repeating this procedure until $\widehat{H}_*(s)$ in the integrand vanishes, we obtain

$$\begin{aligned}
&\mathbf{E}\Big[\widehat{H}_n\Big(\int_0^1 Z(s)\mathrm{d}W_s, \int_0^1 Z(s)^2\mathrm{d}s\Big)\mathcal{E}\Big(\int \dot{f}\mathrm{d}W\Big)_1\Big]\\
&\qquad = \mathbf{E}\Big[\mathcal{E}\Big(\int \dot{f}\,\mathrm{d}W\Big)_1\Big\{\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\widehat{H}_{n-k}\langle Z, f\rangle^k\Big\}\Big].
\end{aligned}$$

□

3.3. Passage to the Cameron-Martin-Maruyama-Girsanov Formula. From Proposition 3.1.2 and Theorem 3.2.1, we will give a new proof of Maruyama-Girsanov formula in the case of $f \in \mathcal{P}$.

LEMMA 3.3.1. *As an operator acting on $\mathcal{P}$,*

$$\sum_{n=1}^{\infty}\frac{1}{n!}L_n^Z = \exp\Big\{\int_0^1 \dot{Z}(t)\mathrm{d}W_t - \frac{1}{2}\int_0^1 \dot{Z}(t)^2\mathrm{d}t\Big\}(1 - \widetilde{\mathrm{e}}^{\,D_Z}).$$

Proof.

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n!}L_n^Z &= 1-\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=1}^{n}\binom{n}{k}\widehat{H}_{n-k}\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s,\int_0^1\dot{Z}(s)^2\mathrm{d}s\Big)D_{-Z}^{\otimes k}\\
&= 1-\sum_{k=1}^{\infty}\Big\{\sum_{n=k}^{\infty}\frac{1}{k!(n-k)!}\widehat{H}_{n-k}\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s,\int_0^1\dot{Z}(s)^2\mathrm{d}s\Big)\Big\}D_{-Z}^{\otimes k}\\
&= 1-\sum_{k=1}^{\infty}\frac{1}{k!}\Big\{\sum_{m=0}^{\infty}\frac{1}{m!}\widehat{H}_{m}\Big(\int_0^1 \dot{Z}(s)\mathrm{d}W_s,\int_0^1\dot{Z}(s)^2\mathrm{d}s\Big)\Big\}D_{-Z}^{\otimes k}\\
&= 1-\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1\sum_{k=1}^{\infty}\frac{1}{k!}D_{-Z}^{\otimes k}\\
&= 1-\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1\sum_{k=0}^{\infty}\frac{1}{k!}D_{-Z}^{\otimes k}+\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1.
\end{aligned}
$$

□

Corollary 3.3.2 (Cameron-Martin-Maruyama-Girsanov formula). *For $f \in \mathcal{P}$, the following formula holds*

$$
\mathbf{E}\Big[\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1 f\Big(W-\int_0^{\cdot}\dot{Z}(s)\mathrm{d}s\Big)\Big]=\mathbf{E}[f(W)]. \tag{1.24}
$$

Proof. By Lemma 3.3.1, we have

$$
\begin{aligned}
&\mathbf{E}\Big[\sum_{n=0}^{\infty}\frac{1}{n!}L_n\big(f(W)\big)\Big] \qquad (1.25)\\
&= \mathbf{E}\Big[f(W)-\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1\sum_{k=0}^{\infty}\frac{1}{k!}D_{-Z}^{\otimes k}f(W)+\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1 f(W)\Big]\\
&= \mathbf{E}\Big[f(W)-\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1\widetilde{\mathrm{e}}^{D_{-Z}}f(W)+\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1 f(W)\Big]\\
&= \mathbf{E}\Big[f(W)-\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1 f\Big(W-\int_0^{\cdot}\dot{Z}(s)\mathrm{d}s\Big)+\mathcal{E}\Big(\int\dot{Z}\mathrm{d}W\Big)_1 f(W)\Big].
\end{aligned}
$$

Then by Theorem 3.2.1, we obtain (1.24). □

4. Another Algebraic Proof for CMMG Formula

As we have mentioned in the introduction, we give an alternative proof which is “purely” algebraic in the sense that we do not use stochastic calculus essentially, though we restrict ourselves in the case of piecewise constant (=finite-dimensional) case.

Let $\mathcal{F} \equiv \{\mathcal{F}_t\}_{0\le t\le 1}$ be the natural filtration of $\mathscr{W}$. Let us consider a simple $\mathcal{F}$-predictable process

$$z(w,t) = \sum_{k=1}^{2^s} 2^{s/2} z_k(w) 1_{(\frac{k-1}{2^s},\frac{k}{2^s}]}(t) \tag{1.26}$$

where z_k, $k = 1,\cdots,2^s$ are $\mathcal{F}_{\frac{k-1}{2^s}}$- measurable random variables. Define $\sigma_k^s \in H$, $k = 1,\cdots,2^s$ by

$$\sigma_k^s(t) := 2^{s/2} \int_0^t 1_{(\frac{k-1}{2^s},\frac{k}{2^s}]}(u)\,\mathrm{d}u.$$

We will suppress the superscript s whenever it is clear from the context. Clearly,

$$D_{\sigma_k} F = 0 \tag{1.27}$$

for any $\mathcal{F}_{\frac{k-1}{2^s}}$-measurable random variable F. Put

$$D_{z_k} := z_k D_{\sigma_k} \text{ and } D^*_{z_k} := z_k D^*_{\sigma_k},$$

for $k = 1,\cdots,2^s$.

Lemma 4.0.3. *For any $n \in \mathbb{N}$ and $f \in \mathcal{P}$, we have*

$$D^n_{z_k} f = \underbrace{z_k D_{\sigma_k} \cdots z_k D_{\sigma_k}}_{n\text{-times}} f = z_k^n D^n_{\sigma_k} f \tag{1.28}$$

and

$$(D^*_{z_k})^n f = \underbrace{z_k D^*_{\sigma_k} \cdots z_k D^*_{\sigma_k}}_{n\text{-times}} f = z_k^n (D^*_{\sigma_k})^n f. \tag{1.29}$$

Proof. These are direct from the following "commutativity":

$$D_{\sigma_j}(z_i f) = z_i D_{\sigma_j} f, \text{ and } D^*_{\sigma_j}(z_i f) = z_i D^*_{\sigma_j} f, \quad \text{if } i \le j$$

for differentiable f. These follows since $D_{\sigma_j}(z_i) = 0$. □

Define the exponentials as

$$\mathrm{e}^{D_{z_k}} := \sum_{n=0}^{\infty} \frac{1}{n!} D^n_{z_k}, \quad k = 1,2,\cdots,N$$

and

$$\mathrm{e}^{D^*_{z_k}} := \sum_{n=0}^{\infty} \frac{1}{n!} (D^*_{z_k})^n, \quad k = 1,2,\cdots N$$

formally. By Lemma 4.0.3 we have

$$\mathrm{e}^{D_{z_k}} = \sum_{n=0}^{\infty} \frac{z_k^n}{n!} D^n_{\sigma_k}$$

and thus we can include $\mathcal{P}$ in the domain of $\mathrm{e}^{D_{z_k}}$.

Let us introduce a subspace $\mathcal{P}_{\mathrm{Haar}}$ of $\mathcal{P}$, which consists of polynomials with respect to $\{[e_i](w)\}$, where $\{e_i\}$ is the Haar system. Note that $\mathcal{P}_{\mathrm{Haar}}$ is also characterized as all the polynomials with respect to $\{[\dot{\sigma}^s_k](w) : k = 1, \cdots, 2^s, s \in \mathbb{N}\}$.

The following is a main result in our program.

Theorem 4.0.4.

(i) *For any $F \in \mathcal{P}_{\mathrm{Haar}}$, we have*

$$\mathrm{e}^{D_{z_{2^s}}} \cdots \mathrm{e}^{D_{z_1}} F(w) = F(w + \int_0^{\cdot} z(w,u)\,\mathrm{d}u). \tag{1.30}$$

(ii) *For any $\mathcal{F}_{(k-1)/2^s}$-measurable random variable F,*

$$\mathrm{e}^{D^*_{z_k}} F = F\,\mathrm{e}^{D^*_{z_k}}(1). \tag{1.31}$$

In particular, the function F is in the domain of $\mathrm{e}^{D^*_{z_k}}$. Furthermore, we have

$$\mathrm{e}^{D^*_{z_{2^s}}} \cdots \mathrm{e}^{D^*_{z_1}}(1) = \exp\Big\{ \int_0^1 z(w,s)\,\mathrm{d}w(s) - \frac{1}{2}\int_0^1 z(w,s)^2\,\mathrm{d}s \Big\}, \tag{1.32}$$

(iii) *Fix $k \in \mathbb{N}$. Let $F \in \mathcal{P}$ and let G be an arbitrary $\mathcal{F}_{(k-1)/2^s}$-measurable integrable function. Then*

$$\mathbf{E}[\,\mathrm{e}^{D_{z_k}}(F)\,G\,] = \mathbf{E}[\,F\,\mathrm{e}^{D^*_{z_k}}(G)\,]. \tag{1.33}$$

Proof. (i) First, notice that $F \in \mathcal{P}_{\mathrm{Haar}}$ is always expressed as a linear combination of $\prod_{k=1}^{2^s} F_k$, where each F_k is a polynomial in

$$\Big\{ [\sigma^t_l](w) : \Big(\frac{l-1}{2^t}, \frac{l}{2^t}\Big] \subset \Big(\frac{k-1}{2^s}, \frac{k}{2^s}\Big] \Big\}, \tag{1.34}$$

so that we can assume that F is of the form

$$F = \sum_{i=1}^{N} \prod_{k=1}^{2^s} F_{k,i},$$

where each $F_{k,i}$ is a polynomial in (1.34). By Proposition 2.2.2 and the definition of D_{σ_k}, we have

$$\mathrm{e}^{D_{z_k}} F_{l,i}(w) = \begin{cases} F_{k,i}(w + z_k\sigma_k) & \text{if } l = k, \\ F_{l,i}(w) & \text{otherwise.} \end{cases}$$

Then by Lemma 2.2.1,

$$\mathrm{e}^{D_{z_k}} \prod_{l=1}^{2^s} F_{l,i}(w) = F_{k,i}(w + z_k\sigma_k) \prod_{l \neq k} F_{l,i}(w).$$

Since z_k is $\mathcal{F}_{t_k}$-measurable, we also have, if $j > k$,

$$\begin{aligned}
&\mathrm{e}^{D_{z_j}}\mathrm{e}^{D_{z_k}}\prod_{l=1}^{2^s}F_{l,i}(w)\\
&= \mathrm{e}^{D_{z_j}}F_{k,i}(w+z_k\sigma_k)\,\mathrm{e}^{D_{z_j}}\prod_{l\neq k}F_{l,i}(w)\\
&= F_{k,i}(w+z_k\sigma_k)F_{j,i}(w+z_j\sigma_j)\prod_{l\neq j,k}F_{l,i}(w).
\end{aligned}$$

Then, inductively we have

$$\mathrm{e}^{D_{z_{2^s}}}\cdots\mathrm{e}^{D_{z_1}}\prod_{l=1}^{2^s}F_{l,i}(w)=\prod_{l=1}^{2^s}F_{l,i}(w+z_l\sigma_l),$$

and by linearity we obtain (1.30) since

$$\sum_{l=1}^{2^s}z_l(w)\sigma_l(t)=\int_0^t z(w,u)\,\mathrm{d}u.$$

(ii) Noting that $D_{\sigma_k}F=0$ for $\mathcal{F}_{(k-1)/2^s}$ - measurable random variable F, we have

$$\begin{aligned}
D^*_{z_k}F &= z_k\Big\{-D_{\sigma_k}+2^{s/2}(w_{k/2^s}-w_{(k-1)/2^s})\Big\}F\\
&= F z_k 2^{s/2}(w_{k/2^s}-w_{(k-1)/2^s}) = F D^*_{z_k}(1)
\end{aligned}$$

since z_k is also $\mathcal{F}_{(k-1)/2^s}$-measurable. Inductively, we then have

$$(D^*_{z_k})^nF=F(D^*_{z_k})^n(1),$$

and hence we have (1.31), which in turn implies (1.32). In fact, we have by induction

$$\mathrm{e}^{D^*_{z_{2^s}}}\cdots\mathrm{e}^{D^*_{z_1}}(1)=\prod_{k=1}^{2^s}\{\mathrm{e}^{D^*_{z_k}}(1)\}$$

since $\mathrm{e}^{D^*_{z_{k-1}}}\cdots\mathrm{e}^{D^*_{z_1}}(1)$ is $\mathcal{F}_{(k-1)/2^s}$-measurable for any k, and for each $i=1,2,\cdots,2^s$, we have

$$\begin{aligned}
\mathrm{e}^{D^*_{z_i}}(1) &= \sum_{n=0}^{\infty}\frac{z_i^n}{n!}(D^*_{\sigma_i})^n(1)=\sum_{n=0}^{\infty}\frac{z_i^n}{n!}H_n[\int_0^1\sigma_k(t)\mathrm{d}w_t]\\
&= \exp\Big\{z_i(w)2^{s/2}(w_{k/2^s}-w_{(k-1)/2^s})-\frac{1}{2}z_i(w)^2\Big\}.
\end{aligned}$$

(iii) Since F is a polynomial,

$$e^{D_{z_k}}F = \sum_{n=0}^{M} \frac{z_k^n}{n!} D_{\sigma_k}^n F$$

for some $M \in \mathbb{N} \cup \{0\}$. Therefore, the left-hand-side of (1.33) is rewritten as

$$\sum_{n=0}^{M} \frac{1}{n!} \mathbf{E}[z_k^n D_{\sigma_k}^n F \cdot G].$$

Since z_k and G are $\mathcal{F}_{(k-1)/2^s}$-measurable, we have, for $n \le M$

$$\begin{aligned}\mathbf{E}[z_k^n D_{\sigma_k}^n F \cdot G] &= \mathbf{E}[F \cdot (D_{\sigma_k}^*)^n z_k^n G] \\ &= \mathbf{E}[F \cdot z_k^n (D_{\sigma_k}^*)^n G] = \mathbf{E}[F \cdot (D_{z_k}^*)^n G].\end{aligned}$$

The relation is valid for $n > M$ since

$$(D_{\sigma_k}^*)^n G = G (D_{\sigma_k}^*)^n (1) = G H_n[\int_0^1 \sigma_k(t)\, dw_t],$$

and the degree of F as a polynomial of $\int_0^1 \sigma_k(t) dw_t$ is less than M, we have

$$\mathbf{E}[z_k^n D_{\sigma_k}^n F \cdot G] = \mathbf{E}[F \cdot D_{z_k}^{*n} G] = 0.$$

Thus we have

$$\mathbf{E}\Big[\sum_{n=0}^{\infty} \frac{1}{n!} D_{z_k}^n F \cdot G\Big] = \mathbf{E}\Big[\sum_{n=0}^{\infty} \frac{1}{n!} F \cdot D_{z_k}^{*n} G\Big],$$

which is the desired relation. □

Remark 4.1. (i) We do not assume smoothness for F in (1.31). (ii) In (1.30) and (1.32), the order of application of the operators is important. If it is changed anywhere, neither holds anymore.

By using the above algebraic results, we can prove the following

Corollary 4.0.5 (Cameron-Martin-Maruyama-Girsanov formula). *For a simple predictable z in (1.26) and $F \in \mathcal{P}_{\mathrm{Haar}}$, it holds*

$$\mathbf{E}\Big[F\Big(w - \int_0^{\cdot} z(w,u)\, du\Big) \exp\Big\{\int_0^1 z(w,t)\, dw_t - \frac{1}{2}\int_0^1 z(w,t)^2\, dt\Big\}\Big] = \mathbf{E}[F]. \tag{1.35}$$

Proof. As a formal series, we have

$$e^{D_{z_k}} e^{-D_{z_k}} = 1,$$

for $k = 1, \cdots 2^s$. Then, for $F \in \mathcal{P}_{\mathrm{Haar}}$, we have

$$F = e^{D_{z_1}} e^{-D_{z_1}} F$$

and since $\mathrm{e}^{-D_{z_1}}F$ is a polynomial, by Theorem 4.0.4 (iii), we have

$$\mathbf{E}[F] = \mathbf{E}[\mathrm{e}^{D_{z_1}}\,\mathrm{e}^{-D_{z_1}}F] = \mathbf{E}[\mathrm{e}^{-D_{z_1}}F \cdot \mathrm{e}^{D^*_{z_1}}(1)]. \tag{1.36}$$

Inductively, since

$$\mathrm{e}^{-\partial_{z_k}} \cdots \mathrm{e}^{-\partial_{z_1}} f(\xi)$$

still is a polynomial in

$$\left\{ [\sigma^t_l](w) : \left(\frac{l-1}{2^t}, \frac{l}{2^t}\right] \subset \left(\frac{k-1}{2^s}, \frac{k}{2^s}\right] \right\},$$

and

$$\mathrm{e}^{D^*_{z_{k-1}}} \cdots \mathrm{e}^{D^*_{z_1}}(1)$$

is $\mathcal{F}_{(k-1)/2^s}$-measurable, we have

$$\begin{aligned} &\mathbf{E}[F] \\ &= \mathbf{E}[\mathrm{e}^{D_{z_k}}\,\mathrm{e}^{-D_{z_k}}\,\mathrm{e}^{-D_{z_{k-1}}} \cdots \mathrm{e}^{-D_{z_1}}F \cdot \mathrm{e}^{D^*_{z_{k-1}}} \cdots \mathrm{e}^{D^*_{z_1}}(1)] \\ &= \mathbf{E}[\mathrm{e}^{-D_{z_k}} \cdots \mathrm{e}^{-D_{z_1}}F \cdot \mathrm{e}^{D^*_{z_k}} \cdots \mathrm{e}^{D^*_{z_1}}(1)]. \end{aligned} \tag{1.37}$$

Combining this with (1.30) and (1.32) in Theorem 1.31, we have the formula (1.35). □

5. Continuity of the Translation

The following lemma extends the translation on the dense subset of polynomials to an operator on L_q to L_p, and hence ensure the MG formula (1.35) for any bounded measurable F.

Lemma 5.0.6. *Let z be a predictable process as (1.26). Suppose that*

$$\mathbf{E}\left[\exp\left\{c\int_0^1 z(t)^2\,\mathrm{d}t\right\}\right] < +\infty \tag{1.38}$$

for some $c > 0$. Then, for $p \in [1,\infty)$, there exists $q \in (p,\infty)$ and a positive constant C_p such that

$$\|\mathrm{e}^{-D_{z_{2^s}}} \cdots \mathrm{e}^{-D_{z_1}}F\|_p \le C_p\|F\|_q$$

for any $F \in \mathcal{P}_{\mathrm{Haar}}$.

Proof. We will denote $Z := \int_0^{\cdot} z(t)\,\mathrm{d}t$ and

$$\mathcal{E}(z) := \exp\left\{\int_0^1 z(t)\,\mathrm{d}w(t) - \frac{1}{2}\int_0^1 z(t)^2\,\mathrm{d}t\right\}.$$

Let $n \geq 1$ be an integer and $p < 2n$. By Hölder's inequality,

$$\begin{aligned}
\mathbf{E}[\,|F(w - Z(w))|^p\,] &= \mathbf{E}\Big[\,|F(w - Z(w))|^p\,\{\mathcal{E}(z)\}^{\frac{p}{2n}}\,\{\mathcal{E}(z)\}^{-\frac{p}{2n}}\Big] \\
&\leq \mathbf{E}\Big[|F(w - Z(w))|^{p\cdot\frac{2n}{p}}\,\{\mathcal{E}(z)\}^{\frac{p}{2n}\cdot\frac{2n}{p}}\Big]^{\frac{p}{2n}} \cdot \mathbf{E}\Big[\{\mathcal{E}(z)\}^{-\frac{p}{2n}\cdot\frac{2n}{2n-p}}\Big]^{\frac{2n-p}{2n}} \\
&= \mathbf{E}\Big[\,|F(w - Z(w))|^{2n}\,\mathcal{E}(z)\Big]^{\frac{p}{2n}} \cdot \mathbf{E}\Big[\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}}\Big]^{\frac{2n-p}{2n}}.
\end{aligned}$$

Since F is a polynomial, so is $|F|^{2n}$. Therefore, we can apply the MG formula for polynomials (1.35) in Corollary 4.0.5, to obtain

$$\mathbf{E}\Big[\,|F(w - Z(w))|^{2n}\,\mathcal{E}(z)\Big]^{\frac{p}{2n}} = \mathbf{E}[\,|F|^{2n}\,]^{\frac{p}{2n}} = \|F\|_{2n}^{p}.$$

Now it suffices to show that

$$\mathbf{E}[\,\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}}\,] < +\infty. \tag{1.39}$$

Let us denote $L_t := \int_0^t z(u)\,\mathrm{d}w(u)$. Then $\langle L\rangle_t = \int_0^t z(u)^2\,\mathrm{d}u$. Now, since we have

$$\begin{aligned}
\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}} &= \exp\left\{-\frac{p}{2n-p}L - \frac{p^2}{(2n-p)^2}\langle L\rangle\right\} \\
&\qquad\times \exp\left\{\left(\frac{p}{2(2n-p)} + \frac{p^2}{(2n-p)^2}\right)\langle L\rangle\right\},
\end{aligned}$$

by Schwartz inequality we have

$$\begin{aligned}
&\mathbf{E}[\,\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}}\,] \\
&\leq \mathbf{E}\Big[\exp\left\{-\frac{2p}{2n-p}L - \frac{2p^2}{(2n-p)^2}\langle L\rangle\right\}\Big]^{1/2} \\
&\qquad\times \mathbf{E}\Big[\exp\left\{\left(\frac{p}{(2n-p)} + \frac{2p^2}{(2n-p)^2}\right)\langle L\rangle\right\}\Big]^{1/2}.
\end{aligned}$$

Clearly, $\frac{p}{(2n-p)} + \frac{2p^2}{(2n-p)^2} \to 0$ as $n \to \infty$, and hence we can take large enough n to have the estimate (1.39) by using the assumption (1.38). □

Remark 5.1. By a similar but easier procedure we can also prove a continuity lemma for e^{D_θ} with $\theta \in \mathscr{H}$, to extend (1.13) in Corollary 2.3.4 to obtain a full version of CM formula.

CHAPTER 2

Ramer-Kusuoka Formula via an Action of Generalized Heisenberg Algebra

This part is based on the joint work [**3**].

1. Introduction

In this chapter, we approach the Ramer-Kusuoka formula from a completely algebraic viewpoint without using stochastic calculus and extract an algebraic structure of the Ramer-Kusuoka formula. We will start with an algebra $\mathcal{D}^*$ over $\mathbb{R}$, a generalization of the Heisenberg algebra, of which the generators ρ_i, ρ_i^* and κ_i's satisfy the commutation relations (2.1), (2.2) and (2.3) from section 2. We will see these calculations are generalizations of calculus with Brownian motion in section 4. We set $\psi_{ij} = ([\rho_i^*, \kappa_j^*])$, $\Psi = (\psi_{ij})_{ij}$, $\rho_\kappa = \sum_i \kappa_i \rho_i$ and $\rho_\kappa^* = \sum_i \kappa_i \rho_i^*$ and further definitions will be explained in section 2. Our main result is the following formula given in Theorem 2.3.5:

$$\det(1 + t\Psi) :\exp t(\rho_\kappa + \rho_\kappa^*): :\exp t(-\rho_\kappa): = 1 + \int_0^t g'(s) :\exp s\rho_\kappa: \mathrm{d}s.$$

where $g(t)$ is defined by (2.8).

In the previous chapter, we approached the Maruyama-Girsanov formula in an algebraic way. There, the predictable process z inducing our transform is assumed to be simple and we used essentially the nilpotency of Dz. The nilpotency of Dz implies that the traces of derived matrices, i.e., Dz, $Dz \wedge Dz$ etc are zero. From this point of view, we will study another representation of the formula given in Theorem 2.3.5 in the latter half of section 2.

In section 3, we represent our $\mathcal{D}^*$-algebra on the classical Wiener space toward on the Ramer-Kusuoka formula. Roughly speaking, ρ_i, ρ_i^* and κ_i are representated by a directional differential operator, it's L^2-adjoint with respect to the Wiener measure and any functional on the Wiener space respectively (Theorem 3.0.9).

In section 4, we explain that, on the classical Wiener space, the formula obtained in Theorem 2.3.5 is the the Ramer-Kusuoka type formula. To do this, we will introduce a vector field D_Z where Z is a measurable process (which may be non-adapted) inducing our transform. Formally, differentiation along the Cameron-Martin subspace can be viewed as a constant section of a bundle of which each fibre is the Cameron-Martin subspace. One may assume that D_Z randomize these constant sections by Z. For getting higher order sections, we will introduce "normal order"-type product $: * :$ (cf. [**36**]) and define a kind of section $:D_Z^n:$, where the relation between $:D_Z^n:$ and the Malliavin derivative D_t is given in section 5, Lemma 5.1.1.

The Ramer-Kusuoka type formula obtained in this algebraic framework is an equation in $\mathbb{R}[[t]]$ (the ring of formal power series in t) rather than $\mathbb{R}$. In section 5, we shall realize our Ramer-Kusuoka type formula as an equation in $\mathbb{R}$ for polynomial functionals on the Wiener space under some integrability condition.

2. A Generalized Heisenberg Algebra

We say an algebra as *$\mathcal{D}^*$-algebra* if it has generators $\{\rho_i, \rho_i^*, \kappa_i : i = 1, 2, \cdots\}$ with their defining relations

$$[\rho_i, \rho_j] = 0, \quad [\rho_i^*, \rho_j^*] = 0, \quad [\kappa_i, \kappa_j] = 0, \tag{2.1}$$

$$[\,[\rho_i^*, \kappa_j], \kappa_k\,] = 0, \quad [\,[\rho_i^*, \kappa_j], [\rho_k^*, \kappa_l]\,] = 0, \tag{2.2}$$

and

$$[\,\rho_i + \rho_i^*, \kappa_i^*\,] = 0, \quad [\,\rho_i + \rho_i^*, [\rho_j^*, \kappa_k]\,] = 0, \quad [\,\rho_i + \rho_i^*, \rho_j + \rho_j^*\,] = 0 \tag{2.3}$$

for every $i, j, k = 1, 2, \cdots$, where $[\cdot, \cdot]$ denotes the commutator with respect to original multiplication of $\mathcal{D}^*$. We fix a natural number N in the following and denote by $\mathcal{D}_N^*$ the subalgebra generated by $\{\rho_i, \rho_i^*, \kappa_i : i = 1, 2, \cdots\}$. $\mathcal{D}_N^*$ is also an $\mathcal{D}^*$-algebra. The subalgebra generated by

$$\left\{\kappa_i, \rho_i + \rho_i^*, [\rho_i^*, \kappa_i] : i = 1, 2, \cdots, N\right\}$$

is the commutative by (2.3) and will be denoted by $\mathscr{F}$.

Let $\mathscr{K}$ be the abelian subalgebra of $\mathcal{D}^*$ generated by $\{\kappa_i, [\rho_i^*, \kappa_j] : i, j = 1, 2, \cdots N\}$. Let $\mathscr{S}$ and $\mathscr{S}^*$ be the subalgebra generated by $\{\rho_i : i = 1, 2, \cdots, N\}$ and $\{\rho_i^* : i = 1, 2, \cdots, N\}$ respectively.

Example 2.1. Let $p(x)$ be a positive smooth function on $\mathbb{R}$. Let ∂ be the derivation: $\partial g = g'$ and let ∂^* be the operator defined by

$$\partial^* g = -\partial g - (\partial \log p) \cdot g$$

for compactly supported smooth function g. Take a compactly supported smooth function f and then $\{\partial, \partial^*, f\}$ generates $\mathcal{D}_1^*$ since

$$[\partial, f\,]g = (\partial f) \cdot g, \quad [\partial, \partial^*\,]g = -(\partial^2 \log p) \cdot g \quad \text{and} \quad [\partial^*, f\,]g = -(\partial f) \cdot g.$$

When $p(x) = e^{-\frac{x^2}{2}}$, the relation among $\partial, \partial^*, 1$ is that of the Heisenberg algebra. More generally for given positive smooth $p(x)$ on $\mathbb{R}^N$, $\{\partial_i, \partial_i^*, f_i\}_{i=1}^N$ generates $\mathcal{D}_N^*$ where

$$\partial_i^* g = -\partial_i g - (\partial_i \log p) \cdot g$$

and f_i is an arbitrary compactly supported smooth function.

For an abelian subalgebra $\mathscr{B}$ in $\mathcal{D}^*$ and $b_{ij} \in \mathscr{B}$, $i, j = 1, 2, \cdots, m$ the determinant of $m \times m$-matrix of $B = (b_{ij})$ can be defined usually as an element of $\mathscr{B}$.

Here we use as a conventional notation "normal order"-type product $:a: \in \mathcal{D}^*$ for $a \in \mathcal{D}^*$ in the following way: (i) $:a:$ is linear in a, (ii) within the colons all the elements commute and (iii) for a monomial it holds

$$\begin{aligned} &: \kappa_{i_1} \cdots \kappa_{i_l} \rho^*_{j_1} \cdots \rho^*_{j_m} \rho_{k_1} \cdots \rho_{k_n} : \\ &\qquad = \kappa_{i_1} \cdots \kappa_{i_l} \rho^*_{j_1} \cdots \rho^*_{j_m} \rho_{k_1} \cdots \rho_{k_n}. \end{aligned}$$

2.1. Formal Series with Coefficients in an Algebra. Let A be an algebra. We denote by $A[\![t]\!]$ the ring of formal series in t with coefficients in A. In this ring, some operation can be defined in obvious way: For $f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_n \in A[\![t]\!]$, differentiation and integration with respect to t are defined by

$$f'(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} f_{n+1} \quad \text{and} \quad \int_0^t f(s)\,\mathrm{d}s := \sum_{n=1}^{\infty} \frac{t^n}{n!} f_{n-1},$$

again each as an elements in $A[\![t]\!]$. In the case of $f(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} f_n$, i.e., $f_0 = 0$ the exponential can be also considered:

$$\exp f(t) = 1 + f(t) + \frac{1}{2} f(t)^2 + \frac{1}{3!} f(t)^3 + \frac{1}{4!} f(t)^4 + \cdots$$

because the right hand side is determined in order of lower degree of t and defines an element in $A[\![t]\!]$.

2.2. The First Algebraic Theorem. We set

$$\rho_\kappa = \sum_{k=1}^{N} \kappa_k \rho_k, \quad \rho^*_\kappa = \sum_{k=1}^{N} \kappa_k \rho^*_k.$$

Let $\mathcal{D}^*[\![t]\!] = \mathcal{D}^*_N[\![t]\!]$ denote the ring of formal series in t, a symbol, over $\mathcal{D}^* = \mathcal{D}^*_N$. For any $a \in \mathcal{D}^*$, we may define

$$\exp ta = \sum_{k=0}^{\infty} \frac{t^k}{k!} a^k \quad \text{and} \quad :\exp ta: = \sum_{k=0}^{\infty} \frac{t^k}{k!} :a^k:$$

as elements in $\mathcal{D}^*[\![t]\!]$. Then, for example, $:\rho_\kappa^2: = \sum_{i,j} \kappa_i\kappa_j \rho_i\rho_j$, $:\rho_\kappa^{*2}: = \sum_{i,j} \kappa_i\kappa_j \rho_i^*\rho_j^*$,

$$:\rho_\kappa^n: = \sum_{i_1,\cdots,i_n} \kappa_{i_1}\cdots\kappa_{i_n}\rho_{i_1}\cdots\rho_{i_n},$$

$$:\rho_\kappa^{*n}: = \sum_{i_1,\cdots,i_n} \kappa_{i_1}\cdots\kappa_{i_n}\rho_{i_1}^*\cdots\rho_{i_n}^*$$

and so on. The following consists of the first half of our algebraic approach.

Theorem 2.2.1. *We have in* $\mathcal{D}^*[\![t]\!]$

$$\Big(:\exp t(\rho_\kappa+\rho_\kappa^*):\Big)\Big(:\exp t(-\rho_\kappa):\Big) = :\exp t\rho_\kappa^*: . \tag{2.4}$$

Remark 2.1. Here we should understand $:(\rho_\kappa+\rho_\kappa^*)^k:$ as

$$\sum_{j=0}^{k} \frac{k!}{(k-j)!j!} :\rho_\kappa^{*j}\rho_\kappa^{k-j}: .$$

Note that $:(\rho_\kappa+\rho_\kappa^*)^k:$ still commutes with $\mathscr{F}$.

Proof. First, note that

$$:\exp t(\rho_\kappa+\rho_\kappa^*): = \sum_{k=0}^{\infty}\frac{1}{k!}:(\rho_\kappa+\rho_\kappa^*)^k:= \sum_{k=0}^{\infty}\frac{1}{k!}:\Big\{\sum_{i=1}^{N}\kappa_i(\rho_i+\rho_i^*)\Big\}^k:$$

$$= \sum_{k=0}^{\infty}\frac{1}{n!}\sum_{i_1,\cdots,i_k}\kappa_{i_1}\cdots\kappa_{i_k}:(\rho_{i_1}+\rho_{i_1}^*)\cdots(\rho_{i_n}+\rho_{i_n}^*):$$

and

$$:\exp t(-\rho_\kappa):= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}:\rho_\kappa^k:= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{i_1,\cdots,i_k}\kappa_{i_1}\cdots\kappa_{i_k}\rho_{i_1}\cdots\rho_{i_k}.$$

Then, the left-hand-side of (2.4) is rewritten as

$$\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}A_kB_{n-k}$$

where

$$A_k = \sum_{i_1,\cdots,i_k}\kappa_{i_1}\cdots\kappa_{i_k}:(\rho_{i_k}+\rho_{i_k}^*)\cdots(\rho_{i_k}+\rho_{i_k}^*):$$

and

$$B_{n-k} = \sum_{i_1,\cdots,i_{n-k}}\kappa_{i_1}\cdots\kappa_{i_{n-k}}(-\rho_{i_i})\cdots(-\rho_{i_{n-k}}).$$

Since $(\rho_k + \rho_k^*)$'s and κ_j's commute, we have by the definition of our normal product

$$\begin{aligned}
&A_k B_{n-k} \\
&= \sum_{i_1,\cdots,i_n} \kappa_{i_1} \cdots \kappa_{i_n} :(\rho_{i_1} + \rho_{i_1}^*) \cdots (\rho_{i_n} + \rho_{i_k}^*)(-\rho_{i_{k+1}}) \cdots (-\rho_{i_n}): \\
&= :(\rho_\kappa + \rho_\kappa^*)^k (-\rho_\kappa)^{n-k}:
\end{aligned}$$

Therefore, the left-hand-side of (2.4) now becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} :(\rho_\kappa + \rho_\kappa^*)^k (-\rho_\kappa)^{n-k}: \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} :(\rho_\kappa + \rho_\kappa^* - \rho_\kappa)^n: = \sum_{n=0}^{\infty} \frac{1}{n!} :(\rho_\kappa^*)^n: .
\end{aligned}$$

This completes the proof. □

2.3. The Second Algebraic Theorem.

Lemma 2.3.1.

$$\sum_{i_1,\cdots,i_k} \mathrm{ad}_{\rho_{i_1}^*} \det \begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_k} \\ [\rho_{i_2}^*, \kappa_{i_1}] & \cdots & [\rho_{i_2}^*, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho_{i_k}^*, \kappa_{i_1}] & \cdots & [\rho_{i_k}^*, \kappa_{i_k}] \end{pmatrix} = \sum_{i_1,\cdots,i_k} \det \begin{pmatrix} [\rho_{i_1}^*, \kappa_{i_1}] & \cdots & [\rho_{i_1}^*, \kappa_{i_k}] \\ [\rho_{i_2}^*, \kappa_{i_1}] & \cdots & [\rho_{i_2}^*, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho_{i_k}^*, \kappa_{i_1}] & \cdots & [\rho_{i_k}^*, \kappa_{i_k}] \end{pmatrix}.$$

Proof. It suffices to prove that

$$\sum_{i_1,\cdots,i_k} \det \begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_k} \\ [\rho_{i_2}^*, \kappa_{i_1}] & \cdots & [\rho_{i_2}^*, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho_{i_1}^*, [\rho_{i_l}^*, \kappa_{i_1}]] & \cdots & [\rho_{i_1}^*, [\rho_{i_l}^*, \kappa_{i_k}]] \\ \vdots & \ddots & \vdots \\ [\rho_{i_k}^*, \kappa_{i_1}] & \cdots & [\rho_{i_k}^*, \kappa_{i_k}] \end{pmatrix} = 0 \tag{2.5}$$

for each $l = 2, \cdots, k$. Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$ denote the row vectors of above matrix in (2.5). Then by skew-symmetry of determinant yields that

$$\det(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k) + \det(\boldsymbol{a}_l, \boldsymbol{a}_1, \cdots, \widehat{\boldsymbol{a}_l}, \cdots, \boldsymbol{a}_k) = 0.$$

But the commutation relation (2.1) and the Jacobi identity imply that $[\rho_{i_1}^*, [\rho_{i_l}^*, \kappa_{i_1}]] = [\rho_{i_l}^*, [\rho_{i_1}^*, \kappa_{i_1}]]$ and $[\rho_{i_1}^*, [\rho_{i_l}^*, \kappa_{i_l}]] = [\rho_{i_l}^*, [\rho_{i_1}^*, \kappa_{i_l}]]$ and hence by taking sum over $i_1, \cdots, i_k$ we obtain the result. □

Proposition 2.3.2.

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i_1,\cdots,i_k}\det\begin{pmatrix}[\rho^*_{i_1},\kappa_{i_1}] & \cdots & [\rho^*_{i_1},\kappa_{i_k}]\\ \vdots & \ddots & \vdots\\ [\rho^*_{i_k},\kappa_{i_1}] & \cdots & [\rho^*_{i_k},\kappa_{i_k}]\end{pmatrix}:\rho_\kappa^{n-k}:$$

$$=\sum_{i_1,\cdots,i_k}\rho^*_{i_1}\Big\{\sum_{k=1}^{n}\binom{n-1}{k-1}\det\begin{pmatrix}\kappa_{i_1} & \cdots & \kappa_{i_k}\\ [\rho^*_{i_2},\kappa_{i_1}] & \cdots & [\rho^*_{i_2},\kappa_{i_k}]\\ \vdots & \ddots & \vdots\\ [\rho^*_{i_k},\kappa_{i_1}] & \cdots & [\rho^*_{i_k},\kappa_{i_k}]\end{pmatrix}:\rho_\kappa^{n-k}:\Big\}.$$

Proof. By Leibniz' rule we find that

$$\rho^*_{i_1}\Big\{\det\begin{pmatrix}\kappa_{i_1} & \cdots & \kappa_{i_k}\\ [\rho^*_{i_2},\kappa_{i_1}] & \cdots & [\rho^*_{i_2},\kappa_{i_k}]\\ \vdots & \ddots & \vdots\\ [\rho^*_{i_k},\kappa_{i_1}] & \cdots & [\rho^*_{i_k},\kappa_{i_k}]\end{pmatrix}\kappa_{i_{k+1}}\cdots\kappa_{i_n}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}\Big\}$$

$$=\Big[\rho^*_{i_1},\det\begin{pmatrix}\kappa_{i_1} & \cdots & \kappa_{i_k}\\ [\rho^*_{i_2},\kappa_{i_1}] & \cdots & [\rho^*_{i_2},\kappa_{i_k}]\\ \vdots & \ddots & \vdots\\ [\rho^*_{i_k},\kappa_{i_1}] & \cdots & [\rho^*_{i_k},\kappa_{i_k}]\end{pmatrix}\Big]\kappa_{i_{k+1}}\cdots\kappa_{i_n}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}$$

$$+\sum_{j=1}^{n-k}\det\begin{pmatrix}\kappa_{i_1} & \cdots & \kappa_{i_k}\\ [\rho^*_{i_2},\kappa_{i_1}] & \cdots & [\rho^*_{i_2},\kappa_{i_k}]\\ \vdots & \ddots & \vdots\\ [\rho^*_{i_k},\kappa_{i_1}] & \cdots & [\rho^*_{i_k},\kappa_{i_k}]\end{pmatrix}\kappa_{i_{k+1}}\cdots[\rho^*_{i_1},\kappa_{i_{k+j}}]\cdots\kappa_{i_n}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}$$

$$+\det\begin{pmatrix}\kappa_{i_1} & \cdots & \kappa_{i_k}\\ [\rho^*_{i_2},\kappa_{i_1}] & \cdots & [\rho^*_{i_2},\kappa_{i_k}]\\ \vdots & \ddots & \vdots\\ [\rho^*_{i_k},\kappa_{i_1}] & \cdots & [\rho^*_{i_k},\kappa_{i_k}]\end{pmatrix}\kappa_{i_{k+1}}\cdots\kappa_{i_n}\rho^*_{i_1}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}.$$

By previous lemma, the right hand side in this proposition is equal to the sum of

$$I_k=\binom{n-1}{k-1}\det\begin{pmatrix}[\rho^*_{i_1},\kappa_{i_1}] & \cdots & [\rho^*_{i_1},\kappa_{i_k}]\\ \vdots & \ddots & \vdots\\ [\rho^*_{i_k},\kappa_{i_1}] & \cdots & [\rho^*_{i_k},\kappa_{i_k}]\end{pmatrix}\kappa_{i_{k+1}}\cdots\kappa_{i_n}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n},$$

$$\mathit{II}_k = (n-k)\binom{n-1}{k-1}\det\begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_k} \\ [\rho^*_{i_2}, \kappa_{i_1}] & \cdots & [\rho^*_{i_2}, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho^*_{i_k}, \kappa_{i_1}] & \cdots & [\rho^*_{i_k}, \kappa_{i_k}] \end{pmatrix} [\rho^*_{i_1}, \kappa_{i_{k+1}}]\kappa_{i_{k+2}}\cdots\kappa_{i_n}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}$$

and

$$\mathit{III}_k = \binom{n-1}{k-1}\det\begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_k} \\ [\rho^*_{i_2}, \kappa_{i_1}] & \cdots & [\rho^*_{i_2}, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho^*_{i_k}, \kappa_{i_1}] & \cdots & [\rho^*_{i_k}, \kappa_{i_k}] \end{pmatrix} \kappa_{i_{k+1}}\cdots\kappa_{i_n}\rho^*_{i_1}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}.$$

We apply to III_{k+1} the cofactor expansion about to the first row to find the sum of III_{k+1} over $i_1, \cdots, i_n$ is equal to the sum of

$$\binom{n-1}{k}\det\begin{pmatrix} [\rho^*_{i_2}, \kappa_{i_1}] & \cdots & [\rho^*_{i_2}, \kappa_{i_{k+1}}] \\ \vdots & \ddots & \vdots \\ [\rho^*_{i_{k+1}}, \kappa_{i_1}] & \cdots & [\rho^*_{i_{k+1}}, \kappa_{i_{k+1}}] \end{pmatrix} \kappa_{i_1}\kappa_{i_{k+2}}\cdots\kappa_{i_n}\rho^*_{i_1}\rho^*_{i_{k+2}}\cdots\rho^*_{i_n}$$

$$-k\binom{n-1}{k}[\rho^*_{i_1}, \kappa_{i_{k+1}}]\det\begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_k} \\ [\rho^*_{i_2}, \kappa_{i_1}] & \cdots & [\rho^*_{i_2}, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho^*_{i_k}, \kappa_{i_1}] & \cdots & [\rho^*_{i_k}, \kappa_{i_k}] \end{pmatrix} \kappa_{i_{k+2}}\cdots\kappa_{i_n}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}.$$

Hence with using formula

$$k\binom{n-1}{k} = (n-k)\binom{n-1}{k-1},$$

it turns out that the sum of $\mathit{III}_{k+1} + \mathit{II}_k$ over $i_1, \cdots, i_n$ is equal to the sum of

$$\binom{n-1}{k}\det\begin{pmatrix} [\rho^*_{i_2}, \kappa_{i_1}] & \cdots & [\rho^*_{i_2}, \kappa_{i_{k+1}}] \\ \vdots & \ddots & \vdots \\ [\rho^*_{i_{k+1}}, \kappa_{i_1}] & \cdots & [\rho^*_{i_{k+1}}, \kappa_{i_{k+1}}] \end{pmatrix} \kappa_{i_1}\kappa_{i_{k+2}}\cdots\kappa_{i_n}\rho^*_{i_1}\rho^*_{i_{k+2}}\cdots\rho^*_{i_n}$$

over $i_1, \cdots, i_n$. Furthermore we use formula

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

to see that sum of $\mathit{III}_{k+1} + I_k + \mathit{II}_k$ equals to the sum of

$$\binom{n}{k}\det\begin{pmatrix} [\rho^*_{i_1}, \kappa_{i_1}] & \cdots & [\rho^*_{i_1}, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho^*_{i_k}, \kappa_{i_1}] & \cdots & [\rho^*_{i_k}, \kappa_{i_k}] \end{pmatrix} \kappa_{i_{k+1}}\cdots\kappa_{i_n}\rho^*_{i_{k+1}}\cdots\rho^*_{i_n}.$$

Now this proposition follows easily. □

We denote by Φ and Ψ the matrices $(\phi_{ij})_{i,j=1}^N$ and $(\psi_{ij})_{i,j=1}^N$ respectively, where

$$\phi_{ij} := \rho_i^* \kappa_j \quad \text{and} \quad \psi_{ij} := [\,\rho_i^*, \kappa_j\,].$$

We note that

$$\kappa_j \rho_i^* = \phi_{ij} - \psi_{ij}.$$

From the knowledge of linear algebra we find that

$$\det(1 + t\Psi) = \sum_{n=0}^{N} \frac{t^n}{n!} \sum_{i_1,\cdots,i_n} \det \begin{pmatrix} \psi_{i_1 i_1} & \cdots & \psi_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ \psi_{i_n i_1} & \cdots & \psi_{i_n i_n} \end{pmatrix} \tag{2.6}$$

as an element in $\mathcal{D}^*[t] \subset \mathcal{D}^*[\![t]\!]$.

Proposition 2.3.3. *As in $\mathcal{D}^*[\![t]\!]$ we have*

$$\det(1 + t\Psi) : \exp t\rho_\kappa^* :$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i_1,\cdots,i_k} \rho_{i_1}^* \Big\{ \sum_{k=1}^{n} \binom{n-1}{k-1} \det \begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_k} \\ [\rho_{i_2}^*, \kappa_{i_1}] & \cdots & [\rho_{i_2}^*, \kappa_{i_k}] \\ \vdots & \ddots & \vdots \\ [\rho_{i_k}^*, \kappa_{i_1}] & \cdots & [\rho_{i_k}^*, \kappa_{i_k}] \end{pmatrix} : \rho_\kappa^{n-k} : \Big\}.$$

Proof. The left hand side of the proposition is equal to

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \sum_{i_1,\cdots,i_n} \det \begin{pmatrix} \psi_{i_1 i_1} & \cdots & \psi_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ \psi_{i_k i_1} & \cdots & \psi_{i_k i_k} \end{pmatrix} \kappa_{i_{k+1}} \cdots \kappa_{i_n} \rho_{i_{k+1}}^* \cdots \rho_{i_n}^*. \tag{2.7}$$

with using (2.6). Hence our claim follows immediately by Proposition 2.3.2. □

We shall set

$$g(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i_1,\cdots,i_n} \rho_{i_1}^* \det \begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_n} \\ \psi_{i_2 i_1} & \cdots & \psi_{i_2 i_n} \\ \vdots & \ddots & \vdots \\ \psi_{i_n i_1} & \cdots & \psi_{i_n i_n} \end{pmatrix} \in \mathcal{D}^*[\![t]\!] \tag{2.8}$$

and then the previous Proposition 2.3.3 implies the next algebraic fact.

Proposition 2.3.4.

$$\det(1 + t\Psi) : \exp t\rho_\kappa^* : \; = 1 + \int_0^t g'(s) : \exp s\rho_\kappa : \mathrm{d}s.$$

Now we combine the first algebraic theorem 2.2.1 and proposition 2.3.4 to deduce

THEOREM 2.3.5.

$$\begin{aligned}\det(1+t\Psi):\exp t(\rho_\kappa+\rho_\kappa^*)::\exp t(-\rho_\kappa):\\ =1+\int_0^t g'(s):\exp s\rho_\kappa: \mathrm{d}s.\end{aligned} \tag{2.9}$$

This formula seems to reveal the algebraic structure of the Ramer-Kusuoka formula. The reason why will be turn out in Section 4 .

2.4. Trace Expression. Before entering the Section 3, where we represent our algebra $\mathcal{D}^*$ on the Wiener space, we shall see and understand more about the expression of (2.9). If we write the left hand side of (2.9) as the form of exponential, how the expressions will appears on the exponential ? The expression should be a series of t and then what and how polynomials will appears as the coefficient of each t^n?

We set $\mathbf{y}=(y_1,y_2,\cdots)$, $\mathbf{z}=(z_1,z_2,\cdots)$ and $\mathbf{w}=(w_1,w_2,\cdots)$ as

$$\mathbf{y}=\Big(\mathrm{tr}(\Phi),\mathrm{tr}(\Phi\Psi),\mathrm{tr}(\Phi\Psi^2),\mathrm{tr}(\Phi\Psi^3),\cdots\Big),$$

$$\mathbf{z}=\Big(\mathrm{tr}(\Psi),\mathrm{tr}(\Psi^2),\mathrm{tr}(\Psi^3),\mathrm{tr}(\Psi^4),\cdots\Big)$$

and

$$\mathbf{w}=\Big(\mathrm{tr}(\Phi),\mathrm{tr}(\Psi\Phi),\mathrm{tr}(\Psi^2\Phi),\mathrm{tr}(\Psi^3\Phi),\cdots\Big).$$

Since $\mathscr{K}$ is abelian, we can define $\wedge^n\Psi$ for $n=0,1,2,\cdots,N$ usually and the knowledge from linear algebra yields

$$\mathrm{tr}(\wedge^n\Psi)=\frac{1}{n!}\sum_{i_1,\cdots,i_n}\det\begin{pmatrix}\psi_{i_1i_1}&\cdots&\psi_{i_1i_n}\\ \vdots&\ddots&\vdots\\ \psi_{i_ni_1}&\cdots&\psi_{i_ni_n}\end{pmatrix}.$$

LEMMA 2.4.1. *For each $n=1,2,\cdots$, we have*

$$\sum_{i_1,\cdots,i_n}\rho_{i_1}^*\det\begin{pmatrix}\kappa_{i_1}&\cdots&\kappa_{i_n}\\ \psi_{i_2i_1}&\cdots&\psi_{i_2i_n}\\ \vdots&\ddots&\vdots\\ \psi_{i_ni_1}&\cdots&\psi_{i_ni_n}\end{pmatrix}=\sum_{k=1}^n(-1)^{k-1}(n-1)\,\mathrm{tr}(\Phi\Psi^{k-1})\,\mathrm{tr}(\wedge^{n-k}\Psi).$$

COROLLARY 2.4.2.

$$g(t)=\sum_{n=0}^\infty\frac{t^n}{n!}\sum_{i_1,\cdots,i_n}\rho_{i_1}^*\det\begin{pmatrix}\kappa_{i_1}&\cdots&\kappa_{i_n}\\ \psi_{i_2i_1}&\cdots&\psi_{i_2i_n}\\ \vdots&\ddots&\vdots\\ \psi_{i_ni_1}&\cdots&\psi_{i_ni_n}\end{pmatrix}\in\mathcal{D}^*[\![t]\!]$$

satisfies

$$g'(t) = \Big(\sum_{n=0}^{\infty} t^n \operatorname{tr}(\Phi(-\Psi)^{n-1})\Big) \cdot \det(1 + t\Psi).$$

Corollary 2.4.3. *For each $n = 1, 2, \cdots$, there exists a polynomial $\widetilde{q}_n(\mathbf{x}_1, \mathbf{x}_2)$ such that*

$$\sum_{i_1,\cdots,i_n} \rho^*_{i_1} \det \begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_n} \\ \psi_{i_2 i_1} & \cdots & \psi_{i_2 i_n} \\ \vdots & \ddots & \vdots \\ \psi_{i_n i_1} & \cdots & \psi_{i_n i_n} \end{pmatrix} = \widetilde{q}_n(\mathbf{y}, \mathbf{z})$$

and $\widetilde{q}_n(\mathbf{y}, \mathbf{z})$ is a linear combination of monomials which consist leading multiple of the form $\operatorname{tr}(\Phi\Psi^k)$, $k = 0, 1, 2, \cdots, n$ and other multiples of the form $\operatorname{tr}(\Psi^k)$, $k = 0, 1, 2, \cdots, n-1$.

Proof of Lemma 2.4.1. Let

$$X_2 := \sum_{i_1,\cdots,i_n} \phi_{i_1 i_2} \det \begin{pmatrix} \psi_{i_2 i_1} & \psi_{i_2 i_3} & \cdots & \psi_{i_2 i_n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{i_n i_1} & \psi_{i_n i_3} & \cdots & \psi_{i_n i_n} \end{pmatrix}$$

and

$$X_k := \sum_{i_1,\cdots,i_n} \phi_{i_1 i_2} \psi_{i_2 i_3} \cdots \psi_{i_{k-1} i_k} \det \begin{pmatrix} \psi_{i_k i_1} & \psi_{i_k i_{k+1}} & \cdots & \psi_{i_k i_n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{i_n i_1} & \psi_{i_n i_{k+1}} & \cdots & \psi_{i_n i_n} \end{pmatrix}$$

for $k = 3, 4, \cdots, n-1$. Applying cofactor expansion to the first column we get

$$\sum_{i_1,\cdots,i_n} \rho^*_{i_1} \det \begin{pmatrix} \kappa_{i_1} & \cdots & \kappa_{i_n} \\ \psi_{i_2 i_1} & \cdots & \psi_{i_2 i_n} \\ \vdots & \ddots & \vdots \\ \psi_{i_n i_1} & \cdots & \psi_{i_n i_n} \end{pmatrix} = \operatorname{tr}(\Phi) \cdot (n-1)! \operatorname{tr}(\wedge^{n-1}\Psi) - (n-1) X_2,$$

In the same way we get

$$X_k = \operatorname{tr}(\Phi\Psi^{k-1}) \cdot (n-k)! \operatorname{tr}(\wedge^{n-k}\Psi) - (n-k) X_{k+1} \tag{2.10}$$

for $k = 2, 3, \cdots, n-2$ by cofactor expansion about the first row in X_k. Using (2.10) and that

$$X_{n-1} = \operatorname{tr}(\Phi\Psi^{n-2}) \operatorname{tr}(\Psi) - \operatorname{tr}(\Phi\Psi^{n-1})$$

we can obtain the result. □

Lemma 2.4.4. *For each $n = 1, 2, \cdots$, we have*

$$:\rho_\kappa^{*n}: \ = \sum_{k=1}^{n} (-1)^{k-1} \frac{(n-1)!}{(n-k)!} \operatorname{tr}(\Psi^{k-1}(\Phi - \Psi)) \ :\rho_\kappa^{*n-k}: .$$

Proof. Since

$$\begin{aligned}
&\kappa_{i_1} \cdots \kappa_{i_n} \rho_{i_1}^* \cdots \rho_{i_n}^* \\
&= \kappa_{i_1} \rho_{i_1}^* \kappa_{i_2} \cdots \kappa_{i_n} \rho_{i_2}^* \cdots \rho_{i_n}^* + \kappa_{i_1} [\kappa_{i_2} \cdots \kappa_{i_n}, \rho_{i_1}^*] \rho_{i_1}^* \cdots \rho_{i_n}^* \\
&= \kappa_{i_1} \rho_{i_1}^* \kappa_{i_2} \cdots \kappa_{i_n} \rho_{i_2}^* \cdots \rho_{i_n}^* + \sum_{k=2}^{n} [\kappa_{i_k}, \rho_{i_1}^*] \kappa_{i_1} \kappa_{i_2} \cdots \widehat{\kappa_{i_k}} \cdots \kappa_{i_n} \rho_{i_2}^* \cdots \rho_{i_n}^*,
\end{aligned}$$

we have

$$\begin{aligned}
:\rho_\kappa^{*n}: \ &= \operatorname{tr}(\Phi) :\rho_\kappa^{*n-1}: + \sum_{i_1, \cdots, i_n} (n-1) [\kappa_{i_2}, \rho_{i_1}^*] \kappa_{i_1} \kappa_{i_3} \cdots \kappa_{i_n} \rho_{i_2}^* \cdots \rho_{i_n}^* \\
&= X_1 + (n-1) Y_1 \quad \text{(say)}.
\end{aligned}$$

Since

$$\begin{aligned}
&[\kappa_{i_2}, \rho_{i_1}^*] \kappa_{i_1} \kappa_{i_3} \cdots \kappa_{i_n} \rho_{i_2}^* \cdots \rho_{i_n}^* \\
&= [\kappa_{i_2}, \rho_{i_1}^*] \kappa_{i_1} \rho_{i_2}^* \kappa_{i_3} \cdots \kappa_{i_n} \rho_{i_3}^* \cdots \rho_{i_n}^* \\
&\qquad + \sum_{k=3}^{n} [\kappa_{i_2}, \rho_{i_1}^*][\kappa_{i_k}, \rho_{i_2}^*] \kappa_{i_1} \kappa_{i_3} \cdots \widehat{\kappa_{i_k}} \cdots \kappa_{i_n} \rho_{i_3}^* \cdots \rho_{i_n}^*,
\end{aligned}$$

Y_1 equals to

$$-\operatorname{tr}(\Psi\Phi) :\rho_\kappa^{*n-2}: + (n-2) \sum_{i_1, \cdots, i_n} [\kappa_{i_2}, \rho_{i_1}^*][\kappa_{i_3}, \rho_{i_2}^*] \kappa_{i_1} \kappa_{i_4} \cdots \kappa_{i_n} \rho_{i_3}^* \cdots \rho_{i_n}^*.$$

So we shall put for $k = 1, 2, \cdots, n-1$,

$$X_k := (-1)^{k-1} \operatorname{tr}(\Psi^{k-1}\Phi) :\rho_\kappa^{*n-k}:$$

and if $k = 1, 2, \cdots, n-2$,

$$Y_k := \sum_{i_1, \cdots, i_n} [\kappa_{i_2}, \rho_{i_1}^*][\kappa_{i_3}, \rho_{i_2}^*] \cdots [\kappa_{i_{k+1}}, \rho_{i_k}^*] \kappa_{i_1} \kappa_{i_{k+2}} \cdots \kappa_{i_n} \rho_{i_{k+1}}^* \cdots \rho_{i_n}^*$$

and

$$Y_{n-1} := \sum_{i_1, \cdots, i_n} [\kappa_{i_2}, \rho_{i_1}^*][\kappa_{i_3}, \rho_{i_2}^*] \cdots [\kappa_{i_n}, \rho_{i_{n-1}}^*] \kappa_{i_1} \rho_{i_n}^* = (-1)^{n-1} \operatorname{tr}(\Psi^{n-1}\Phi).$$

Then by the same way as above we see that

$$\begin{aligned}
&Y_{k-1} = -X_k + (n-k) Y_k \quad \text{for} \quad k = 2, 3, \cdots, n \quad \text{and} \\
&Y_{n-2} = -X_{n-1} + Y_{n-1}.
\end{aligned}$$

Then by using this we obtain the result. □

Corollary 2.4.5. *For each* $n = 1, 2, \cdots$, *there exists a polynomial* $r_n(\mathbf{x}_1, \mathbf{x}_2)$ *such that* $:\rho_\kappa^{*n}: = r_n(\mathbf{y}, \mathbf{w})$.

Corollary 2.4.6.

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} :\rho_\kappa^{*n}: \in \mathcal{D}^*[\![t]\!]$$

satisfies

$$f'(t) = \sum_{n=0}^{\infty} t^n \operatorname{tr}((-\Psi^n)(\Phi - \Psi)) \cdot f(t).$$

Corollary 2.4.7.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} :\rho_\kappa^{*n}: = f(t) = \exp\Big\{ \sum_{n=1}^{\infty} \frac{t^n}{n} \operatorname{tr}((-\Psi)^n) + \sum_{n=1}^{\infty} \frac{t^n}{n} \operatorname{tr}((-\Psi)^{n-1}\Phi) \Big\}.$$

Finally we can give answer to our interests.

Theorem 2.4.8. *There exist polynomials* $q_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, $q_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, $\cdots$ *such that*

$$\det(1 + t\Psi) :\exp t\rho_\kappa^*: = \exp\Big\{ t\, q_1(\mathbf{y}, \mathbf{z}, \mathbf{w}) + t^2 q_2(\mathbf{y}, \mathbf{z}, \mathbf{w}) + \cdots \Big\}.$$

as an element in $\mathcal{D}^*[\![t]\!]$ *and each* $q_k(\mathbf{y}, \mathbf{z}, \mathbf{w})$ *is a linear combination of monomials whose leading factors are of the form* $\operatorname{tr}(\Phi\Psi^l)$, $l = 0, 1, 2, \cdots, k-1$.

Proof. We set $\xi(t, \mathbf{x})$ and $\{p_i(\mathbf{x}) : i \in \mathbb{N}\}$ by

$$\xi(t, \mathbf{x}) = t x_1 + t^2 x_2 + t^3 x_3 + \cdots$$

and

$$\mathrm{e}^{\xi(t,\mathbf{x})} = 1 + p_1(\mathbf{x})\, t + p_2(\mathbf{x})\, t^2 + p_3(\mathbf{x})\, t^3 + \cdots$$

inductively. Note that we don't assume commutativity of x_i's. For example, we have

$$\begin{aligned}
& p_1(\mathbf{x}) = x_1, \quad p_2(\mathbf{x}) = \frac{x_1^2}{2} + x_2, \\
& p_3(\mathbf{x}) = \frac{x_1^3}{6} + \frac{x_1 x_2 + x_2 x_1}{2} + x_3, \\
& p_4(\mathbf{x}) = \frac{x_1^4}{24} + \frac{x_1^2 x_2 + x_1 x_2 x_1 + x_2 x_1^2}{6} + \frac{x_1 x_3 + x_2^2 + x_3 x_1}{2} + x_4
\end{aligned} \tag{2.11}$$

and so on. Now we have only to determine x_i's when we assume that

$$\det(1 + t\Psi) :\exp t\rho_\kappa^*: = \mathrm{e}^{\xi(t,\mathbf{x})}.$$

By proposition 2.3.2, we see that

$$p_1(\mathbf{x}) = -\sum_{k=1}^{N} \rho_k^*(-\kappa_k) = \mathrm{tr}(\Phi)$$

and hence we obtain $q_1(\mathbf{y},\mathbf{z},\mathbf{w}) = y_1$. Assume that $q_2(\mathbf{y},\mathbf{z},\mathbf{w}),\cdots,q_m(\mathbf{y},\mathbf{z},\mathbf{w})$ are given so that the assumption are satisfied and then can find $q_{m+1}(\mathbf{y},\mathbf{z},\mathbf{w})$ with using proposition 2.3.2 and corollaries 2.4.3, 2.4.5. □

In fact, with a little computation together with (2.11), we can give $q_1(\mathbf{y},\mathbf{z},\mathbf{w})$, $q_2(\mathbf{y},\mathbf{z},\mathbf{w})$, $q_3(\mathbf{y},\mathbf{z},\mathbf{w})$ and $q_4(\mathbf{y},\mathbf{z},\mathbf{w})$ explicitly:

$$\begin{aligned} q_1(\mathbf{y},\mathbf{z},\mathbf{w}) &= \mathrm{tr}(\Phi), \quad q_2(\mathbf{y},\mathbf{z},\mathbf{w}) = -\frac{\mathrm{tr}(\Phi\Psi)}{2}, \\ q_3(\mathbf{y},\mathbf{z},\mathbf{w}) &= \frac{\mathrm{tr}(\Phi\Psi^2)}{3} - \frac{1}{12}[\,\mathrm{tr}(\Phi\Psi),\mathrm{tr}(\Phi)\,], \\ q_4(\mathbf{y},\mathbf{z},\mathbf{w}) &= -\frac{\mathrm{tr}(\Phi\Psi^3)}{4} + \frac{1}{12}[\,\mathrm{tr}(\Phi\Psi^2),\mathrm{tr}(\Phi)]. \end{aligned}$$

3. Representation of $\mathcal{D}^*$-algebra on Path Space

In this section, to understand Theorem 2.3.5 with using a standard setting of stochastic analysis, we construct a representation of $\mathcal{D}^*$-algebra on the Wiener space.

Let $(\mathscr{W}, H, \mathbf{P})$ be the classical Wiener space on closed interval $[0,1]$. For $\xi \in H$, we denote by D_ξ the differentiation in the direction of ξ: for a function F on $\mathscr{W}$,

$$D_\xi F(w) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{F(w+\varepsilon\xi) - F(w)\Big\} \quad \text{if it exists.}$$

We define an operator D_ξ^* by

$$D_\xi^* F(w) := -D_\xi F(w) + \int_0^1 \dot{\xi}(t)\,\mathrm{d}w(t)\cdot F(w)$$

for a differentiable function F defined on $\mathscr{W}$.

We shall fix an orthonormal basis $\xi_1, \xi_2, \cdots$ of H and D_k, D_k^* denotes the corresponding differentiations: $D_k = D_{\xi_k}$ and $D_k^* = D_{\xi_k}^*$ for $k = 1, 2, \cdots$. Recall that we have fixed a natural number N. Let $\mathcal{P} = \mathcal{P}_N = \mathbb{R}[\int_0^1 \dot{\xi}_k(t)\mathrm{d}w(t) : k = 1, 2, \cdots, N]$ and then D_ξ is a linear mapping from $\mathcal{P}$ to $\mathcal{P}$ and it extends to a linear mapping from $\mathbb{D}_\infty$ to $\mathbb{D}_\infty$ where $\mathbb{D}_\infty$ stands for Meyer-Watanabe's test function space. Note that $\mathbb{D}_\infty$ forms an algebra.

Example 3.1. If we set for $k = 1, 2, \cdots, N$

$$\kappa_k := 0, \quad \rho_k := D_k \quad \text{and} \quad \rho_k^* := D_k^*$$

then we obtain $\mathcal{D}^*$-algebra generated by $\{\rho_k, \rho_k^* : k = 1, 2, \cdots, N\}$, here we take the multiplication as the composition of operators, which satisfies the Heisenberg commutation relations

$$[\rho_i, \rho_j] = 0, \quad [\rho_i, \rho_j^*] = \delta_{ij} \quad \text{and} \quad [\rho_i^*, \rho_j^*] = 0. \tag{2.12}$$

Now we fix $Z_1, Z_2, \cdots \in \mathbb{D}_\infty$ which act on $\mathbb{D}_\infty$ as multiplication operators. Let $\mathscr{D} = \mathscr{D}_N$ be the algebra generated by $\{D_k, D_k^*, Z_k\}_{k=1}^N$ The "normal order"-type product $: x :\in \mathscr{D}$ for $x \in \mathscr{D}$ can be considered in the following way: (i) $: a :$ is linear in a, (ii) within the colons all the elements commute and (iii) for a monomial it holds

$$\begin{aligned} &: Z_{i_1} \cdots Z_{i_l} D_{j_1}^* \cdots D_{j_m}^* D_{k_1} \cdots D_{k_n} : \\ &\qquad = Z_{i_1} \cdots Z_{i_l} D_{j_1}^* \cdots D_{j_m}^* D_{k_1} \cdots D_{k_n}. \end{aligned}$$

Let $\varphi : \mathcal{D}_N^* \to \mathscr{D}_N \subset \mathfrak{gl}(\mathbb{D}_\infty)$ be the homomorphism of $\mathcal{D}_N^*$ determined by

$$\varphi(\rho_k) = D_k, \quad \varphi(\rho_k^*) = D_k^* \quad \text{and} \quad \varphi(\kappa_k) = Z_k \tag{2.13}$$

for each $k = 1, 2, \cdots, N$.

Theorem 3.0.9. *φ gives a representation of the algebra $\mathcal{D}^*$.*

Proof. It is enough to prove that

$$[D_i, D_j] = 0, \quad [D_i^*, D_j^*] = 0, \quad [Z_i, Z_j] = 0,$$

$$[[D_i^*, Z_j], Z_k] = 0, \quad [[D_i^*, Z_j], [D_k^*, Z_l]] = 0,$$

and

$$[D_i + D_i^*, Z_i] = 0, \quad [D_i + D_i^*, [D_j^*, Z_k]] = 0, \quad [D_i + D_i^*, D_j + D_j^*] = 0.$$

But since for every $f \in \mathbb{D}_\infty$, $[D_i^*, f]$ and $[f, D_i]$ act on $\mathbb{D}_\infty$ as the multiplication by $-D_i f$, the first seven identities are clear. The last identity follows by the Heisenberg commutation relations. □

4. Reduction to the Ramer-Kusuoka Formula

In this section, we will understand that, on the Wiener space, Theorem 2.3.5 means the Ramer-Kusuoka formula with using the representation of $\mathcal{D}^*$-algebra we have constructed in previous section.

For this, we first summarize some fundamental facts below.

Proposition 4.0.10. *Let $\xi \in H$ and $F, G \in \mathbb{D}_\infty$. Then*

(1) *we have*

$$D_\xi^{*n} 1(w) = H_n\Big[\int_0^1 \dot{\xi}(t)\, \mathrm{d}w(t)\Big] \quad \text{if } \xi \text{ is of length one} \tag{2.14}$$

where $H_n[x]$ is the n-th Hermite polynomial defined by

$$e^{\lambda x - \frac{\lambda^2}{2}} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n[x], \tag{2.15}$$

so that

$$\mathbf{E}[\, D_\xi^* F\,] = 0. \tag{2.16}$$

(ii) *We have*

$$D_\xi^*(FG) = (D_\xi^* F)\,G - F D_\xi G \tag{2.17}$$

and hence with taking the expectations of both sides we have

$$\mathbf{E}[\,(D_\xi^* F)\,G\,] = \mathbf{E}[\,F D_\xi G\,]. \tag{2.18}$$

Proof. (i) First we prove (2.14) by induction. It is clear if $n = 1$. We shall assume (2.14) is true in the case of n and then with using the assumption of length of ξ, we have

$$\begin{aligned} D_\xi^{*(n+1)} 1(w) &= D_\xi^* H_n[\int_0^1 \dot\xi(t) \mathrm{d}w(t)\,] \\ &= \int_0^1 \dot\xi(t) \mathrm{d}w(t)\, H_n[\int_0^1 \dot\xi(t) \mathrm{d}w(t)\,] - D_\xi H_n[\int_0^1 \dot\xi(t) \mathrm{d}w(t)\,] \\ &= \int_0^1 \dot\xi(t) \mathrm{d}w(t)\, H_n[\int_0^1 \dot\xi(t) \mathrm{d}w(t)\,] - n H_{n-1}[\int_0^1 \dot\xi(t) \mathrm{d}w(t)\,] \end{aligned}$$

which is $H_{n+1}[\int_0^1 \dot\xi(t)\mathrm{d}w(t)]$ itself by the property of Hermite polynomials. Every $F \in \mathcal{P}$ can be written as a linear combination of product of Hermite polynomials in $\int_0^1 \dot\theta_1(t)\mathrm{d}w(t), \cdots, \int_0^1 \dot\theta_k(t)\mathrm{d}w(t)$ for some orthonormal system $\theta_1, \cdots, \theta_k \in H$. Since $\int_0^1 \dot\theta_1(t)\mathrm{d}w(t), \cdots, \int_0^1 \dot\theta_k(t)\mathrm{d}w(t)$ are independent and $\mathbf{E}[\,H_n[\int_0^1 \dot\theta_k(t)\mathrm{d}w(t)]\,] = 0$ for every k and n we have (2.16) for every $F \in \mathcal{P}$. Since $\mathcal{P}$ is dense in $\mathbb{D}_\infty$, (2.16) holds for every $F \in \mathbb{D}_\infty$

(ii) The formula (2.17) is clear from Leibnitz' formula for D_ξ and then with taking expectation of both sides in (2.17) and using (2.16) we have the adjointness (2.18). □

We define formal series in t by

$$\exp tD_\xi := \sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^n \quad \text{and} \quad \exp tD_\xi^* := \sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^{*n}. \tag{2.19}$$

Both of them act on $\mathbb{D}_\infty$ in the weak sense of

$$(\exp tD_\xi)F = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^n F \in \mathbb{D}_\infty[\![t]\!] \tag{2.20}$$

and

$$(\exp tD_\xi^*)F = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^{*n} F \in \mathbb{D}_\infty[\![t]\!] \tag{2.21}$$

for $F \in \mathbb{D}_\infty$. The above quantities need not necessary converge in general. But if F lies in $\mathcal{P}$ then the former is actually a finite sum and hence $(\exp tD_\xi)F \in \mathcal{P}$.

Proposition 4.0.11.

$$(\exp tD_\xi)F(w) = F(w + t\xi) \quad \textit{for} \quad F \in \mathcal{P}.$$

Proof. Since

$$\begin{aligned}
(\exp tD_\xi)F \cdot (\exp tD_\xi)G &= \Big(\sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^n F \Big)\Big(\sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^n G \Big) \\
&= FG + \Big\{ D_\xi F \cdot G + F D_\xi G \Big\} t \\
&\quad + \Big\{ \frac{1}{2!} D_\xi^2 F \cdot G + D_\xi F \cdot D_\xi G + F \frac{1}{2!} D_\xi^2 G \Big\} t^2 + \cdots \\
&= FG + tD_\xi(FG) + \frac{t^2}{2!} D_\xi^2 (FG) + \cdots = (\exp tD_\xi)(FG)
\end{aligned}$$

for every F and $G \in \mathcal{P}$, to prove this proposition, it is enough to prove in the case where F is of the form $f[\int_0^1 \dot{\xi}_k(t) \mathrm{d}w(t)]$ for some polynomial function f defined on $\mathbb{R}$. In this case we have

$$\begin{aligned}
(\exp tD_\xi)F(w) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^n f\Big[\int_0^1 \dot{\xi}_k(s) \mathrm{d}w(s) \Big] \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}\Big[\int_0^1 \dot{\xi}(s) \mathrm{d}w(s) \Big] \Big\{ \Big(\int_0^1 \dot{\xi}_k(s) \mathrm{d}w(s) + \langle \xi, \xi_k \rangle \Big) - \int_0^1 \dot{\xi}_k(s) \mathrm{d}w(s) \Big\}^n \\
&= f\Big[\int_0^1 \dot{\xi}_k(s)\, \mathrm{d}w(s) + \langle \xi, \xi_k \rangle \Big] = F(w + \xi).
\end{aligned}$$

□

So we shall define $\tau_{t\xi} : \mathbb{D}_\infty \to \mathbb{D}_\infty[\![t]\!]$ by

$$\tau_{t\xi} F := (\exp tD_\xi)F = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_\xi^n F$$

for each $F \in \mathbb{D}_\infty$ and we extend this to the mapping $\tau_{t\xi} : \mathbb{D}_\infty[\![t]\!] \to \mathbb{D}_\infty[\![t]\!]$ by

$$\tau_{t\xi}\Big(\sum_{n=0}^{\infty} t^n F_n\Big) = \sum_{n=0}^{\infty} t^n \, \tau_{t\xi} F_n.$$

Then one can see easily that for every $F \in \mathbb{D}_\infty$ and $\xi, \eta \in H$ it follows that

$$\tau_{t\xi}\, \tau_{t\eta} F = \tau_{t(\xi+\eta)} F = \tau_{t\eta}\, \tau_{t\xi} F. \tag{2.22}$$

On the other hand, by (2.14) in proposition 4.0.10 we have

Proposition 4.0.12.

$$(\exp tD^*_\xi)1(w) = \exp\Big\{t\int_0^1 \dot{\xi}(s)\mathrm{d}w(s) - \frac{t^2}{2}\int_0^1 \dot{\xi}(s)^2\mathrm{d}s\Big\}. \tag{2.23}$$

More generally, we have for $F \in \mathbb{D}_\infty$

$$(\exp tD^*_\xi)F(w) = (\tau_{-t\xi}F)\,\exp\Big\{t\int_0^1 \dot{\xi}(s)\mathrm{d}w(s) - \frac{t^2}{2}\int_0^1 \dot{\xi}(s)^2\mathrm{d}s\Big\}. \tag{2.24}$$

as an element in $\mathbb{D}_\infty[\![t]\!]$.

Proof. (2.23) is obvious from proposition 4.0.10. To prove (2.24), it needs only to prove that

$$\frac{d^n}{dt^n}\bigg|_{t=0}(\tau_{-t\xi}F)\,\exp\Big\{t\int_0^1 \dot{\xi}(s)\mathrm{d}w(s) - \frac{t^2}{2}\int_0^1 \dot{\xi}(s)^2\mathrm{d}s\Big\} = D^{*n}_\xi F(w).$$

By proposition 4.0.11 and (2.23), we already know that the left hand side of above is equal to

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k D^k_\xi F(w)\cdot D^{*(n-k)}_\xi 1(w).$$

On the other hand, with using the rule (2.17), the right hand side is

$$\begin{aligned}
D^*_\theta F &= D^*_\theta 1\cdot F - D_\theta F,\\
D^{*2}_\theta F &= D^{*2}_\theta 1\cdot F - D^*_\theta 1\cdot D_\theta F - D^*_\theta 1\cdot D_\theta F + D^2_\theta F\\
&= D^{*2}_\theta 1\cdot F - 2D^*_\theta 1\cdot D_\theta F + D^2_\theta F,\\
D^{*3}_\theta F &= D^{*3}_\theta 1\cdot F - D^{*2}_\theta 1\cdot D_\theta F\\
&\qquad - 2D^{*2}_\theta 1\cdot D_\theta F + 2D^*_\theta 1\cdot D^2_\theta F\\
&\qquad\quad + D^*_\theta 1\cdot D^2_\theta F - D^3_\theta F\\
&= D^{*3}_\theta 1\cdot F - 3D^{*2}_\theta 1\cdot D_\theta F + 3D^*_\theta 1\cdot D^2_\theta F - D^3_\theta F
\end{aligned}$$

and our claim is shown inductively. □

COROLLARY 4.0.13. $:\exp t(D_k + D_k^*):$ *acts as a mapping from* $\mathbb{D}_\infty$ *to* $\mathbb{D}_\infty[\![t]\!]$ *as the multiplication by*

$$\exp\Big\{t\int_0^1 \dot{\xi}_k(s)\mathrm{d}w(s) - \frac{t^2}{2}\int_0^1 \dot{\xi}_k(s)^2\mathrm{d}s\Big\}$$

for each $k = 1, 2, \cdots, N$.

PROOF. Let φ_k be the representation which can be obtained from φ when we set $z_k = 1, z_l = 0$ for $l \neq k$ in the way of (2.13). Then by Theorem 3.0.9, Proposition 4.0.11 and the first algebraic theorem 2.2.1 we have for $F \in \mathbb{D}_\infty$,

$$\begin{aligned}
&:\exp t(D_k + D_k^*): F(w)\\
&\quad= \Big([:\exp t(D_k + D_k^*): \exp t(-D_k)]\,\tau_{t\xi}F\Big)(w)\\
&\quad= \Big(\varphi_k[:\exp t(\rho_k + \rho_k^*): :\exp t(-\rho_k):]\,\tau_{t\xi}F\Big)(w)\\
&\quad= \Big(\varphi_k[:\exp t\rho_k^*:]\,\tau_{t\xi}F\Big)(w)\\
&\quad= \Big((\exp tD_k^*)\,\tau_{t\xi}F\Big)(w)\\
&\quad= F(w)\exp\Big\{t\int_0^1 \dot{\xi}(s)\mathrm{d}w(s) - \frac{t^2}{2}\int_0^1 \dot{\xi}(s)^2\mathrm{d}s\Big\}.
\end{aligned}$$

□

For $F(t) = \sum_{n=0}^\infty t^n F_n \in \mathbb{D}_\infty[\![t]\!]$ we define its expectation as an element in $\mathbb{R}[\![t]\!]$ by the formula

$$\mathbf{E}[F(t)] := \sum_{n=0}^\infty t^n \mathbf{E}[F_n] \in \mathbb{R}[\![t]\!].$$

COROLLARY 4.0.14. *We can obtain the Cameron-Martin formula in this framework.*

PROOF. For each k and $F \in \mathcal{P}$ we have

$$\begin{aligned}
&\mathbf{E}\Big[F(w - t\xi_k)\exp\Big\{t\int_0^1 \dot{\xi}_k(s)\mathrm{d}w(s) - \frac{t^2}{2}\int_0^1 \dot{\xi}_k(s)^2\mathrm{d}s\Big\}\Big]\\
&\quad= \mathbf{E}\Big[\Big(:\exp t(D_k + D_k^*):\ \exp t(-D_k)\Big)F(w)\Big]\\
&\quad= \mathbf{E}\Big[(\exp tD_k^*)F(w)\Big] = \mathbf{E}[F] + \sum_{n=1}^\infty \frac{t^n}{n!}\mathbf{E}[D_k^{*n}F] = \mathbf{E}[F].
\end{aligned}$$

□

We set a measurable process $Z(t, w)$ by

$$Z(t, w) = \sum_{k=0}^N Z_k(w)\xi_k(t)$$

for $0 \le t \le 1$ and $w \in \mathscr{W}$. As a function of w, z determines a mapping $Z : \mathscr{W} \to H$ which defines a trace class operator $DZ(w) : H \to H$ for almost all $w \in \mathscr{W}$ by

$$\Big\langle DZ(w)(h), \xi \Big\rangle = D_\xi \Big\langle Z(w), h \Big\rangle$$

In the coordinates $\xi_1, \xi_2, \cdots$, DZ can be viewed as a $N \times N$-matrix whose (i, j)-th component is given by $D_i Z_j = -[\,D_i^*, Z_j\,]$.

We also set D_Z and D_Z^* by

$$D_Z := \sum_{k=1}^{N} Z_k D_k \quad \text{and} \quad D_Z^* := \sum_{k=1}^{N} Z_k D_k^*$$

which lie in $\mathscr{D} = \mathscr{D}_N$. Note that $\varphi(\rho_\kappa) = D_Z$ and $\varphi(\rho_\kappa^*) = D_Z^*$. Each of D_Z and D_Z^* behaves like a vector field on $\mathscr{W}$, a section of the bundle whose fibres are the Cameron-Martin space H in what follows.

Proposition 4.0.15. *For each $F \in \mathbb{D}_\infty$ we have*

$$:\exp tD_Z\!: F(w) = (\tau_{tZ(w)}F)(w) \tag{2.25}$$

and

$$\begin{aligned} &:\exp tD_Z^*\!: F(w) \\ &= (\tau_{-tZ(w)}F)(w)\, \mathrm{e}^{t\,\cdot\,\mathrm{tr} DZ(w)} \exp\Big\{ t \int_0^1 \dot{Z}(s)\,\delta w(s) - \frac{t^2}{2} \int_0^1 \dot{Z}(s)^2\,\mathrm{d}s \Big\}. \end{aligned} \tag{2.26}$$

Proof. We first prove (2.25). With using (2.22),

$$\begin{aligned} :\exp tD_Z : F(w) &= \Big(:\exp t \sum_{k=1}^{N} Z_k D_k\!: \Big) F(w) \\ &= \Big(\exp t \sum_{k=1}^{N} Z_k(w) D_k \Big) F(w) \\ &= \Big(\exp(tZ_N(w)D_N) \cdots \exp(tZ_1(w)D_1) F \Big)(w) \\ &= (\tau_{tZ(w)}F)(w). \end{aligned}$$

Next we shall prove (2.26). Likewise we have

$$\begin{aligned} &:\exp tD_Z^* :\ F(w) \\ &= \Big(\exp(tZ_N(w)D_N^*) \cdots \exp(tZ_1(w)D_1^*) F \Big)(w) \\ &= (\tau_{-tZ(w)}F)(w) \exp\Big\{ t \sum_{k=1}^{N} Z_k(w) \int_0^1 \dot{\xi}_k(s)\mathrm{d}w(s) - \frac{t^2}{2} \int_0^1 \dot{\xi}_k(s)^2 \mathrm{d}s \Big\}. \end{aligned}$$

We use the integration by parts formula(e.g. see **[15]**, pp 36)

$$Z_k(w)\int_0^1 \dot{\xi}_k(s)\,\mathrm{d}w(s) = \int_0^1 Z_k(w)\dot{\xi}_k(s)\,\delta w(s) + \int_0^1 \dot{\xi}_k(s) D_s Z_k(w)\,\mathrm{d}s$$

to get the relation between Ogawa integral and Skorohod integral:

$$\sum_{k=1}^N Z_k(w)\int_0^1 \dot{\xi}_k(s)\,\mathrm{d}w(s) = \int_0^1 \dot{Z}(s)\,\delta w(s) + \mathrm{tr}DZ(w)$$

since $\int_0^1 \dot{\xi}_k(s) D_s Z_k(w)\mathrm{d}s = \langle D_k Z, \xi_k\rangle = D_k Z_k$. Hence (2.26) follows. □

In the same way as corollary 4.0.13, we can prove

Corollary 4.0.16. $:\exp t\,(D_Z + D_Z^*):$ *acts as a mapping from* $\mathbb{D}_\infty$ *to* $\mathbb{D}_\infty[\![t]\!]$ *as the multiplication by*

$$\mathrm{e}^{t\,\cdot\,\mathrm{tr}DZ(w)}\exp\Big\{t\int_0^1 \dot{Z}(s)\,\delta w(s) - \frac{t^2}{2}\int_0^1 \dot{Z}(s)^2\,\mathrm{d}s\Big\}.$$

Corollary 4.0.17. *Let* $F \in \mathbb{D}_\infty$*. Then as an element of* $\mathbb{R}[\![t]\!]$*, we have the Ramer-Kusuoka formula*

$$\begin{aligned}\mathbf{E}[F] = \mathbf{E}\Big[&(\tau_{-tZ(w)}F)(w)\det(1 - tDZ(w)) \\ &\times \mathrm{e}^{t\,\cdot\,\mathrm{tr}DZ(w)}\exp\Big\{t\int_0^1 \dot{Z}(s)\,\delta w(s) - \frac{t^2}{2}\int_0^1 \dot{Z}(s)^2\,\mathrm{d}s\Big\}\Big].\end{aligned} \tag{2.27}$$

Proof. By Corollary 4.0.17, Theorem 3.0.9 and Theorem 2.3.5, for each $F \in \mathbb{D}_\infty$ we have

$$\begin{aligned}&\mathbf{E}\Big[(\tau_{-tZ(w)}F)(w)\det(1 - tDZ(w)) \\ &\qquad \times \mathrm{e}^{t\,\cdot\,\mathrm{tr}DZ(w)}\exp\Big\{t\int_0^1 \dot{Z}(s)\,\delta w(s) - \frac{t^2}{2}\int_0^1 \dot{Z}(s)^2\,\mathrm{d}s\Big\}\Big] \\ &= \mathbf{E}\Big[\det(1 - tDZ(w))\Big(:\exp t\,(D_Z + D_Z^*)::\exp t\,(-D_Z):\Big)F(w)\Big] \\ &= \mathbf{E}\Big[\varphi\Big(\det(1 - t\Psi):\exp t\,(\rho_\kappa + \rho_\kappa^*)::\exp t\,(-\rho_\kappa):\Big)F(w)\Big] \\ &= \mathbf{E}\Big[\varphi\Big(1 + \int_0^t g'(s):\exp s\rho_\kappa:\mathrm{d}s\Big)F(w)\Big] \\ &= \mathbf{E}[F] + \int_0^t \mathbf{E}\Big[\Big(\varphi(g'(s)):\exp sD_Z:\Big)F(w)\Big]\mathrm{d}s.\end{aligned}$$

Since $\varphi(g'(t))$ is of the form

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{i_1,\cdots,i_n} D^*_{i_1}\det\begin{pmatrix}\varphi(\kappa_{i_1}) & \cdots & \varphi(\kappa_{i_n})\\ \varphi(\psi_{i_2i_1}) & \cdots & \varphi(\psi_{i_2i_n})\\ \vdots & \ddots & \vdots\\ \varphi(\psi_{i_ni_1}) & \cdots & \varphi(\psi_{i_ni_n})\end{pmatrix}\in\mathscr{D}[\![t]\!], \tag{2.28}$$

by using Proposition 4.0.10-(2.16) the second term of above is zero. □

5. Analytic Observations

In our framework, Z is actually a finite sum if we expand it in H with respect to the orthonormal basis $\xi_1, \xi_2, \cdots$ which we have chosen. To justify our Corollary 4.0.17 as an identity in $\mathbb{R}$ rather than $\mathbb{R}[\![t]\!]$ even in the case of Z expands as an infinite sum in H, we shall need some analytic observations.

5.1. Relation Between $:D_Z^n:$ and the Malliavin Derivative. The notion of vector field D_Z is also proposed in [**31**]. In [**31**], pp.10, the following identity is given:

$$D_Z F = \int_0^1 \dot{Z}(t) D_t F \,\mathrm{d}t \tag{2.29}$$

with using the Malliavin derivative D_t. We shall see that our $:D_Z^n:$ generalize (2.29) in the sense of the next lemma.

Lemma 5.1.1. *If* $\int_0^1 \dot{Z}(t)^2\,\mathrm{d}t < +\infty$ *then for every* $F \in \mathbb{D}_\infty$ *we have*

$$:D_Z^n\colon F = \int_0^1\cdots\int_0^1 \dot{Z}(t_1)\cdots\dot{Z}(t_n)\, D_{t_1}\cdots D_{t_n} F\,\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

Proof. Since

$$D_{t_1}\cdots D_{t_n}F = \sum_{k_1,\cdots,k_n} D_{\xi_{k_1}}\cdots D_{\xi_{k_n}}F\cdot\dot{\xi}_{k_n}(t_n)\cdots\dot{\xi}_{k_1}(t_1)$$

and

$$\dot{Z}(t_1)\cdots\dot{Z}(t_n) = \sum_{k_1,\cdots,k_n} Z_{k_1}\cdots Z_{k_n}\dot{\xi}_{k_n}(t_n)\cdots\dot{\xi}_{k_1}(t_1),$$

it follows that

$$\begin{aligned}
&\int_0^1\cdots\int_0^1 \dot{Z}(t_1)\cdots\dot{Z}(t_n)\,D_{t_1}\cdots D_{t_n}F\,\mathrm{d}t_1\cdots\mathrm{d}t_n\\
&=\sum_{k_1,\cdots,k_n}\int_0^1\cdots\int_0^1 Z_{k_1}\cdots Z_{k_n}\,D_{\xi_{k_1}}\cdots D_{\xi_{k_n}}F\,\dot{\xi}_{k_n}(t_n)^2\cdots\dot{\xi}_{k_1}(t_1)^2\,\mathrm{d}t_1\cdots\mathrm{d}t_n\\
&=\sum_{k_1,\cdots,k_n} Z_{k_1}\cdots Z_{k_n}\,D_{\xi_{k_1}}\cdots D_{\xi_{k_n}}F=\,:D_Z^n:F.
\end{aligned}$$

□

5.2. Some Estimates. In this subsection, we assume that Z has the expansion

$$Z=\sum_{k=1}^{\infty} Z_k\xi_k$$

in H where $Z_k\in\mathbb{D}_\infty$ for every $k=1,2,\cdots$. We denote by $Z^{(N)}$ the measurable process

$$Z^{(N)}=\sum_{k=1}^{N} Z_k\xi_k$$

which we have treated so far. Note that for any $Z^{(N)}$, we have already defined $D_{Z^{(N)}}, D^*_{Z^{(N)}}$ and so forth. We assume that $DZ(w)$ belongs to the trace class for a.a. w.

Proposition 5.2.1. *If* $\mathbf{E}[\int_0^1 \dot{Z}(t)^{2np}\mathrm{d}t]<+\infty$ *for some* $p>1$ *then for each* $F\in\mathbb{D}_\infty$ *we can define* $:D_Z^n:F$ *as an element in* $L^2(\mathscr{W})$ *so that it holds that*

$$:D_Z^n:F=\int_0^1\cdots\int_0^1 \dot{Z}(t_1)\cdots\dot{Z}(t_n)\,D_{t_1}\cdots D_{t_n}F\,\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

Proof. Set $:D_Z^n:F=\limsup_N :D^n_{Z^{(N)}}:F$. Since

$$\begin{aligned}
:D^n_{Z^{(N)}}:F&=\int_0^1\cdots\int_0^1 \dot{Z}^{(N)}(t_1)\cdots\dot{Z}^{(N)}(t_n)\,D_{t_1}\cdots D_{t_n}F\,\mathrm{d}t_1\cdots\mathrm{d}t_n\\
&\le\Big\{\int_0^1 \dot{Z}^{(N)}(t)^2\,\mathrm{d}t\Big\}^{n/2}\Big\{\int_0^1\cdots\int_0^1(D_{t_1}\cdots D_{t_n}F)^2\,\mathrm{d}t_1\cdots\mathrm{d}t_n\Big\}^{1/2},
\end{aligned}$$

and $\int_0^1 \dot{Z}^{(N)}(t)^2\mathrm{d}t\uparrow\int_0^1\dot{Z}(t)^2\mathrm{d}t$ as $N\to\infty$ it follows that

$$(:D^n_{Z^{(N)}}:F)^2\le\int_0^1\dot{Z}(t)^{2n}\mathrm{d}t\int_0^1\cdots\int_0^1(D_{t_1}\cdots D_{t_n}F)^2\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$

for any N. Let G_F be the second factor of the right hand side in the last line. Since $F \in \mathbb{D}_\infty$, G_F is q-th integrable where q is the conjugate of p, i.e., they satisfy $1/p + 1/q = 1$. By Hölder's inequality we have

$$\mathbf{E}[\int_0^1 \dot{Z}(t)^{2n} \mathrm{d}t \int_0^1 \cdots \int_0^1 (D_{t_1} \cdots D_{t_n} F)^2 \,\mathrm{d}t_1 \cdots \mathrm{d}t_n] \leq \mathbf{E}[\int_0^1 \dot{Z}(t)^{2np} \mathrm{d}t]^{1/p} \,\mathbf{E}[G_F^q]^{1/q}.$$

Since $\mathbf{E}[G_F^q]^{1/q}$ is finite, by Lebesgue's dominated convergence theorem we can obtain the result. □

Remark 5.1. From the proof of Proposition 5.2.1, if F is bounded and has bounded derivatives, $:D_Z^n\colon F$ can be defined even in the case of $p = 1$.

We define

$$:\exp tD_Z\colon F = \limsup_N \,:\exp tD_{Z^{(N)}}\colon F$$

and

$$\begin{aligned} :\exp tD_Z^*\colon F &= \limsup_N \,:\exp tD_{Z^{(N)}}^*\colon F \\ &= (:\exp tD_{-Z}^*\colon F)\, \exp\Big\{ t \int_0^1 \dot{Z}(s)\,\delta w(s) - \frac{t^2}{2} \int_0^1 \dot{Z}(s)^2 \,\mathrm{d}s \Big\} \end{aligned}$$

Lemma 5.2.2. *If* $\mathbf{E}[\int_0^1 \dot{Z}(t)^{2n} \mathrm{d}t] < +\infty$ *for every* $n = 1, 2, \cdots$ *then for every* $F \in \mathbb{D}_\infty$ *we have*

$$\begin{aligned} &\mathbf{E}[\,|:\exp tD_Z^n\colon F|\,] \\ &\quad \leq \sup_n \mathbf{E}\Big[\int_0^1 \cdots \int_0^1 (D_{t_1} \cdots D_{t_n} F)^2 \,\mathrm{d}t_1 \cdots \mathrm{d}t_n \Big]^{1/2} \Big(\mathbf{E}\Big[\exp\Big\{ \int_0^1 \dot{Z}(t)^2 \mathrm{d}t \Big\} \Big] + \mathrm{e} \Big). \end{aligned}$$

Proof. We have

$$\begin{aligned} &|:\exp tD_{Z^{(N)}}\colon F| \\ &\quad \leq \sup_N \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^1 \cdots \int_0^1 |\dot{Z}^{(N)}(t_1) \cdots \dot{Z}^{(N)}(t_n)\, D_{t_1} \cdots D_{t_n} F| \,\mathrm{d}t_1 \cdots \mathrm{d}t_n \end{aligned}$$

for every N and by using Schwartz' inequality,

$$\begin{aligned} &\mathbf{E}\Big[\sup_N \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^1 \cdots \int_0^1 |\dot{Z}^{(N)}(t_1) \cdots \dot{Z}^{(N)}(t_n)\, D_{t_1} \cdots D_{t_n} F| \,\mathrm{d}t_1 \cdots \mathrm{d}t_n \Big] \\ &\quad \leq \sum_{n=0}^\infty \frac{t^n}{n!} \mathbf{E}\Big[\Big\{ \int_0^1 \dot{Z}(t)^2 \mathrm{d}t \Big\}^n \Big]^{1/2} \mathbf{E}\Big[\int_0^1 \cdots \int_0^1 (D_{t_1} \cdots D_{t_n} F)^2 \,\mathrm{d}t_1 \cdots \mathrm{d}t_n \Big]^{1/2}. \end{aligned}$$

This is dominated by

$$\sup_n \mathbf{E}\Big[\int_0^1 \cdots \int_0^1 (D_{t_1} \cdots D_{t_n} F)^2 \, \mathrm{d}t_1 \cdots \mathrm{d}t_n\Big]^{1/2} \Big(\mathbf{E}\Big[\exp\Big\{\int_0^1 \dot{Z}(t)^2 \mathrm{d}t\Big\}\Big] + \mathrm{e}\Big)$$

from which we can obtain immediately the result by using the Lebesgue dominated convergence theorem. □

Proposition 5.2.3. *If*

$$\mathbf{E}\Big[\exp\Big\{t \int_0^1 \dot{Z}(s)^2 \mathrm{d}s\Big\}\Big] < +\infty$$

then for every $F \in \mathcal{P}$, $:\exp tD_Z\colon F$ is in $L^p(\mathscr{W})$ for any $p \geq 1$.

Proof. Since $(:\exp tD_{Z^{(N)}}\colon F)^p = (:\exp tD_{Z^{(N)}}:)(F^p)$ and F is polynomial, our assertion follows by Lemma 3.17. □

Proposition 5.2.4. *Let $p \geq 1$. If*

$$\mathbf{E}\Big[\exp\Big\{t \int_0^1 \dot{Z}(s) \mathrm{d}s\Big\} + \exp\Big\{ptu \int_0^1 \dot{Z}(s)\, \delta w(s)\Big\}\Big] < +\infty$$

for some $u > 1$ then for every $F \in \mathcal{P}$, $:\exp tD_Z^*\colon F$ is in $L^p(\mathscr{W})$.

Proof. With noting that $|:\exp tD_{-Z}\colon F|^p$ is integrable from Proposition 5.2.3,

$$\begin{aligned}
&\mathbf{E}[\,|:\exp tD_Z^*\colon F|^p\,] \\
&= \mathbf{E}\Big[|:\exp tD_{-Z}\colon F|^p \exp\Big\{pt\int_0^1 \dot{Z}(s)\,\delta w(s) - \frac{pt^2}{2}\int_0^1 \dot{Z}(s)^2\,\mathrm{d}s\Big\}\Big] \\
&\leq \mathbf{E}\Big[|:\exp tD_{-Z}\colon F|^{pv}\Big]^{1/v} \mathbf{E}\Big[\exp\Big\{ptu\int_0^1 \dot{Z}(s)\,\delta w(s)\Big\}\Big]^{1/u}.
\end{aligned}$$

□

Proposition 5.2.5. *If*

$$\mathbf{E}\Big[\exp\Big\{t \int_0^1 \dot{Z}(s)^2 \mathrm{d}s\Big\} + \exp\Big\{2pt \sup_N \int_0^1 \dot{Z}^{(N)}(s)\,\delta w(s)\Big\}\Big] < +\infty$$

for some $p > 1$ then for $F \in \mathcal{P}$ we have

$$:\exp tD_{Z^{(N)}}^*\colon F \overset{n\to\infty}{\to} :\exp tD_Z^*\colon F \quad \textit{in } L^1(\mathscr{W}).$$

PROOF. Since

$$
\begin{aligned}
&|:\exp tD^*_{Z^{(N)}}: F| \\
&\le |\exp tD_{-Z^{(N)}}F| \exp\Big\{t\int_0^1 \dot{Z}^{(N)}(s)\,\delta w(s)\Big\} \\
&\le \sup_N \sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^1\cdots\int_0^1 |\dot{Z}^{(N)}(t_1)\cdots\dot{Z}^{(N)}(t_n)\,D_{t_1}\cdots D_{t_n}F|\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\
&\qquad\qquad \times \exp\Big\{t\sup_N\int_0^1 \dot{Z}^{(N)}(s)\,\delta w(s)\Big\}
\end{aligned}
$$

we have by Hölder's inequality

$$
\begin{aligned}
&\mathbf{E}[|:\exp tD^*_{Z^{(N)}}: F|] \\
&\le \Big(\mathbf{E}\Big[\exp\Big\{t\int_0^1 \dot{Z}(s)^2\mathrm{d}s\Big\}\Big] + \mathrm{e}\Big) \\
&\qquad \times \sup_n \mathbf{E}\Big[\int_0^1\cdots\int_0^1 (D_{t_1}\cdots D_{t_n}F)^2\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\
&\qquad\qquad\qquad\qquad \times \exp\Big\{2t\sup_N\int_0^1 \dot{Z}^{(N)}(s)\,\delta w(s)\Big\}\Big]^{1/2} \\
&\le \Big(\mathbf{E}\Big[\exp\Big\{t\int_0^1 \dot{Z}(s)^2\,\mathrm{d}s\Big\}\Big] + \mathrm{e}\Big) \\
&\qquad \times \sup_n \mathbf{E}\Big[\Big\{\int_0^1\cdots\int_0^1 (D_{t_1}\cdots D_{t_n}F)^2\,\mathrm{d}t_1\cdots\mathrm{d}t_n\Big\}^q\Big]^{1/q} \\
&\qquad\qquad\qquad\qquad \times \mathbf{E}\Big[\exp\Big\{2pt\sup_N\int_0^1 \dot{Z}(s)\,\delta w(s)\Big\}\Big]^{1/2p}
\end{aligned}
$$

where q is conjugate to p. □

THEOREM 5.2.6. *If*

$$
\mathbf{E}\Big[\exp\Big\{t\int_0^1 \dot{Z}(s)^2\,\mathrm{d}s\Big\} + \exp\Big\{2pt\cdot\mathrm{tr}|DZ| + 2pt\sup_N\int_0^1 \dot{Z}^{(N)}(s)\,\delta w(s)\Big\}\Big] < +\infty
$$

for some $p > 1$, then for every $F \in \mathcal{P}$ we have the Ramer-Kusuoka formula (2.27) as a equation in $\mathbb{R}$.

PROOF. Since

$$\begin{aligned}
&|\det(1-tDZ^{(N)}):\exp tD^*_{Z^{(N)}}:F| \\
&\le \sup_N \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 \cdots \int_0^1 |\dot Z^{(N)}(t_1)\cdots \dot Z^{(N)}(t_n) D_{t_1}\cdots D_{t_n} F| \,\mathrm{d}t_1\cdots \mathrm{d}t_n \\
&\qquad\qquad \times \exp\Big\{ t\,\mathrm{tr}|DZ| + t \sup_N \int_0^1 \dot Z^{(N)}(s)\,\delta w(s) \Big\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\Big[\sup_N \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 \cdots \int_0^1 |\dot Z^{(N)}(t_1)\cdots \dot Z^{(N)}(t_n) D_{t_1}\cdots D_{t_n} F| \,\mathrm{d}t_1\cdots \mathrm{d}t_n \\
&\qquad\qquad\qquad \times \exp\Big\{ t\,\mathrm{tr}|DZ| + t \sup_N \int_0^1 \dot Z^{(N)}(s)\,\delta w(s) \Big\}\Big] \\
&\le \Big(\mathbf{E}\Big[\exp\Big\{ t \int_0^1 \dot Z(s)^2\,\mathrm{d}s \Big\}\Big] + \mathrm{e} \Big) \\
&\qquad \times \sup_n \mathbf{E}\Big[\int_0^1 \cdots \int_0^1 (D_{t_1}\cdots D_{t_n}F)^2 \,\mathrm{d}t_1\cdots\mathrm{d}t_n \\
&\qquad\qquad \times \exp\Big\{ 2t\,\mathrm{tr}|DZ| + 2t \sup_N \int_0^1 \dot Z^{(N)}(s)\,\delta w(s) \Big\}\Big]^{1/2}
\end{aligned}$$

which is dominated by

$$\begin{aligned}
&\Big(\mathbf{E}\Big[\exp\Big\{ t \int_0^1 \dot Z(s)^2\,\mathrm{d}s \Big\}\Big] + \mathrm{e} \Big) \\
&\quad \times \sup_n \mathbf{E}\Big[\Big\{ \int_0^1 \cdots \int_0^1 (D_{t_1}\cdots D_{t_n}F)^2 \,\mathrm{d}t_1\cdots\mathrm{d}t_n \Big\}^q\Big]^{1/q} \\
&\qquad \times \mathbf{E}\Big[\exp\Big\{ 2pt\,\mathrm{tr}|DZ| + 2pt \sup_N \int_0^1 \dot Z^{(N)}(s)\,\delta w(s) \Big\}\Big]^{1/2p} < +\infty,
\end{aligned}$$

where q is conjugate to p. Hence by Lebesgue's dominated theorem we have

$$\begin{aligned}
&\mathbf{E}[\det(1-tDZ):\exp tD^*_Z: F] \\
&\qquad = \lim_{N\to\infty} \mathbf{E}[\det(1-tDZ^{(N)}):\exp tD^*_{Z^{(N)}}: F].
\end{aligned}$$

Again by Lebesgue's dominated theorem we have

$$\mathbf{E}[\det(1-tDZ^{(N)}):\exp tD^*_{Z^{(N)}}: F] = \mathbf{E}[F] + \sum_{n=1}^{\infty} \frac{t^n}{n!} \mathbf{E}[G_n^{(N)}]$$

where $G_n^{(N)}$ is defined by

$$\det(1 - tDZ^{(N)}) : \exp tD^*_{Z^{(N)}} \colon F = F + \sum_{n=1}^{\infty} \frac{t^n}{n!} G_n^{(N)}.$$

From the proof of Corollary 4.0.17 one can find that $\mathbf{E}[G_n^{(N)}] = 0$ for every n and N. Hence we have

$$\mathbf{E}[\det(1 - tDZ) : \exp tD^*_Z \colon F] = \mathbf{E}[F].$$

□

Part 2

Discrete-Time Clark-Ocone Formulae

CHAPTER 3

Discrete-Time Clark-Ocone Formula for Wiener Functionals

This part is based on the joint work [**2**].

1. Introduction

Let $T > 0$, $(W_t)_{0\le t\le T}$ be a Brownian motion starting from 0, and $(\mathcal{G}_t)_{0\le t\le T}$ be its natural filtration. Let $X \in L^2(\mathcal{G}_T)$ be differentiable in the sense of Malliavin, for which we may write $X \in \mathbb{D}_{2,1}$ (see e.g.[**24**]). Then, it holds that

$$X = \mathbf{E}[X] + \int_0^T \mathbf{E}[\,D_s X\,|\,\mathcal{G}_s\,]\,\mathrm{d}W_s, \tag{3.1}$$

where D_s means the Malliavin derivative (evaluated at s).

The formula (3.1) is originally obtained by Clark in [**12**] for "well-behaved" Fréchet differentiable functions F, in which D_tF meant essentially the Fréchet derivative of F. Haussmann [**22**] extended the formula (3.1) to functionals of solutions for stochastic differential equations in the framework of Clark, and Ocone [**41**] obtained (3.1) for F in $\mathbb{D}_{1,2}$ (see [**24**] for its definition) by using Malliavin calculus. Once the formula (3.1) was established in the framework of Malliavin calculus, subsequent researches around the Clark-Ocone formula have been done mainly with using Malliavin calculus or its generalization: for example, Üstünel [**56**] extended (3.1) for F in $\mathbb{D}_{-\infty}$ and Aase-Øksendal-Privault-Ubøe [**1**] gave a white noise generalization which is also a generalization of the result by Üstünel [**56**]. There are many variants for (3.1) (see e.g., [**42**]) which we omit here.

In the context of mathematical finance, the formula gives an alternative description of the hedging portfolio in terms of Malliavin derivatives. However, explicit expressions of the Malliavin derivatives of a Wiener functional are not available in general (except for some special cases: see [**49**]). In this chapter, we will introduce a *finite dimensional approximation* of (3.1) and discuss the "order of the convergence" in a finance-oriented mode. Actually, this kind of finite-dimensional approximation or something similar is commonly used in financial practice. Hence the results presented in this chapter might be more insightful and useful for the practitioners in the field.

Let us be more precise. Put $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$ for $k \in \mathbb{N}$, where $\Delta t = T/N$ and $N \in \mathbb{N}$. Then the random variable $(\Delta W_1, \cdots, \Delta W_N)$ is distributed as

$N(0, \Delta t I)$. Let $\mathcal{G}_k^N$, $k = 1, \cdots N$, be the σ-algebra generated by $(\Delta W_1, \cdots, \Delta W_k)$. Note that $\mathcal{G}^N := (\mathcal{G}_k^N)_{k=0}^N$ is a filtration, and

$$L^2(\mathcal{G}_N^N, P) \simeq L^2(\mathbb{R}^N, \mu^N),$$

where

$$\mu^N(dx) = \frac{1}{(2\pi\Delta t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{2\Delta t}} \mathrm{d}x.$$

With the filtration $\mathcal{G}^N$, we can discuss "stochastic integral" (which is in fact a Riemannian sum) with respect to the process (random walk) $W^{\Delta t} = \sum \Delta W$. On the other hand, we can naturally define (a precise formulation will be given in section 2.1) a finite dimensional version of the Malliavin derivative D_s by the weak partial derivatives such as

$$\partial_l X(x_1, \cdots, x_N)|_{x_k = \Delta W_k, k=1,\cdots,N}.$$

Then one might well guess that a discrete version of the Clark-Ocone formula could be

$$X \stackrel{?}{=} \mathbf{E}[X] + \sum_{l=1}^N \mathbf{E}[\partial_l X | \mathcal{G}_{l-1}^N] \Delta W_l$$

but this is not true since the random walk $W^{\Delta t}$ does not have the martingale representation property. In fact, if the martingale representation property holds for a random walk, then we can establish a precise discrete-time Clark-Ocone formula if we define "differentiation" properly. For the binary case, N. Privault **[45]** has made a detailed study on the discrete Clark-Ocone formula and related discrete Malliavin calculus.

We should instead ask how much the (martingale representation) error,

$$\text{Mart.Err} := X - \mathbf{E}[X] - \sum_{l=1}^N \mathbf{E}[\partial_l X | \mathcal{G}_{l-1}^N] \Delta W_l,$$

(which we will also denote by Err_N) measured by a norm, (actually we concentrate on the estimation with respect to $L^2(\mathbb{R}^N)$-one), is. Further, its asymptotic behaviour as $N \to \infty$ with $N\Delta t$ = time horizon T. This is closely related to the problem of so-called *tracking error of the delta hedge*. If one has a nice finite dimensional approximation X^N of a Wiener functional X, both defined on the same probability space, then the tracking error can be controlled by the (supremum in N of) Mart.Err plus the error caused by the discretization (finite-dimensional approximation) as we see from :

$$\begin{aligned}
\text{Tra.Err} &:= X - \mathbf{E}[X] - \sum_{l=1}^{N} \mathbf{E}[\, D_{(l\Delta t)} X \,|\, \mathcal{G}_{l\Delta t}\,] \Delta W_l \\
&= X - X^N + \mathbf{E}[\, X - X^N\,] \\
&- \sum_{l=1}^{N} \Big(\mathbf{E}[\, D_{(l\Delta t)} X \,|\, \mathcal{G}_{(l\Delta t)}\,] - \mathbf{E}[\, \partial_l X^N \,|\, \mathcal{G}^N_{l-1}\,] \Big) \Delta W_l + \text{Mart.Err} \\
&=: \text{Disc.Err} + \text{Mart.Err}.
\end{aligned}$$

There are considerably many studies on the subject of the tracking error as well. It at least dates back to the paper by Rootzen [**51**], where the weak convergence of the scaled error was studied. The problem is reformulated as "tracking error of the delta hedge" in Bertsimas, Kogan, and Lo [**7**], where the error was also measured by L^2-norm. Hayashi and Mykland [**23**] further developed the argument from financial perspectives.

Notable results in this topic are summarized as follows. Although the underlying continuous process X and the definition of the error may differ, one has roughly the following results:

- Convergence in law of the normalized error:

$$\sqrt{N} \cdot \text{Tra.Err} \overset{|\Delta| \to 0}{\to} \frac{1}{\sqrt{2}} \int_0^T \mathbf{E}[\, D^2_{t,t} F \,|\, \mathcal{G}_t\,] dB_t \quad \text{in law} \tag{3.2}$$

  where $F = f(X_T)$ where $X = (X_t)_{0 \le t \le T}$ is a diffusion defined via a stochastic differential equation driven by a Wiener process $W = (W_t)_{0 \le t \le T}$ in [**7**] and general Itô processes in [**23**] and $B = (B_t)_{0 \le t \le T}$ is a Brownian motion independent of W (Here actually the differentiability is not required. The expression $\mathbf{E}[\, D_s X \,|\, \mathcal{G}_s\,]$ should be understood as simply the integrand of the martingale representation of X and the meaning of $\mathbf{E}[\, D^2_s X \,|\, \mathcal{G}_s\,]$ will be clarified later).
- L^2-convergence of the error: When treating only *equidistant* partitions Δ,

$$\|\text{Tra.Err}\|_{L^2} \to 0 \quad \text{as } |\Delta| = 1/N \to 0 \tag{3.3}$$

  with the order
  — $O(N^{-1/2})$ when $F = \max\{X_T - K, 0\}$ (the pay-off for call option), $\max\{K - X_T, 0\}$ (the pay-off for put option) or $F = f(X_T)$ where f is *absolutely continuous* with a polynomial growth and where $X = (X_t)_{0 \le t \le T}$ is a diffusion process defined by a stochastic differential equation in [**62**]

— $O(N^{-1/4})$ when $F = 1_{[K,+\infty)}(X_T)$ (the pay-off for digital option) which is more irregular than the above is shown in **[19]** and **[55]**,

— $O(N^{\theta-1/2})$ when $F = f(X_T)$ with f belonging to a fractional Sobolev-type space indexed by $\theta \in [0, 1/2)$ in **[17]**, where they revealed the reason why the absolute continuity assumption on f was needed to get the $O(N^{-1/2})$-convergence (which is the best possible) with equidistant time partitions.

— $O(N^{-s/2})$ when $F \in \mathbb{D}_{2,s}(\mathbb{R})$ with a finite dimensionality in **[2]** for $0 \le s \le 1$.

Notably, it is shown in **[17]** $O(N^{-1})$-convergence rather than $O(N^{-1/2})$ by taking suitable deterministic and non-equidistant time partitions.

In this chapter, we shall establish the corresponding results for the Mart.Err, which almost parallel with the above.

After introducing the Discrete Clark-Ocone formula (Theorem 2.2.1, section 2.2), we will show, by using the formula, a multi-level central limit theorem for the error (Theorem 3.2.1). This corresponds to the result (3.2) above. Since we will be working on a sequence of *discrete* Wiener functionals unlike the situations concerning tracking error, we need to some discussions on the finite-dimensionality. An answer is given in section 3.3, and under the condition it is proven that the convergence order is related to a fractional smoothness (Theorem 3.3.1). This corresponds to the result (3.3) above. Section 3.5 is devoted to a study of the asymptotics of the error of the additive functionals. As a case study, we give a detailed estimate of the martingale representation error of the Riemann-sum approximation of Brownian occupation time (Theorem 3.6.2).

The proofs given in this chapter are largely based on elementary calculus with a bit of classical Fourier analysis and distribution theory, but nonetheless our methods can be, in spirit, a finite-dimensional reduction of Malliavin-Watanabe's distribution theory. Some detailed discussions on this point of view will be given in sections 2.1, 2.3, and 3.1. We have restricted ourselves to one-dimensional Wiener space case, but this is only for simplicity for the notations.

2. A Discrete Version of Clark-Ocone Formula

2.1. Generalized Wiener Functional in Discrete Time. Throughout this section we fix $N \in \mathbb{N}$ and work on the canonical probability space $(\mathbb{R}^N, \mathfrak{B}(\mathbb{R}^N), \mu^N)$ though we will abuse the notations like ΔW as the coordinate map.

Let $\mathcal{S}_N \equiv \mathcal{S}(\mathbb{R}^N)$ be the Schwartz space; the space of all rapidly decreasing functions and $\mathcal{S}'_N$ be its dual; the space of all tempered distributions (see, e.g. **[52]**). We (may) call $X \in \mathcal{S}'_N$ a "discrete generalized Wiener functional" and its generalized expectation is defined to be the coupling ${}_{\mathcal{S}'_N}\langle X, p^N \rangle_{\mathcal{S}_N}$, where p^N is the density of μ^N, which is of course in $\mathcal{S}_N$.

The conditional expectation $\mathbf{E}[X|\mathcal{G}_k^N]$ for $X \in \mathcal{S}'_N$ is then defined as follows. We first note that the inclusion $\mathcal{G}_k^N \subset \mathcal{G}_N^N$ induces those of $\mathcal{S}(\mathbb{R}^k) \subset \mathcal{S}(\mathbb{R}^N)$ and $\mathcal{S}'(\mathbb{R}^k) \subset \mathcal{S}'(\mathbb{R}^N)$. In this sense we write $\mathcal{S}_k$ and $\mathcal{S}'_k$ for the Schwartz space and the space of generalized Wiener functionals with respect to $\mathcal{G}_k^N$, $k = 1, \cdots, N$. Then $Y = \mathbf{E}[X|\mathcal{G}_k^N]$ in $\mathcal{S}'_k$ is defined in terms of the relation

$$\mathbf{E}[XZ] = \mathbf{E}[YZ], \quad \forall Z \in \mathcal{S}_k,$$

which should be understood as

$$_{\mathcal{S}'_N}\langle X, Zp^N\rangle_{\mathcal{S}_N} = {}_{\mathcal{S}'_k}\langle Y, Zp^k\rangle_{\mathcal{S}_k}, \quad \forall Z \in \mathcal{S}_k.$$

In particular, we see that the conditional expectation is well-defined by du Bois-Reymond lemma (see e.g. [**52**]). Note that this generalized conditional expectation reduces to the standard one on $L^1(\mu^N)$, which is included in $\mathcal{S}'_N$ unlike the L^1 space with respect to the Lebesgue measure. Furthermore, differentiations of $X \in \mathcal{S}'_N$ are defined as usual, namely,

$$\partial_k X = Y \iff {}_{\mathcal{S}'_N}\langle Y, Z\rangle_{\mathcal{S}_N} = -{}_{\mathcal{S}'_N}\langle X, \partial_k Z\rangle_{\mathcal{S}_N} \quad \forall Z \in \mathcal{S}_N,$$

which imply

$$\mathbf{E}[\partial_k X] = \mathbf{E}[X \partial_k \log p^N],$$

and so on.

2.2. Clark-Ocone Formula in Discrete Time. We have the following series expansion in Δt:

Theorem 2.2.1 (A Discrete Version of Clark-Ocone Formula). *For $X \in L^2(\mathcal{G}_N^N) \simeq L^2(\mu^N)$, we have the following L^2-convergent series expansion:*

$$X - \mathbf{E}[X] = \sum_{m=1}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{m/2}}{\sqrt{m!}} \mathbf{E}[\partial_l^m X | \mathcal{G}_{l-1}^N] H_m\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right) \tag{3.4}$$

where H_m is the m-th Hermite polynomial for $m \in \mathbb{Z}_+$;

$$H_m(x) = \frac{(-1)^m}{\sqrt{m!}} e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}} \quad (m \in \mathbb{Z}_+). \tag{3.5}$$

Here the differentiations are understood in the distribution sense, as explained in the previous section.

Proof. Since $\left\{\prod_{i=1}^N H_{k_i}(\frac{\Delta W_i}{\sqrt{\Delta t}})\right\}_{k_1,\ldots,k_n \in \mathbb{Z}_+}$ is an orthonormal basis of $L^2(\mathbb{R}^N, \mu^N)$, we have the following orthogonal expansion of $X \in L^2(\mathbb{R}^N, \mu^N)$:

$$X(\Delta W_1, \ldots, \Delta W_N) = \sum_{k_1,\ldots,k_N} c_{(k_1,\ldots,k_N)} \prod_{i=1}^N H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right). \tag{3.6}$$

where we denote

$$c_{(k_1,\dots,k_N)} := \Big\langle X, \prod_{i=1}^{N} H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\Big\rangle = \mathbf{E}\Big[\, X \prod_{i=1}^{N} H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\Big].$$

Let us "sort" the series according as the "highest" non-zero k_i;

$$\begin{aligned} &X(\Delta W_1, \dots, \Delta W_N) \\ &\quad = \mathbf{E}[X] + \sum_{l=1}^{N} \sum_{k_1,\dots,k_{l-1}} \sum_{k_l \geq 1} c_{(k_1,\dots,k_l,0,\dots,0)} \prod_{i=1}^{l} H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right). \end{aligned} \tag{3.7}$$

Here we claim that

$$\sum_{l=1}^{N} \sum_{k_1,\dots,k_{l-1}} c_{(k_1,\dots,k_l,0,\dots,0)} \prod_{i=1}^{l-1} H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right) = \mathbf{E}[\, X H_{k_l}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right) | \mathcal{G}^N_{l-1}\,]. \tag{3.8}$$

In fact, from the expansion (3.6) we have

$$\begin{aligned} &\mathbf{E}[\, X H_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right) | \mathcal{G}^N_{l-1}\,] \\ &= \mathbf{E}\Big[\sum_{k'_1,\dots,k'_N} c_{(k'_1,\dots,k'_N)} H_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right) \prod_{i=1}^{N} H_{k'_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right) \Big| \mathcal{G}^N_{l-1}\,\Big] \\ &= \sum_{k'_1,\cdots,k'_N;} c_{(k'_1,\dots,k'_N)} \prod_{i=1}^{l-1} H_{k'_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right) \mathbf{E}\Big[H_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right) \prod_{i=l}^{N} H_{k'_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\Big], \end{aligned}$$

and we confirm the claim since $\mathbf{E}[H_{k_l}(\frac{\Delta W_l}{\sqrt{\Delta t}}) \prod_{i=l}^{n} H_{k'_i}(\frac{\Delta W_i}{\sqrt{\Delta t}})] = 0$ unless $k'_l = k_l$ and $k'_i = 0$ for $i > l$.

We further claim that

$$\mathbf{E}[\, X H_{k_l}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right) | \mathcal{G}^N_{l-1}\,] = \frac{(\Delta t)^{k/2}}{\sqrt{k!}} \mathbf{E}[\, \partial^k_l X | \mathcal{G}^N_{l-1}\,], \tag{3.9}$$

which, together with (3.7) and (3.8), will prove the expansion (3.4) in the L^2 case. Here, the conditional expectation should be understood in the generalized sense. Following the definition we have made, it suffices to show that

$$\mathbf{E}\left[X H_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right) f(\Delta W_1, \cdots, \Delta W_{l-1}) \right] = \frac{(\Delta t)^{k/2}}{\sqrt{k!}} \mathbf{E}[\, (\partial^k_l X) f\,]$$

for any $f \in \mathcal{S}_{l-1}$ but this is easy to see if we write down the generalized expectation as the coupling of $\mathcal{S}$ and $\mathcal{S}'$:

$$
\begin{aligned}
{}_{\mathcal{S}'}\Big\langle X, H_k(x/\sqrt{\Delta t}) f p^N \Big\rangle_{\mathcal{S}} &= {}_{\mathcal{S}'}\Big\langle X, f(-1)^k \frac{(\Delta t)^{k/2}}{\sqrt{k!}} \partial_l^k p^N \Big\rangle_{\mathcal{S}} \\
&= \frac{(\Delta t)^{k/2}}{\sqrt{k!}} {}_{\mathcal{S}'}\langle \partial_l^k X, f p^N \rangle_{\mathcal{S}}.
\end{aligned}
$$

□

2.3. Comment on Discrete Generalized Wiener Functionals. In this subsection, we remark that our discrete generalized Wiener functionals is slightly broader than that of the direct finite dimensional reduction; there is a gap. For simplicity, we let $\Delta t = 1$ in this subsection.

We know that (see e.g. [**48**, Appendix to V.3]) the orthogonal expansion in $L^2(\mathbb{R}^N, \text{Leb})$ with respect to the Hermite functions:

$$
\phi_N(x) := \frac{1}{\sqrt{N!}} H_N(x)(p^N)^{1/2}
$$

gives so-called $\mathcal{N}$-representation of $\mathcal{S}$ and $\mathcal{S}'$; the series for $f \in \mathcal{S}$ (resp. $\in \mathcal{S}'$)

$$
\sum {}_{\mathcal{S}'}\langle f, \phi_N \rangle_{\mathcal{S}} \phi_N
$$

converges to f in $\mathcal{S}$ (resp. in $\mathcal{S}'$). In our context, it then follows that if $X(p^N)^{1/2} \in \mathcal{S}$ (resp. $\in \mathcal{S}'$), then the convergence of the expansion (3.4) is in $\mathcal{S}$ (resp. in $\mathcal{S}'$) as well. It should be further noted that we have the following equivalences:

Proposition 2.3.1. *It holds that*

$$
X(p^N)^{1/2} \in \mathcal{S} \iff X \in \mathbb{D}^{(N)}_{2,\infty} = \cap_{s>0} \mathbb{D}^{(N)}_{2,s} \tag{3.10}
$$

and

$$
X(p^N)^{1/2} \in \mathcal{S}' \iff X \in \mathbb{D}^{(N)}_{2,-\infty} = \cup_{s<0} \mathbb{D}^{(N)}_{2,s}, \tag{3.11}
$$

where $\mathbb{D}^{(N)}_{2,s}$ is the completion of $L^2(\mu^N)$ by the norm $\|f\|_{2,s} = \|(1+L)^{s/2} f\|_{L^2(\mu^N)}$. Here L is the Ornstein-Uhlenbeck operator on $\mathbb{R}^N$;

$$
L = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}.
$$

Proof. Let $\{\phi_n : n \in \mathbb{Z}\}$ be norms defined by

$$
\phi_n(f) = \|(1+S)^n f\|_{L^2(\text{Leb})},
$$

where S is the following Schrödinger operator of the harmonic oscillator:

$$S := -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{1}{4}|x|^2 - \frac{1}{2}.$$

We know that $\mathcal{S}$ is a Fréchet space by the seminorms $\{\phi_n\}$. In fact, both L and S are the number operators respectively in that;

$$L \prod_{i=1}^{N} H_{k_i}(x_i) = (\sum_{i=1}^{N} k_i) \prod_{i=1}^{N} H_{k_i}(x_i)$$

and

$$S \prod_{i=1}^{N} \phi_{k_i}(x_i) = (\sum_{i=1}^{N} k_i) \prod_{i=1}^{N} \phi_{k_i}(x_i).$$

We also have

$$L(f)(p^N)^{1/2} = S(f(p^N)^{1/2}),$$

which implies

$$\|f\|_{2,n} = \phi_n(f).$$

This proves (3.10).

The equivalence (3.11) follows from the following equivalence of the duality:

$${}_{\mathbb{D}^{(N)}_{2,-\infty}}\langle X, Y\rangle_{\mathbb{D}^{(N)}_{2,\infty}} = {}_{\mathcal{S}'}\langle X(p^N)^{1/2}, Y(p^N)^{1/2}\rangle_{\mathcal{S}}.$$

□

Corollary 2.3.2. *For $X \in \mathbb{D}^{(N)}_{2,s}$, $s \in \mathbb{R}$, the convergence of (3.4) is also attained in $\mathbb{D}^{(N)}_{2,s}$.*

Proof. It follows from the fact that, by the assumption, the partial sums

$$X_n := \sum_{k_1+\cdots+k_N \le n} c_{(k_1,\cdots,k_N)} \prod_{i=1}^{N} H_{k_i}(x_i), \quad n \in \mathbb{N}$$

form a Cauchy sequence in $\mathbb{D}^{(N)}_{2,s}$. □

3. Asymptotic Analysis of Martingale Representation Errors

In this section, we will consider the asymptotic behavior of the error term when $N \to \infty$ with $N\Delta t = T$. Let $t_k :=: t^{(N)}_k := \frac{kT}{N}$ for each $k = 0, 1, \cdots, N$. We recall that $\Delta W_k = W_{t^{(N)}_k} - W_{t^{(N)}_{k-1}}$ for each k and N, and $\mathcal{G}^N_k := \sigma(\Delta W_l; l = 1, \cdots, k)$. Further, to facilitate the discussion in the limit, we implement our discrete Malliavin-Watanabe calculus into the classical one in the first subsection.

3.1. Consistency with the Classical Malliavin Calculus. First, we review briefly the Malliavin calculus over the one-dimensional classical Wiener space to introduce notations which we will use in the following sections devoted to asymptotic analyses, and then will show how our framework, established in the previous sections, is "embedded" to the classical Malliavin calculus (Proposition 3.2.1).

Let $(\mathscr{W}, \mathbf{P})$ be the one-dimensional Wiener space on $[0, T]$. We consider the canonical process $w = (w(t))_{0 \le t \le T}$ starting from zero a.s. In this context, the Hilbert space

$$H = \left\{ h \in \mathscr{W} : \begin{array}{l} h(0) = 0 \text{ and } h \text{ is absolutely continuous} \\ \text{with square-integrable derivative} \end{array} \right\}$$

equipped with the inner product defined by

$$\langle h_1, h_2 \rangle_H = \int_0^T \dot{h}_1(t) \dot{h}_2(t) \, \mathrm{d}t, \quad h_1, h_2 \in H$$

is called the *Cameron-Martin subspace* of $\mathscr{W}$. For each complete orthonormal system (CONS, in short) $\{h_i\}_{i=1}^\infty$ of H, it is known that $\Big\{ \prod_{i=1}^\infty H_{a_i}\Big(\int_0^T \dot{h}_i(t) \mathrm{d}w(t) \Big) : \boldsymbol{a} \in \boldsymbol{\Lambda} \Big\}$ forms a CONS in $L^2(\mathscr{W})$ (see e.g., **[24]** Proposition 8.1), where $\boldsymbol{\Lambda}$ is the set of all sequence $\boldsymbol{a} = (a_i)_{i=1}^\infty$ of nonnegative integers except for a finite number of i's and H_n is the n-th Hermite polynomial defined in (2.2.1). We also denote by $J_n : L^2(\mathscr{W}) \to C_n$ the orthogonal projection, where C_n is the $L^2(\mathscr{W})$-closure of the subspace spanned by $\Big\{ \prod_{i=1}^\infty H_{a_i}\Big(\int_0^T \dot{h}_i(t) \mathrm{d}w(t) \Big) : \sum_{i=1}^\infty a_i = n \Big\}$ over $\mathbb{R}$. Each C_n is called the subspace of n-th *Wiener's homogeneous chaos*.

For each $s \in \mathbb{R}$, a Sobolev-type Hilbert space $\mathbb{D}_{2,s} = \mathbb{D}_{2,s}(\mathbb{R})$ is defined as the completion of $\{F \in L^2(\mathscr{W}) : \|F\|_{\mathbb{D}_{2,s}} < +\infty\}$ under the seminorm $\| \cdot \|_{\mathbb{D}_{2,s}}$ on $L^2(\mathscr{W})$ defined by

$$\|F\|^2_{\mathbb{D}_{2,s}} = \sum_{n=0}^\infty (1+n)^s \, \| J_n F \|^2_{L^2}, \quad F \in L^2(\mathscr{W})$$

which may be infinite in general.

In the following, for any two separable Hilbert space H_1 and H_2, we denote by $H_1 \otimes H_2$ the completion of the algebraic tensor product of H_1 and H_2 under the Hilbert-Schmidt norm.

It is known that one can define a (continuous) linear operator $D : \mathbb{D}_{2,1} \to L^2(\mathscr{W}) \otimes H$ such that

$$\langle DF, h \rangle_H = D_h F \in L^2(\mathscr{W})$$

for every $h \in H$ and $F \in \mathbb{D}_{2,1}$, where $D_h F$ is defined by

$$(D_h F)(w) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big\{F(w + \varepsilon h) - F(w)\Big\} \quad \text{for a.e. } w \in \mathscr{W},$$

which is well-defined due to the Cameron-Martin theorem (see e.g., [**24**] Theorem 8.5). For each $t \in [0, T]$, let $e_t : \mathscr{W} \to \mathbb{R}$ denote the evaluation map defined by $e_t(w) = w(t)$. Then a linear operator $D_t : \mathbb{D}_{2,1} \to L^2(\mathscr{W})$ is defined by

$$D_t F = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{id}_{L^2(\mathscr{W})} \otimes e_t\Big)(DF), \quad F \in \mathbb{D}_{2,1} \tag{3.12}$$

for a.a. $t \in [0, T]$.

Under these notations, we can state the relationship between our framework established in section 2 and that of Malliavin calculus. We omit the proof because it is immediate from the definition.

Proposition 3.1.1. *For each $X \in \mathbb{D}^{(N)}_{2,1}$, we have*

$$(D_t X)(w) = \sum_{l=1}^{N} 1_{\{t_{l-1} \le t < t_l\}} (\partial_l X)(w)$$

for a.a. $(t, w) \in [0, T] \times \mathscr{W}$.

For each $F \in \mathbb{D}_{2,1}$, one can prove that $\mathbf{E}[F|\mathcal{G}^N_N] \in \mathbb{D}^{(N)}_{2,1}$ and $\lim_{N\to\infty} \mathbf{E}[F|\mathcal{G}^N_N] = F$ in $\mathbb{D}_{2,1}$ (consult e.g., [**31**] Theorem 1.10). By using also the fact that $e_t(h) = \langle 1_{[0,t)}, h\rangle_H$ for each $h \in H$, one can obtain

$$(D_t F)(w) = \lim_{N\to\infty} \sum_{l=1}^{N} 1_{\{t_{l-1} \le t < t_l\}} \partial_l \mathbf{E}[F|\mathcal{G}^N_N](w) \tag{3.13}$$

for a.a. $(t, w) \in [0, T] \times \mathscr{W}$. Note that in [**31**], the derivative D on the path space $\mathscr{W}$ is defined directly by (3.13) with $N = 2^n$. Following this approach in [**31**], we define $D^k_\cdot X \in L^2[0, T] \otimes L^2(\mathbf{P})$ as the L^2-limit of the sequence $(D^k_\cdot \mathbf{E}[X|\mathcal{G}^N_N])^\infty_{N=1}$ if it exists (see [**31**], Theorem 1.10 to consult what condition is enough to get this limit).

By the above discussions, we may write

$$D^k_t X := \partial^k_l X \quad \text{if } t_{l-1} \le t < t_l$$

for $X \in \mathbb{D}^{(N)}_{2,n}$, $t \in [0, T]$, and $k = 1, 2, \cdots, n$.

3.2. A Central Limit Theorem for the Errors. Suppose that we are given a sequence $(X^N)^\infty_{N=1}$ of finite dimensional Wiener functionals $X^N \in L^2(\mathcal{G}^N_N)$ for each N.

We put, for $n \geq 0$,

$$\mathrm{Err}_N(n) := X^N - \sum_{m=0}^{n} \sum_{l=1}^{N} \frac{(\Delta t)^{m/2}}{\sqrt{m!}} \mathbf{E}[\, D^m_{lT/N} X^N \,|\, \mathcal{G}^N_{l-1}\,] H_m\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right).$$

Theorem 3.2.1. *Let $n \in \mathbb{N}$. Suppose that $X^N \in \mathbb{D}^{(N)}_{2,n+2}$ for each $N = 1, 2, \cdots$ and for some Wiener functional $X \in \mathbb{D}_{2,n+1}(\mathbb{R})$, we have*

- $X^N \to X$ *in* $L^2(\mathbf{P})$,
- $\displaystyle\int_0^T \| D^{p+1}_t X^N - D^{p+1}_t X \|^2_{L^2} \mathrm{d}t \to 0$

as $N \to \infty$ for each $p = 0, 1, \cdots, n$ and

- $\displaystyle\sup_N \int_0^T \| D^{n+2}_t X^N \|^2_{L^2} \mathrm{d}t < +\infty.$

Then we have

$$\begin{pmatrix} \mathrm{Err}_N(0) \\ (\Delta t)^{-1/2}\mathrm{Err}_N(1) \\ \vdots \\ (\Delta t)^{-n/2}\mathrm{Err}_N(n) \end{pmatrix} \to \begin{pmatrix} \displaystyle\int_0^T \mathbf{E}[\, D_t X \,|\, \mathcal{G}_t\,] \mathrm{d}W_t \\ \displaystyle\frac{1}{\sqrt{2}} \int_0^T \mathbf{E}[\, D^2_t X \,|\, \mathcal{G}_t\,] \mathrm{d}B^1_t \\ \vdots \\ \displaystyle\frac{1}{\sqrt{(n+1)!}} \int_0^T \mathbf{E}[\, D^{n+1}_t X \,|\, \mathcal{G}_t\,] \mathrm{d}B^n_t \end{pmatrix}$$

in probability on an extended probability space as $N \to \infty$, where $(B^1, \cdots, B^n) = (B^1_t, \cdots, B^n_t)_{0 \leq t \leq T}$ is an n-dimensional Brownian motion independent of $W = (W_t)_{0 \leq t \leq T}$.

Remark 3.1. Although the Brownian motion $B = (B^1, \cdots, B^n)$ above is not adapted to the filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$, the above stochastic integrals make sense because it is an $\left(\mathcal{G}_t \vee \sigma(B_s : 0 \leq s \leq t)\right)_{0 \leq t \leq T}$-Brownian motion.

Proof. By Theorem 2.2.1, we have,

$$(\Delta t)^{-p/2}\mathrm{Err}_N(p) = \sum_{m=p+1}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{(m-p)/2}}{\sqrt{m!}} \mathbf{E}[\, D^m_{lT/N} X^N \,|\, \mathcal{G}^N_{l-1}\,] H_m\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right).$$

For $m \geq p+2$, by using the integration by parts formula (3.9), we see that

$$
\begin{aligned}
&\left\| \sum_{m=p+2}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{(m-p)/2}}{\sqrt{m!}} \mathbf{E}[\, D^m_{lT/N} X^N \,|\, \mathcal{G}^N_{l-1}\,] H_m\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right) \right\|^2_{L^2} \\
&= (\Delta t)^2 \sum_{k=0}^{\infty} \sum_{l=1}^{N} \frac{k!}{(k+p+2)!} \left\| \mathbf{E}[\, \left(D^{p+2}_{lT/N} X^N \right) H_k\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right) |\, \mathcal{G}^N_{l-1}\,] \right\|^2_{L^2} \\
&\leq (\Delta t) \sum_{k=1}^{\infty} \frac{1}{k^{p+2}} \times \sum_{l=1}^{N} \| D^{p+2}_{lT/N} X^N \|^2_{L^2} \Delta t \\
&= (\Delta t) \sum_{k=1}^{\infty} \frac{1}{k^{p+2}} \times \int_0^T \| D^{p+2}_t X^N \|^2_{L^2} \mathrm{d}t
\end{aligned}
\tag{3.14}
$$

which goes to zero as $N \to \infty$ for each $p = 0, 1, \cdots, n$ by the assumption.

Let us consider the case $m = p+1$. For each $p = 0, 1, \cdots, n$, we define a right-continuous process $L^{p,N} = (L^{p,N}_t)_{0 \leq t \leq T}$ with left-hand side limits by

$$
L^{p,N}_t := \sum_{l=1}^{k} H_{p+1}\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right) \quad \text{if } t_{k-1} \leq t < t_k
$$

for $k = 1, 2, \cdots, N$, and $L^{p,N}_T := L^{p,N}_{t_{N-1}}$.

Since $H_{p+1}\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right)$, $l = 1, 2, \cdots, N$ are i.i.d. random variables and $H_{p+1}\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right)$, $p = 0, 1, \cdots, n$ are orthogonal to each other for each $l = 1, 2, \cdots, N$, the central limit theorem of finite dimensional distributions of $(\Delta t)^{1/2} L^{p,N}$, $N = 1, 2, \cdots$ follows as for each $0 \leq s < t$, with taking $t_{j-1} \leq s < t_j$ and $t_{k-1} \leq t < t_k$,

$$
\begin{aligned}
&\lim_{N \to \infty} \mathbf{E}\Big[\mathrm{e}^{ i \sum_{p=0}^{n} \xi_p \left\{ (\Delta t)^{1/2} L^{p,N}_t - (\Delta t)^{1/2} L^{p,N}_s \right\} } \Big| \mathcal{F}^{L^{0,N}}_s \vee \mathcal{F}^{L^{1,N}}_s \vee \cdots \mathcal{F}^{L^{n,N}}_s \Big] \\
&= \lim_{N \to \infty} \prod_{l=j+1}^{k} \mathbf{E}\Big[\mathrm{e}^{ i \sum_{p=0}^{n} \left(\xi_p \sqrt{t_k - t_j} \right) \cdot (k-j)^{-1/2} H_{p+1}\left(\frac{\Delta W^N_l}{\sqrt{\Delta t}}\right) } \Big] \\
&= \lim_{N \to \infty} \prod_{l=j+1}^{k} \left\{ 1 - \frac{|\xi|^2}{2(k-j)} (t_k - t_j) + o\left(\frac{|\xi|^2}{k-j} \right) \right\} = \mathrm{e}^{-\frac{\xi^2}{2}(t-s)}.
\end{aligned}
$$

for each $\xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, where $(\mathcal{F}^Z_t)_{0 \leq t \leq T}$ denotes the filtration generated by a stochastic process $Z = (Z_t)_{0 \leq t \leq T}$ and the little-o-notation is with respect to the asymptotics when $N \to +\infty$ (so that $k - j \to +\infty$). This implies that every finite dimensional distribution of $(n+1)$-dimensional process

$((\Delta t)^{1/2} L^{p,N})_{p=0}^{n}$ converges to that of an $(n+1)$-dimensional Brownian motion $(B^0, B^1, \cdots, B^n) = (B_t^0, B_t^1, \cdots, B_t^n)_{0 \le t \le T}$.

Besides, using Kolmogorov's inequality, we have for each $p = 0, 1, \cdots, n$,

$$\lim_{K\to\infty} \limsup_{N\to\infty} \mathbf{P}\Big(\sup_{0\le t\le T} \big|(\Delta t)^{1/2} L_t^{p,N}\big| \ge K\Big) \le \lim_{K\to\infty} \frac{(\Delta t) \| L_T^{p,N} \|_{L^2}^2}{K^2} = 0$$

and for each $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbf{P}\Big(\inf_{\substack{\{s_j\}_j \subset [0,T]:\\ |s_j - s_{j+1}| > \delta}} \max_j \sup_{t,s\in[s_{j-1},s_j)} (\Delta t)^{1/2} | L_t^{p,N} - L_s^{p,N} | \ge \varepsilon \Big) \\
&\le \limsup_{N\to\infty} \mathbf{P}\Big(\max_{j=1,2,\cdots,N} \sup_{t,s\in[s_{j-1},s_j)} (\Delta t)^{1/2} | L_t^{p,N} - L_s^{p,N} | \ge \varepsilon \Big) \\
&= \limsup_{N\to\infty} \mathbf{P}(0 \ge \varepsilon) = 0.
\end{aligned}$$

They imply the tightness of $\{(\Delta t)^{1/2} L^{p,N}\}_{N=1}^{\infty}$(see Billingsley [**9**], Theorem 13.2). Therefore,

$$\Big\{ \Big((\Delta t)^{1/2} L^{0,N}, (\Delta t)^{1/2} L^{1,N}, \cdots, (\Delta t)^{1/2} L^{n,N} \Big) \Big\}_{N=1}^{\infty}$$

also forms a tight family. Hence we have

$$(\sqrt{\Delta t} L^{0,N}, \sqrt{\Delta t} L^{1,N}, \cdots, \sqrt{\Delta t} L^{n,N}) \to (B^0, B^1, \cdots, B^n)$$

in law as $N \to \infty$. By the Skorohod representation theorem (see Ikeda-Watanabe [**24**], Theorem 2.7 and we remark that on the space of all right-continuous functions with left-hand side limits, one can endow so-called the *Skorohod topology* which is metrizable and makes the space a complete separable metric space. For details, see Billingsley [**9**], Chapter 5.), we may assume that the above convergence is realized as an almost sure convergence on an extended probability space. Note that on the probability space we still have $B^0 = W$ a.s.

Hence we have

$$\begin{aligned}
&\frac{(\Delta t)^{((p+1)-p)/2}}{\sqrt{(p+1)!}} \sum_{l=1}^{N} \mathbf{E}[D_{lT/N}^{p+1} X^N | \mathcal{G}_{l-1}^N] H_{p+1}\left(\frac{\Delta W_l^N}{\sqrt{\Delta t}} \right) \\
&= \frac{1}{\sqrt{(p+1)!}} \sum_{l=1}^{N} \mathbf{E}[D_{t_l}^{p+1} X^N | \mathcal{G}_{t_{l-1}}] \Big\{ (\Delta t)^{1/2} L_{t_l}^{p,N} - (\Delta t)^{1/2} L_{t_{l-1}}^{p,N} \Big\} \\
&\to \frac{1}{\sqrt{(p+1)!}} \int_0^T \mathbf{E}[D_t^{p+1} X | \mathcal{G}_t] \mathrm{d}B_t^p \quad \text{in probability as } N \to \infty
\end{aligned}$$

simultaneously for $p = 0, 1, \cdots, n$. □

Substituting $p = 0$ into the inequality (3.14) in the proof of Theorem 3.2.1, we also obtain the following

Corollary 3.2.2. *If* $\sup_N \int_0^T \| D_t^2 X^N \|_{L^2}^2 \mathrm{d}t < +\infty$ *then we have*

$$\left\| X^N - \left\{ \mathbf{E}[X^N] + \sum_{l=1}^{N} \mathbf{E}[\, D_{lT/N} X^N \,|\, \mathcal{G}_{l-1}^N \,] \Delta W_l^N \right\} \right\|_{L^2} = O(N^{-1/2})$$

as $N \to \infty$.

3.3. The Cases with "Finite Dimensional" Functionals. We have seen that the martingale representation error is of an order 1/2 for a smooth functional. In this section, we will observe that for a non-smooth functional, the order is related to its fractional differentiability if it behaves eventually like a finite dimensional functional. This parallels with the corresponding results in the cases of the tracking error as we have pointed out in Introduction.

Let us start with one-dimensional cases. Let $F \in L^2(\mathbb{R}, \mu_T)$, where μ_T is the Gaussian measure with variance $T > 0$. Then, since

$$\frac{\partial^k}{\partial x_l^k} F(x_1 + \cdots + x_N) = F^{(k)}(x_1 + \cdots + x_N),$$

we have, for $k_1 + \cdots + k_N = n$,

$$\begin{aligned} \mathbf{E}[\, D^{k_1}_{t_1^{(N)}} \cdots D^{k_l}_{t_l^{(N)}} F(W_T)\,]^2 &= \mathbf{E}[\, F^{(n)}(W_T)\,]^2 \\ &= \frac{n!}{T^n} \mathbf{E}[\, F(W_T) H_n \left(\frac{W_T}{\sqrt{T}} \right)]^2 = \frac{n!}{T^n} \| J_n F(W_T) \|_{L^2}^2, \end{aligned}$$

irrespective of l and N. Here J_n is the projection to the n-th chaos. With this observation in mind, we understand the following property as a finite-dimensionality of a sequence; let $\{F^N\}$ be such that each F^N being $\mathcal{G}_N^N$-measurable and that

$$\sup_{k_1 + \cdots + k_N = n} \left(\mathbf{E}[\, D^{k_1}_{t_1^{(N)}} \cdots D^{k_N}_{t_N^{(N)}} F \,] \right)^2 = O\left(\frac{n! \, \| J_n F^N \|^2}{T^n} \right) \tag{3.15}$$

$$\text{uniformly in } n = 2, 3, \cdots \text{ as } N \to \infty.$$

Note that a sequence composed of a one dimensional functional $F(W_T)$ satisfies the above property trivially.

Theorem 3.3.1. *Suppose that we are given a sequence of* $F^N \in \mathbb{D}^{(N)}_{2,-\infty}$, $N = 1, 2, \cdots$ *satisfying*

$$\sup_N \| F^N \|_{\mathbb{D}_{2,s}}^2 < +\infty$$

for some $0 \le s \le 1$ *and the "finite-dimensional property" (3.15). Then*

$$\| \text{1-Mart.Err}(F^N) \|_{L^2}^2 = O(N^{-s/2}) \quad \textit{as } N \to \infty.$$

Proof. By observing (3.7), we notice that

$$\begin{aligned}
&\|\,1\text{-Mart.Err}(F^N)\,\|_{L^2}^2 \\
&= \sum_{l=1}^{N} \sum_{\substack{k_1+\cdots+k_l=n \\ k_l\geq 2}} \frac{n!}{k_1!\cdots k_l!}(\Delta t)^n \mathbf{E}[\,\partial_1^{k_1}\cdots\partial_l^{k_l}F^N\,]^2
\end{aligned}$$

for each $n=2,3,\cdots$. By the assumption, there is a constant $C>0$ such that

$$\sup_{k_1+\cdots+k_l=n} \mathbf{E}[\,\partial_1^{k_1}\cdots\partial_l^{k_l}F^N\,]^2 \leq C\frac{n!\,\|\,J_nF^N\,\|^2}{T^n}$$

for each $n=2,3,\cdots$ and $N=1,2,\cdots$ and the multinomial theorem yields that

$$\sum_{\substack{k_1+k_2+\cdots+k_l=n \\ k_l\geq 2}} \frac{n!}{k_1!\cdots k_l!}(\Delta t)^n = \left(\frac{lT}{N}\right)^n - \left(\frac{(l-1)T}{N}\right)^n - n\frac{T}{N}\left(\frac{(l-1)T}{N}\right)^{n-1}.$$

Putting them together, we have

$$\begin{aligned}
&\|\,1\text{-Mart.Err}(F^N)\,\|_{L^2}^2 \\
&\leq C\sum_{n=2}^{\infty}\Big\{1-n\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{l}{N}\right)^{n-1}\Big\}\|\,J_nF^N\,\|_{L^2}^2 \qquad (3.16)\\
&= CN^{-s}\sum_{n=2}^{\infty}\frac{N^s}{n^{s-1}}\Big\{\frac{1}{n}-\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{l}{N}\right)^{n-1}\Big\}n^s\,\|\,J_nF^N\,\|_{L^2}^2
\end{aligned}$$

for each $s\in\mathbb{R}$.

On the other hand, since we have

$$\begin{aligned}
I_{n,N} &:= \frac{1}{n}-\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{l}{N}\right)^{n-1} \\
&= \sum_{l=0}^{N-1}\int_{l/N}^{(l+1)/N}\Big\{x^{n-1}-\Big(\frac{l}{N}\Big)^{n-1}\Big\}\mathrm{d}x > 0,
\end{aligned}$$

$I_{n,N}\leq 1/n$, and

$$I_{n,N} \leq \frac{1}{N}\sum_{l=0}^{N-1}\Big\{\left(\frac{l+1}{N}\right)^{n-1}-\left(\frac{l}{N}\right)^{n-1}\Big\} = \frac{1}{N},$$

we notice that

$$I_{n,N} = (I_{n,N})^s(I_{n,N})^{1-s} \leq \left(\frac{1}{N}\right)^s\left(\frac{1}{n}\right)^{1-s} \qquad (3.17)$$

for every $0\leq s\leq 1$.

By (3.16) and (3.17), we finally have

$$\begin{aligned}&\|1\text{-Mart.Err}(F^N)\|_{L^2}^2\\ &\le CN^{-s}\sum_{n=2}^{\infty} n^s \|J_n F^N\|_{L^2}^2 \le CN^{-s}\sup_N \|F^N\|_{\mathbb{D}_{2,s}^{(N)}}^2.\end{aligned}$$

□

3.4. The Case with One Dimensional Functionals in Multi-Dimensional Brownian Motion. Let $W = (W_t^1, \cdots, W_t^d)_{0\le t\le T}$ be a d-dimensional Brownian motion starting from zero. Although the framework discussed so far is for one-dimensional Brownian motion, it obviously extends to multi-dimensional case.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary Borel function. We denote

$$\mathrm{Err}_N := f(W_T) - \Big\{\mathbf{E}[f(W_T)] + \sum_{i=1}^{d}\sum_{l=1}^{N} \mathbf{E}[(\partial_i f)(W_T)|\mathcal{G}_{l-1}^N]\Delta W_l^i\Big\},$$

where $\mathcal{G}_0^N :=$ the trivial σ-field, and $\mathcal{G}_l^N := \sigma(\Delta W_1^i, \cdots, \Delta W_l^i : i = 1, \cdots, d)$ for $l = 1, \cdots, N$.

Theorem 3.4.1. *If $f(W_T) \in \mathbb{D}_{2,s}$ for some $0 \le s \le 1$, then we have $\|\mathrm{Err}_N\|_{L^2} = O(N^{-s/2})$ as $N \to \infty$.*

Proof. A straightforward extension of the discrete Clark-Ocone formula (3.4) to the multi-dimensional case yields that

$$\begin{aligned}\mathrm{Err}_N = &\sum_{n=2}^{\infty}\sum_{l=1}^{N}\sum_{k_1+\cdots+k_d=n}\frac{(\Delta t)^{k_1/2}\cdots(\Delta t)^{k_d/2}}{\sqrt{k_1!\cdots k_d!}}\\ &\times \mathbf{E}[(\partial_1^{k_1}\cdots\partial_d^{k_d} f)(W_T)|\mathcal{G}_{l-1}^N] H_{k_1}\Big(\frac{\Delta W_l^1}{\sqrt{\Delta t}}\Big)\cdots H_{k_1}\Big(\frac{\Delta W_l^d}{\sqrt{\Delta t}}\Big).\end{aligned}$$

Then Parseval's identity can be applied to obtain

$$\|\mathrm{Err}_N\|_{L^2}^2 = \sum_{n=2}^{\infty}\sum_{l=1}^{N}\sum_{k_1+\cdots+k_d=n}\frac{(\Delta t)^n}{k_1!\cdots k_d!}\mathbf{E}\Big[\mathbf{E}[(\partial_1^{k_1}\cdots\partial_d^{k_d} f)(W_T)|\mathcal{G}_{l-1}^N]^2\Big]. \tag{3.18}$$

Since $\mathbf{E}[(\partial_1^{k_1}\cdots\partial_d^{k_d}f)(W_T)\,|\,\mathcal{G}_{l-1}^N]$ is a function of $W_{t_{l-1}}$, it can be expanded by the chaos in $W_{t_{l-1}}$ as

$$\begin{aligned}&\mathbf{E}[(\partial_1^{k_1}\cdots\partial_d^{k_d}f)(W_T)\,|\,\mathcal{G}_{l-1}^N]\\&=\sum_{m=0}^{\infty}\sum_{j_1+\cdots+j_d=m}\frac{(t_{l-1})^{j_1/2}\cdots(t_{l-1})^{j_d/2}}{\sqrt{j_1!\cdots j_d!}}\\&\qquad\times\mathbf{E}[(\partial_1^{j_1+k_1}\cdots\partial_d^{j_d+k_d}f)(W_T)]H_{j_1}\Big(\frac{W_{t_{l-1}}^1}{\sqrt{t_{l-1}}}\Big)\cdots H_{j_d}\Big(\frac{W_{t_{l-1}}^d}{\sqrt{t_{l-1}}}\Big),\end{aligned}$$

so that

$$\begin{aligned}&\mathbf{E}\Big[\mathbf{E}[(\partial_1^{k_1}\cdots\partial_d^{k_d}f)(W_T)\,|\,\mathcal{G}_{l-1}^N]^2\Big]\\&=\sum_{m=0}^{\infty}\sum_{j_1+\cdots+j_d=m}\frac{(t_{l-1})^m}{j_1!\cdots j_d!}\mathbf{E}[(\partial_1^{j_1+k_1}\cdots\partial_d^{j_d+k_d}f)(W_T)]^2.\end{aligned}\tag{3.19}$$

Combining (3.18) and (3.19), we have

$$\begin{aligned}&\|\mathrm{Err}_N\|_{L^2}^2\\&=\sum_{l=1}^{N}\sum_{n=2}^{\infty}\sum_{m=0}^{\infty}\sum_{k_1+\cdots+k_d=n}\sum_{j_1+\cdots+j_d=m}\frac{(\Delta t)^n}{k_1!\cdots k_d!}\frac{(t_{l-1})^m}{j_1!\cdots j_d!}\mathbf{E}[(\partial_1^{j_1+k_1}\cdots\partial_d^{j_d+k_d}f)(W_T)]^2.\end{aligned}$$

To compute this, first we note that this series is absolutely convergent, therefore we can change freely the order of the summation. We first change the variable as $\boldsymbol{j}\mapsto\boldsymbol{i}:=\boldsymbol{k}+\boldsymbol{j}$ for each $\boldsymbol{k}$, where $\boldsymbol{k}=(k_1,\cdots,k_d)$ and similarly for $\boldsymbol{i}$ and $\boldsymbol{j}$, and after that, we again change the variable as $m\mapsto p:=n+m$ for each n. As the result, the above equals to

$$\sum_{l=1}^{N}\sum_{n=2}^{\infty}\sum_{p=n}^{\infty}\sum_{k_1+\cdots+k_d=n}\sum_{\substack{i_1+\cdots+i_d=p\\ i_r\ge k_r\text{ for all }r}}\frac{(\Delta t)^n}{k_1!\cdots k_d!}\frac{(t_{l-1})^{p-n}}{(i_1-k_1)!\cdots(i_d-k_d)!}\mathbf{E}[(\partial_1^{i_1}\cdots\partial_d^{i_d}f)(W_T)]^2.$$

Next, we change the order of the summation with respect to (p,n) as

$$\sum_{l=1}^{N}\sum_{p=2}^{\infty}\sum_{n=2}^{p}\sum_{k_1+\cdots+k_d=n}\sum_{\substack{i_1+\cdots+i_d=p\\ i_r\ge k_r\text{ for all }r}}\frac{(\Delta t)^n}{k_1!\cdots k_d!}\frac{(t_{l-1})^{p-n}}{(i_1-k_1)!\cdots(i_d-k_d)!}\mathbf{E}[(\partial_1^{i_1}\cdots\partial_d^{i_d}f)(W_T)]^2$$

and then, since $\sum_{k_1+\cdots+k_d\ge 2}\sum_{\substack{i_1+\cdots+i_d=p\\ i_r\ge k_r\text{ for all }r}}=\sum_{i_1+\cdots+i_d=p}\sum_{\substack{k_1+\cdots+k_d\ge 2\\ k_r\le i_r\text{ for all }r}}$, we see that the above is

$$\sum_{l=1}^{N}\sum_{p=2}^{\infty}\sum_{i_1+\cdots+i_d=p}\sum_{\substack{k_1+\cdots+k_d\ge 2\\ k_r\le i_r\text{ for all }r}}\frac{(\Delta t)^{k_1+\cdots+k_d}}{k_1!\cdots k_d!}\frac{(t_{l-1})^{p-(k_1+\cdots+k_d)}}{(i_1-k_1)!\cdots(i_d-k_d)!}\mathbf{E}[(\partial_1^{i_1}\cdots\partial_d^{i_d}f)(W_T)]^2.$$

By integration by parts, we see that

$$\|\mathrm{Err}_N\|_{L^2}^2 = \sum_{l=1}^{N}\sum_{p=2}^{\infty}\frac{1}{T^p}\sum_{i_1+\cdots+i_d=p}\sum_{\substack{k_1+\cdots+k_d\geq 2\\ k_r\leq i_r \text{ for all } r}}\prod_{r=1}^{d}\binom{i_r}{k_r}(\Delta t)^{k_r}(t_{l-1})^{i_r-k_r}$$
$$\times \mathbf{E}[f(W_T)H_{i_1}\Big(\frac{W_T^1}{\sqrt{T}}\Big)\cdots H_{i_d}\Big(\frac{W_T^d}{\sqrt{T}}\Big)]^2,$$

in which, the summation with respect to $\boldsymbol{k}$ can be computed, with uses of the binomial theorem, as

$$\sum_{\substack{k_1+\cdots+k_d\geq 2\\ k_r\leq i_r \text{ for all } r}}\prod_{r=1}^{d}\binom{i_r}{k_r}(\Delta t)^{k_r}(t_{l-1})^{i_r-k_r} = (t_l)^p-(t_{l-1})^p - N(\Delta t)(t_{l-1})^{p-1},$$

where $p = i_1+\cdots+i_d$. With noticing that

$$\|J_p f(W_T)\|_{L^2}^2 = \sum_{i_1+\cdots+i_d=p}\mathbf{E}[f(W_T)H_{i_1}\Big(\frac{W_T^1}{\sqrt{T}}\Big)\cdots H_{i_d}\Big(\frac{W_T^d}{\sqrt{T}}\Big)]^2,$$

we have

$$\|\mathrm{Err}_N\|_{L^2}^2 = \sum_{p=2}^{\infty} p\, I_{p,N}\,\|J_p f(W_T)\|_{L^2}^2,$$

where $I_{p,N} = 1/p - (1/N)\sum_{l=0}^{N-1}(l/N)^{p-1}$. Since $I_{p,N}\leq N^{-s}p^{s-1}$ for $0\leq s\leq 1$ as in (3.17), we finally conclude that

$$\|\mathrm{Err}_N\|_{L^2}^2 \leq N^{-s}\sum_{p=2}^{\infty} p^s\,\|J_p f(W_T)\|_{L^2}^2 \leq N^{-s}\,\|f(W_T)\|_{\mathbb{D}_{2,s}}^2.$$

□

3.5. A Study on Additive Functionals. Let $W=(W_t)_{0\leq t\leq T}$ be a one-dimensional Brownian motion starting from zero. In this subsection, we study sequences of "additive functionals",

$$F^N := \sum_{i=1}^{N} f_N(t_i, W_{t_i^{(N)}})\Delta t \tag{3.20}$$

where $f_N(t_i,\cdot)$, $i=1,\cdots N$ is a sequence in $\mathbb{D}^{(1)}_{2,-\infty}$.

We are interested in the conditions for the sequence to be "finite-dimensional" in the sense of (3.15).

We define an index to control the finite-dimensionality. Let

$$A_l := \Big(\sum_{i=l}^{N} i^{-n/2} \mathbf{E}[\, f_N(t_i, W_{t_i}) H_n(W_{t_i}/\sqrt{t_i})\,] \Big)^2$$

and

$$\alpha_{N,n}(F^N) := \begin{cases} 0 & \text{if } \sum_{l=1}^N A_l \{ l^n - (l-1)^n \} = 0, \\ \dfrac{N^n \sup A_l}{\sum_{l=1}^N A_l \{ l^n - (l-1)^n \}} & \text{otherwise.} \end{cases}$$

Then, we have the following criterion.

Proposition 3.5.1. *The sequence $\{F_N\}$ of (3.20) satisfies (3.15) if and only if*

$$\sup_n \sup_N \alpha_{n,N}(F_N) < +\infty.$$

Proof. For arbitrary non-negative integers $k_1, \cdots, k_N$ with $k_1 + \cdots + k_N = n$, we have

$$\begin{aligned}\mathbf{E}[\, D_{t_1}^{k_1} \cdots D_{t_N}^{k_N} F^N\,] &= \sum_{i=1}^{N} 1_{\{k_{i+1} = \cdots = k_N = 0\}} \mathbf{E}[\, f_N^{(n)}(t_i, W_{t_i})\,] \Delta t \\ &= (n!)^{1/2} (\Delta t)^{(2-n)/2} \sum_{i=1}^{N} 1_{\{k_{i+1} = \cdots = k_N = 0\}} i^{-n/2} \mathbf{E}[\, f_N(t_i, W_{t_i}) H_n(W_{t_i}/\sqrt{t_i})\,].\end{aligned}$$

If further $k_l \geq 1$ and $k_{l+1} = \cdots k_N = 0$ for some l, then

$$\begin{aligned}&\mathbf{E}[\, D_{t_1}^{k_1} \cdots D_{t_l}^{k_l} F^N\,] \\ &= (n!)^{1/2} (\Delta t)^{(2-n)/2} \sum_{i=l}^{N} i^{-n/2} \mathbf{E}[\, f_N(t_i, W_{t_i}) H_n(W_{t_i}/\sqrt{t_i})\,] \\ &= (n!)^{1/2} (\Delta t)^{(2-n)/2} A_l^{1/2}.\end{aligned}$$

Therefore,

$$\sup_{k_1 + \cdots + k_N = n} \Big(\mathbf{E}[\, D_{t_1}^{k_1} \cdots D_{t_N}^{k_N} F^N\,] \Big)^2 = n!\, (\Delta t)^{(2-n)} \sup_{l = 1, \cdots, N} A_l \tag{3.21}$$

On the other hand, we have

$$\begin{aligned}
&\| J_n F^N \|_{L^2}^2 \\
&= \sum_{l=1}^{N} \sum_{\substack{k_1+\cdots+k_l=n \\ k_l \geq 1}} \Big(\mathbf{E}[F^N H_{k_1}(\Delta W_1/\sqrt{\Delta t}) \cdots H_{k_l}(\Delta W_l/\sqrt{\Delta t})] \Big)^2 \\
&= \sum_{l=1}^{N} \sum_{\substack{k_1+\cdots+k_l=n \\ k_l \geq 1}} \frac{(\Delta t)^n}{k_1! \cdots k_l!} \Big(\mathbf{E}[D_{t_1}^{k_1} \cdots D_{t_l}^{k_l} F^N] \Big)^2 \qquad (3.22) \\
&= \sum_{l=1}^{N} A_l \sum_{\substack{k_1+\cdots+k_l=n \\ k_l \geq 1}} \frac{(\Delta t)^2 n!}{k_1! \cdots k_l!} = (\Delta t)^2 \sum_{l=1}^{N} A_l \{ l^n - (l-1)^n \}.
\end{aligned}$$

Putting (3.21) and (3.22) together, we have

$$\begin{aligned}
\sup_{k_1+\cdots+k_N=n} \frac{\big(\mathbf{E}[D_{t_1}^{k_1} \cdots D_{t_N}^{k_N} F^N] \big)^2}{\| J_n F^N \|_{L^2}^2} &= \frac{n!}{T^n} \frac{N^n \sup A_l}{\sum_{l=1}^{N} A_l \{ l^n - (l-1)^n \}} \\
&= \frac{n!}{T^n} \alpha_{N,n}(F^N).
\end{aligned}$$

Note that $\| J_n F^N \|_{L^2}^2 = 0$ implies both $\alpha_{N,n}(F^N) = 0$ and

$$\sup_{k_1+\cdots+k_N=n} \Big(\mathbf{E}[D_{t_1}^{k_1} \cdots D_{t_N}^{k_N} F^N] \Big)^2 = 0.$$

□

Corollary 3.5.2. *If*

$$\sup_N \frac{\sup_l A_l}{\inf_l A_l} < \infty,$$

then $\{F^N\}$ *is finite-dimensional.*

Proof. Since

$$\sum_{l=1}^{N} A_l \{ l^n - (l-1)^n \} \geq \inf_l A_l \sum_{l=1}^{N} \{ l^n - (l-1)^n \} = N^n \inf_l A_l,$$

we see that

$$\alpha_{n,N}(F^N) \leq \frac{\sup_l A_l}{\inf_l A_l}.$$

□

3.6. Asymptotic Analysis of the Martingale Representation Error of a Discretization of Brownian Occupation Time. The sequence of Riemann sum approximations

$$F^N := \sum_{i=1}^{N} 1_{[0,\infty)}(W_{t_i})\Delta t, \quad N \in \mathbb{N} \tag{3.23}$$

of the Brownian occupation time $\int_0^T 1_{[0,\infty)}(W_s)\,\mathrm{d}s$ is an interesting example where an explicit calculation is possible. We first prove that the sequence is not finite-dimensional in the sense of (3.15). However, it is rather difficult to check if the condition for Corollary 3.2.2 is satisfied. Instead, by a direct calculation the martingale representation error of the sequence is proven to be of order 1/2.

Proposition 3.6.1. *The index $\alpha_{n,N}(F^N)$ of the sequence (3.23) is not bounded.*

Proof. First, we observe that

$$\begin{aligned} A_l &= \left(\sum_{i=l}^{N} i^{-n/2}\mathbf{E}[\,1_{[0,\infty)}(W_{t_i})H_n(W_{t_i}/\sqrt{t_i})\,]\right)^2 \\ &= \left(\sum_{i=l}^{N} i^{-n/2}t_i^{1/2}n^{-1/2}\mathbf{E}[\,\delta_0(W_{t_i})H_{n-1}(W_{t_i}/\sqrt{t_i})\,]\right)^2 \\ &= (2\pi n)^{-1}\Big(H_{n-1}(0)\Big)^2\left(\sum_{i=l}^{N} i^{-n/2}\right)^2. \end{aligned}$$

Then, we now see that

$$\alpha_{n,N}(F^N) = \frac{N^n\left(\sum_{i=1}^N i^{-n/2}\right)^2}{\sum_{l=1}^N\left(\sum_{i=l}^N i^{-n/2}\right)^2\{\,l^n-(l-1)^n\,\}}. \tag{3.24}$$

First, we estimate the numerator of (3.24). We let $n \geq 5$. Then

$$\begin{aligned} N^n\left(\sum_{i=1}^{N} i^{-n/2}\right)^2 &= N^2\left(\sum_{i=1}^{N}\left(\frac{i}{N}\right)^{-n/2}\frac{1}{N}\right)^2 \\ &\geq N^2\left(\int_{1/N}^{1} x^{-n/2}\mathrm{d}x\right)^2 = N^2\left\{\frac{2}{n-2}(N^{(n-2)/2}-1)\right\}^2. \end{aligned} \tag{3.25}$$

Next, the denominator is estimated as follows:

$$
\begin{aligned}
&\sum_{l=1}^{N}\left(\sum_{i=l}^{N} i^{-n/2}\right)^2 \{l^n-(l-1)^n\} \\
&= N^2\sum_{l=1}^{N}\left(\sum_{i=l}^{N}\left(\frac{i}{N}\right)^{-n/2}\frac{1}{N}\right)^2\left\{\left(\frac{l}{N}\right)^n-\left(\frac{l-1}{N}\right)^n\right\} \\
&\leq N^2\sum_{l=1}^{N}\left(\int_{l/N}^{1} x^{-n/2}\mathrm{d}x+\left(\frac{l}{N}\right)^{-n/2}\frac{1}{N}\right)^2 \\
&\qquad\qquad \times\left\{\left(\frac{l}{N}\right)^n-\left(\frac{l-1}{N}\right)^n\right\} \\
&\leq N^2\sum_{l=1}^{N}\left(\frac{2}{n-2}\left\{\left(\frac{l}{N}\right)^{(2-n)/2}-1\right\}+\left(\frac{l}{N}\right)^{-n/2}\frac{1}{N}\right)^2 \\
&\qquad\qquad \times\left\{\left(\frac{l}{N}\right)^n-\left(\frac{l-1}{N}\right)^n\right\} \\
&= N^2\left\{J_1^N+J_2^N+J_3^N\right\}
\end{aligned}
$$

where

$$
J_1^N := (n-2)^{-2}\sum_{l=1}^{N}\left\{\left(\frac{l}{N}\right)^{(2-n)/2}-1\right\}^2\left\{\left(\frac{l}{N}\right)^n-\left(\frac{l-1}{N}\right)^n\right\},
$$

$$
J_2^N := \frac{2(n-2)^{-1}}{N}\sum_{l=1}^{N}\left\{\left(\frac{l}{N}\right)^{1-n}-\left(\frac{l}{N}\right)^{-n/2}\right\}\left\{\left(\frac{l}{N}\right)^n-\left(\frac{l-1}{N}\right)^n\right\}
$$

and

$$
J_3^N := \frac{1}{N^2}\sum_{l=1}^{N}\left(\frac{l}{N}\right)^{-n}\left\{\left(\frac{l}{N}\right)^n-\left(\frac{l-1}{N}\right)^n\right\}.
$$

It is easy to see that $\sup_N J_2^N < \infty$ and $\lim_{N\to\infty} J_3^N = 0$. Since J_1^N behaves like

$$
(n-2)^{-2}\int_0^1\left\{x^{(2-n)/2}-1\right\}^2 nx^{n-1}\mathrm{d}x < +\infty
$$

as $N\to\infty$, it is also seen that $\sup_N J_1^N < \infty$. Therefore, there is a constant C_n independent of N but possibly dependent on n such that

$$
\sum_{l=1}^{N}\left(\sum_{i=l}^{N} i^{-n/2}\right)^2 \{l^n-(l-1)^n\} \leq N^2 C_n. \tag{3.26}
$$

From (3.25) and (3.26), we see that $\sup_N \alpha_{n,N} = \infty$. □

Our main result in this subsection is the following.

THEOREM 3.6.2. *It holds that*

$$\|1\text{-Mart.Err}(F^N)\|_{L^2} = O(N^{-1/2}).$$

PROOF. By Theorem 2.2.1, we have

$$\begin{aligned}
&\|\mathrm{Err}_N\|_{L^2}^2 \\
&= \sum_{l=1}^{N}\sum_{k=2}^{\infty}\mathbf{E}\Big[\mathbf{E}\Big[\sum_{i=1}^{N}1_{[0,\infty)}(W_{t_i})\Delta t\, H_k\left(\frac{\Delta W_l^N}{\sqrt{\Delta t}}\right)\Big|\mathcal{G}_{l-1}^N\Big]^2\Big] \\
&= \sum_{l=1}^{N}\sum_{k=2}^{\infty}\frac{(\Delta t)^k}{k!}\mathbf{E}\Big[\mathbf{E}\Big[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}(W_{t_i})\Delta t\Big|\mathcal{G}_{l-1}^N\Big]^2\Big].
\end{aligned}\tag{3.27}$$

For $l \geq 2$, by the Hermite expansion in $L^2(\mathbf{R}, \mu_{t_{l-1}})$,

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}(W_{t_i})\Delta t\Big|\mathcal{G}_{l-1}^N\Big] \\
&= \sum_{n=0}^{\infty}\frac{(t_{l-1})^{n/2}}{\sqrt{n!}}\mathbf{E}\Big[\sum_{i=l}^{N}1_{[0,\infty)}^{(n+k)}(W_{t_i})\Delta t\Big]H_n\left(\frac{W_{t_{l-1}}}{\sqrt{t_{l-1}}}\right),
\end{aligned}$$

and by Parseval's identity we have

$$\begin{aligned}
&E\left[\mathbf{E}\Big[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}(W_{t_i})\Delta t\Big|\mathcal{G}_{l-1}^N\Big]^2\right] \\
&= \sum_{n=0}^{\infty}\frac{(t_{l-1})^{n}}{n!}\mathbf{E}\Big[\sum_{i=l}^{N}1_{[0,\infty)}^{(n+k)}(W_{t_i})\Delta t\Big]^2.
\end{aligned}\tag{3.28}$$

Note that (3.28) is also valid for $l = 1$ with the conventions $t_0 = 0$ and $t_0^0 = 1$. Plugging (3.28) into (3.27), we have

$$\begin{aligned}
&\|\mathrm{Err}_N\|_{L^2}^2 \\
&= \sum_{l=1}^{N}\sum_{k=2}^{\infty}\sum_{n=0}^{\infty}\frac{(\Delta t)^k}{k!}\frac{(t_{l-1})^{n}}{n!}\mathbf{E}\Big[\sum_{i=l}^{N}1_{[0,\infty)}^{(n+k)}(W_{t_i})\Delta t\Big]^2.
\end{aligned}$$

By the renumbering $(n+k, n) \mapsto (k, n)$, we have

$$\begin{aligned}
&\|\mathrm{Err}_N\|_{L^2}^2 \\
&= \sum_{l=1}^{N}\sum_{k=2}^{\infty}\frac{1}{k!}\mathbf{E}\Big[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}(W_{t_i})\Delta t\Big]^2\sum_{n=0}^{k-2}\frac{k!}{(k-n)!n!}(\Delta t)^k(t_{l-1})^n,
\end{aligned}$$

by keeping the conventions on t_0. With a use of the binomial theorem,

$$\begin{aligned}
&\|\mathrm{Err}_N\|_{L^2}^2 \\
&= \sum_{l=1}^{N}\sum_{k=2}^{\infty}\frac{1}{k!}\mathbf{E}\Big[\sum_{i=l}^{N} 1_{[0,\infty)}^{(k)}(W_{t_i})\Delta t\Big]^2 \\
&\qquad\qquad \times\Big\{(t_l)^k-(t_{l-1})^k-k(\Delta t)(t_{l-1})^{k-1}\Big\}.
\end{aligned}$$

Then, on one hand, for $l \geq 1$ and $k \geq 2$,

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{i=l}^{N} 1_{[0,\infty)}^{(k)}(W_{t_{i-1}})\Delta t\Big]^2 \\
&=\Big\{\sum_{i=l}^{N}\frac{\sqrt{(k-1)!}}{(t_i)^{(k-1)/2}}\mathbf{E}\Big[\delta_0(W_{t_i})H_{k-1}\left(\frac{W_{t_i}}{\sqrt{t_i}}\right)\Big]\Delta t\Big\}^2 \\
&=\Big\{\sum_{i=l}^{N}\frac{\sqrt{(k-1)!}}{(t_i)^{(k-1)/2}}H_{k-1}(0)\frac{1}{\sqrt{2\pi t_i}}\Delta t\Big\}^2 \\
&= k!\cdot\frac{H_{k-1}(0)^2}{2\pi k}\Big\{\sum_{i=l}^{N}\frac{\Delta t}{(t_i)^{n/2}}\Big\}^2.
\end{aligned}$$

By a similar argument, we find

$$\mathbf{E}\Big[1_{[0,\infty)}(W_T)H_k\left(\frac{W_T}{\sqrt{T}}\right)\Big]=\frac{H_{k-1}(0)}{\sqrt{2\pi k}}$$

and therefore

$$\|\mathrm{Err}_N\|_{L^2}^2=\sum_{k=2}^{\infty}Z_{N,k}\mathbf{E}\Big[1_{[0,\infty)}(W_T)H_k\left(\frac{W_T}{\sqrt{T}}\right)\Big]^2$$

where

$$Z_{N,k} := \sum_{l=1}^{N}\Big\{\sum_{i=l}^{N}\frac{\Delta t}{(t_i)^{k/2}}\Big\}^2\Big\{(t_l)^k-(t_{l-1})^k-k(\Delta t)(t_{l-1})^{k-1}\Big\}. \tag{3.29}$$

On the other hand, by Lemma 3.6.3 below, we know that there exists a constant $K > 0$ such that

$$Z_{N,k} \leq \frac{K}{N}$$

for each $k = 2, 3, \cdots$ and $N = 3, 4, \cdots$. Hence we have

$$\|\,\mathrm{Err}_N\,\|_{L^2}^2 \leq \frac{2K}{N}\|1_{[0,\infty)}(W_T)\|_{L^2}^2.$$

□

LEMMA 3.6.3. *For $k \geq 2$, it holds that*

$$Z_{N,k} \leq \frac{9T^2}{N}. \tag{3.30}$$

where $Z_{N,k}$ is given as above in (3.29).

PROOF. We may write

$$\begin{aligned} Z_{N,k} = \sum_{l=1}^{N} \Big[\Big\{ \sum_{i=l}^{N} \Big(\frac{t_l}{t_i}\Big)^{k/2} \Delta t \Big\}^2 - \Big\{ \sum_{i=l}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} \Delta t \Big\}^2 \Big] \\ - k \sum_{l=1}^{N} \Big\{ \sum_{i=l}^{N} \frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}} \Delta t \Big\}^2 \Delta t. \end{aligned}$$

For $l \geq 2$, we have

$$\begin{aligned} &\Big\{ \sum_{i=l}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} \Delta t \Big\}^2 \\ &= \Big\{ \sum_{i=l-1}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} \Delta t \Big\}^2 - 2 \sum_{i=l-1}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} (\Delta t)^2 + (\Delta t)^2, \end{aligned}$$

and therefore,

$$\begin{aligned} &\sum_{l=2}^{N} \Big[\Big\{ \sum_{i=l}^{N} \Big(\frac{t_l}{t_i}\Big)^{k/2} \Delta t \Big\}^2 - \Big\{ \sum_{i=l}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} \Delta t \Big\}^2 \Big] \\ &= \sum_{l=2}^{N} \Big[\Big\{ \sum_{i=l}^{N} \Big(\frac{t_l}{t_i}\Big)^{k/2} \Delta t \Big\}^2 - \Big\{ \sum_{i=l-1}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} \Delta t \Big\}^2 \Big] \\ &\qquad + 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} (\Delta t)^2 - N(\Delta t)^2. \end{aligned}$$

Using this,

$$\begin{aligned} Z_{N,k} &= (\Delta t)^2 + 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{n/2} (\Delta t)^2 \\ &\qquad - N(\Delta t)^2 - k \sum_{l=2}^{N} \Big\{ \sum_{i=l}^{N} \frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}} \Delta t \Big\}^2 \Delta t \\ &\leq 2 \sum_{l=2}^{N} \sum_{i=l-1}^{N} \Big(\frac{t_{l-1}}{t_i}\Big)^{k/2} (\Delta t)^2 - k \sum_{l=1}^{N} \Big\{ \sum_{i=l}^{N} \frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}} \Delta t \Big\}^2 \Delta t. \end{aligned} \tag{3.31}$$

We observe that

$$2\sum_{l=2}^{N}\sum_{i=l-1}^{N}\left(\frac{t_{l-1}}{t_i}\right)^{k/2}(\Delta t)^2$$

behaves like

$$2\int_0^T\int_t^T\left(\frac{t}{s}\right)^{k/2}\mathrm{d}s\mathrm{d}t$$

and

$$k\sum_{l=1}^{N}\Big\{\sum_{i=l}^{N}\frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}}\Delta t\Big\}^2\Delta t$$

behaves like

$$k\int_0^T\Big\{\int_t^T\frac{t^{(k-1)/2}}{s^{k/2}}\mathrm{d}s\Big\}^2\mathrm{d}t$$

as $N\to\infty$ respectively. We note that

$$2\int_0^T\int_t^T\left(\frac{t}{s}\right)^{k/2}\mathrm{d}s\mathrm{d}t = n\int_0^T\Big\{\int_t^T\frac{t^{(k-1)/2}}{s^{k/2}}\mathrm{d}s\Big\}^2\mathrm{d}t = \begin{cases}\dfrac{T^2}{2} & \text{if } k=2,\\ \dfrac{2T^2}{k+2} & \text{if } k\geq 2.\end{cases}$$

Based on the observations, we estimate $Z_{N,k}$ by separating it into two terms;

$$Z_{N,k}\leq Z^1_{N,k}+Z^2_{N,k}$$

where

$$Z^1_{N,k} := 2\sum_{l=2}^{N}\sum_{i=l-1}^{N}\left(\frac{t_{l-1}}{t_i}\right)^{k/2}(\Delta t)^2 - 2\int_0^T\int_t^T\left(\frac{t}{s}\right)^{k/2}\mathrm{d}s\mathrm{d}t,$$
$$Z^2_{N,k} := k\int_0^T\Big\{\int_t^T\frac{t^{(k-1)/2}}{s^{k/2}}\mathrm{d}s\Big\}^2\mathrm{d}t - k\sum_{l=1}^{N}\Big\{\sum_{i=l}^{N}\frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}}\Delta t\Big\}^2\Delta t.$$

We estimate each of them. Firstly, we have

$$
\begin{aligned}
Z^1_{N,n} &\leq 2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\int_{t_{l-2}}^{t_{l-1}}\int_{t_i}^{t_{i+1}}\Big\{\Big(\frac{t_{l-1}}{t_i}\Big)^{k/2}-\Big(\frac{t}{s}\Big)^{k/2}\Big\}\mathrm{d}s\mathrm{d}t \\
&\qquad + 2\sum_{l=2}^{N}\Big(\frac{t_{l-1}}{t_N}\Big)^{k/2}(\Delta t)^2 \\
&\leq 2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\int_{t_{l-2}}^{t_{l-1}}\int_{t_i}^{t_{i+1}}\Big\{\Big(\frac{t_{l-1}}{t_i}\Big)^{k/2}-\Big(\frac{t_{l-2}}{t_{i+1}}\Big)^{k/2}\Big\}\mathrm{d}s\mathrm{d}t \\
&\qquad + 2\sum_{l=2}^{N}\Big(\frac{t_{l-1}}{t_N}\Big)^{k/2} \\
&= 2(\Delta t)^2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\Big\{\Big(\frac{l-1}{i}\Big)^{k/2}-\Big(\frac{l-2}{i}\Big)^{k/2}\Big\} \\
&\qquad + 2(\Delta t)^2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\Big\{\Big(\frac{l-2}{i}\Big)^{k/2}-\Big(\frac{l-2}{i+1}\Big)^{k/2}\Big\} \\
&\qquad + 2\sum_{l=2}^{N}\Big(\frac{t_{l-1}}{t_N}\Big)^{k/2}(\Delta t)^2.
\end{aligned}
\tag{3.32}
$$

By a bit of algebra, the last term in (3.32) is seen to be

$$
2(\Delta t)^2\sum_{l=2}^{N}\Big\{1+\Big(\frac{l-1}{l}\Big)^{k/2}\Big\},
\tag{3.33}
$$

which is bounded above by $4T^2/N$.

Next, we estimate $Z^2_{N,k}$. We set

$$
\begin{aligned}
I = &\sum_{l=1}^{N}\int_{t_{l-1}}^{t_l} t^{k-1}\Big\{\int_t^T\frac{\mathrm{d}s}{s^{k/2}}\Big\}^2\mathrm{d}t \\
&\qquad -\sum_{l=1}^{N}\int_{t_{l-1}}^{t_l} t^{k-1}\Big\{\sum_{i=l}^{N}\frac{\Delta t}{(t_i)^{k/2}}\Big\}^2\mathrm{d}t
\end{aligned}
$$

and

$$\mathit{II} = \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} t^{k-1} \Big\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \Big\}^2 \mathrm{d}t \\ - \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} (t_{l-1})^{k-1} \Big\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \Big\}^2 \mathrm{d}t.$$

Note that $Z_{N,k}^2 = k(I + \mathit{II})$. For $t_{l-1} \le t \le t_l$, $l = 1, \cdots, N$, we have

$$\begin{aligned} &\int_t^T \frac{\mathrm{d}s}{s^{k/2}} - \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \\ &= \sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left(\frac{1}{s^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) \mathrm{d}s + \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} - \frac{\Delta t}{(t_l)^{k/2}} \ge 0, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} &\sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left(\frac{1}{s^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) \mathrm{d}s \\ &\quad \le \sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left(\frac{1}{(t_{i-1})^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) \mathrm{d}s \\ &\quad\quad = \Delta t \left(\frac{1}{(t_l)^{k/2}} - \frac{1}{(t_N)^{k/2}} \right). \end{aligned}$$

Combining these two, we have

$$\begin{aligned} &\left\{ \int_t^T \frac{\mathrm{d}s}{s^{k/2}} \right\}^2 - \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 \\ &\le \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} \left(\int_t^T \frac{\mathrm{d}s}{s^{k/2}} + \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right) \le 2 \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} \left(\int_t^T \frac{\mathrm{d}s}{s^{k/2}} \right) \\ &= \begin{cases} \dfrac{4}{k-2} (t^{1-\frac{k}{2}} - T^{1-\frac{k}{2}}) \displaystyle\int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} \le \frac{4}{k-2} t^{1-\frac{k}{2}} \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} & \text{if } k \ge 3, \\ 2 \displaystyle\int_t^{t_l} \frac{\mathrm{d}s}{s} \log \frac{T}{t} & \text{if } k = 2. \end{cases} \end{aligned}$$

Then for $k \geq 3$,

$$
\begin{aligned}
I &\leq \frac{4}{k-2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t}^{t_l} \left(\frac{t}{s}\right)^{k/2} \mathrm{d}s\mathrm{d}t \\
&\leq \frac{4}{k-2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t}^{t_l} \mathrm{d}s\mathrm{d}t = \frac{2}{k-2} \sum_{l=1}^{N} (t_l - t_{l-1})^2 = \frac{2}{k-2} \frac{T^2}{N}
\end{aligned} \tag{3.35}
$$

and for $k = 2$, we have

$$
\begin{aligned}
I &\leq 2 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t}^{t_l} \frac{t}{s} \log \frac{T}{t} \mathrm{d}s\mathrm{d}t \leq 2 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t}^{t_l} \mathrm{d}s \, \log \frac{T}{t} \, \mathrm{d}t \\
&\leq 2\Delta t \sum_{l=1}^{N} \left\{ \Delta t \log T - \Big[t \log t - t \Big]_{t=t_{l-1}+0}^{t=t_l} \right\} = \frac{2T^2}{N}.
\end{aligned} \tag{3.36}
$$

Now we turn to the estimate of $I\!I$. By (3.34), for $k \geq 3$,

$$
\begin{aligned}
I\!I &\leq \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left\{ t^{k-1} - (t_{l-1})^{k-1} \right\} \left(\int_{t}^{T} \frac{\mathrm{d}s}{s^{k/2}} \right)^2 \mathrm{d}t \\
&\leq \frac{4}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left\{ t^{k-1} - (t_{l-1})^{k-1} \right\} t^{2-k} \mathrm{d}t \\
&= \frac{4(k-1)}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t} \left(\frac{s}{t}\right)^{k-2} \mathrm{d}s\mathrm{d}t \\
&\leq \frac{4(k-1)}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t} \mathrm{d}s\mathrm{d}t = \frac{2(k-1)}{(k-2)^2} \frac{T^2}{N}.
\end{aligned} \tag{3.37}
$$

For $k = 2$, we have

$$
\begin{aligned}
I\!I &\leq \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} (t - t_{l-1}) \left(\int_{t}^{T} \frac{\mathrm{d}s}{s} \right)^2 \mathrm{d}t \\
&\leq \Delta t \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left(\log \frac{T}{t} \right)^2 \mathrm{d}t = \frac{2T^2}{N}.
\end{aligned} \tag{3.38}
$$

By (3.35), (3.36), (3.37) and (3.38), we have

$$
Z_{N,k}^2 \leq \frac{5T^2}{N}. \tag{3.39}
$$

Combining (3.33) and (3.39), we obtained (3.30).

□

REMARK 3.2. A result by Ngo-Ogawa ([38], Theorem 2.2.) tells us that the sequence of processes

$$\Big\{n^{3/4}\Big(\frac{1}{N}\sum_{i=0}^{[Nt]} 1_{[0,\infty)}(X_{i/N}) - \int_0^t 1_{[0,\infty)}(X_s)\mathrm{d}s\Big)\Big\}_{t\geq 0}$$

is tight for a diffusion $X = (X_t)_{t\geq 0}$ although their results are more general. Moreover they say that this is optimal in L^2-sense in the case where X is the standard Brownian motion (see [38], Proposition 2.3).

3.7. Error with Euler-Maruyama Approximation. We shall consider the following stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t, \\ X_0 = x_0 \end{cases} \tag{3.40}$$

where $W = (W_t)_{0\leq t\leq T}$ is a one-dimensional Brownian motion starting from zero. In the following, we assume that the stochastic differential equation (3.40) has a unique strong solution which we denote by $X = (X_t)_{0\leq t\leq T}$.

For each partition $\Delta = \Delta^{(N)} : 0 = t_0 < t_1 < \cdots < t_N = T$ where $t_k = \frac{kT}{N}$ for $k = 0, 1, \cdots, N$, we introduce the Euler-Maruyama approximation $X^N = (X_t^N)_{0\leq t\leq N}$ of the stochastic differential equation (3.40), which is defined by

$$\begin{cases} \mathrm{d}X_t^N = \sigma(X^N_{\phi_N(t)})\mathrm{d}W_t + b(X^N_{\phi_N(t)})\mathrm{d}t, \\ X_0^N = x_0 \end{cases}$$

where $\phi_N : [0, T] \to [0, T]$ is defined by

$$\phi_N(t) = t_{k-1} \quad \text{if } t_{k-1} \leq t < t_k$$

for $k = 1, 2, \cdots, N$. We will also write $X_t^N = X_t^N(x_0)$ when we want to emphasize the initial state x.

Suppose we are given a function $f : \mathbb{R} \to \mathbb{R}$. Let us assume that $(f(X_T^N))_{N=1}^{\infty}$ is a finite dimensional approximation of $f(X_T)$ (though in general we would have to impose certain conditions on f, σ and b).

This motivates our investigation of

$$\begin{aligned} \mathrm{Err}_N := {} & f(X_T^N) \\ & - \Big\{\mathbf{E}[\,f(X_T^N)\,] + \sum_{k=1}^{N} \mathbf{E}[\,(\partial_k g_N)(\Delta W_1, \cdots, \Delta W_N)\,|\,\mathcal{G}_{k-1}^N\,]\Delta W_k\Big\} \end{aligned}$$

where $g_N : \mathbb{R}^N \to \mathbb{R}$ is such that $f(X_T^N) = g_N(\Delta W_1, \cdots, \Delta W_N)$ and can be computed explicitly.

Let us further prepare some notations. We denote by $q_n^{\Delta}(x,y)$ the n-step transition density of the Markov chain $(X_{t_n}^N)_{n=0}^N$:

$$q_n^{\Delta}(x,y) := \int_{-\infty}^{+\infty} q_1^{\Delta}(x,z_1)\mathrm{d}z_1 \int_{-\infty}^{+\infty} q_1^{\Delta}(z_1,z_2)\mathrm{d}z_2 \cdots \int_{-\infty}^{+\infty} q_1^{\Delta}(z_{n-2},z_{n-1})\mathrm{d}z_{n-1} q_1^{\Delta}(z_{n-1},y)$$

where for each $x, y \in \mathbb{R}$,

$$q_1^{\Delta}(x,y) := \frac{\exp\Big\{-\dfrac{\big(y-(x+b(x)\Delta t)\big)^2}{2\sigma(x)^2\Delta t}\Big\}}{\sqrt{2\pi\sigma(x)^2\Delta t}}.$$

Additionally, for $v>0$, $t>0$ and $x, y \in \mathbb{R}$ we set

$$p_t^v(x,y) := \frac{\mathrm{e}^{-\frac{(y-x)^2}{2v^2t}}}{\sqrt{2\pi v^2 t}}$$

which is the transition density function of $(vW_t)_{0\le t\le T}$.

Assumption 3.7.1. *There are constants $c>0$ and $C\ge 1$ such that*

$$C^{-1}p_{t_n}^{c^{-1}}(x,y) \le q_n^{\Delta}(x,y) \le C p_{t_n}^{c}(x,y) \tag{3.41}$$

for every $x, y \in \mathbb{R}$, $n = 1,2,\cdots,N$ and every partition $\Delta : 0 = t_0 < \cdots < t_n \equiv \frac{nT}{N} < \cdots < t_N = T$.

This assumption is assured if, for example, σ is uniformly elliptic and uniformly η-Holder continuous for some $\eta>0$ and b is bounded (see Lemaire-Menozzi [**28**], Theorem 2.1.).

We shall keep the symbols $c>0$ and $C\ge 1$ given in the above assumption. In the following, we can, without loss of generality, assume that $c\ge 1$ as one can always modify the constant C if necessary.

Theorem 3.7.2. *Define $W_t^{x_0} := x_0 + W_t$ for $0\le t\le T$. Under Assumption 3.7.1, suppose that there exists a constant $K>0$ such that*

$$\begin{aligned}&\frac{1}{(t_{l-1})^n}\int_{-\infty}^{+\infty} \mathbf{E}[f\Big(X_{T-t_l}^N(cW_{t_{l-1}}^{x_0}+cy)\Big)H_n\left(\frac{W_{t_{l-1}}}{\sqrt{t_{l-1}}}\right)]^2\frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}\mathrm{d}y\\ &\le \frac{K}{(T)^n}\int_{-\infty}^{+\infty}\mathbf{E}[f(cW_T^{x_0}+cy)H_n\left(\frac{W_T}{\sqrt{T}}\right)]^2\frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}\mathrm{d}y\end{aligned} \tag{3.42}$$

for each $l=1,2,\cdots,N$, $n=2,3,\cdots$ and $N=1,2,\cdots$. If

$$\limsup_{N\to\infty}\int_{-\infty}^{+\infty}\|f(cW_T^{x_0}+cy)\|_{\mathbb{D}_{2,s}}^2\frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}\mathrm{d}y < +\infty$$

for some $0 \leq s \leq 1$ *then*

$$\| \mathrm{Err}_N \|_{L^2} = O(N^{-s/2}) \quad \textit{as } N \to \infty.$$

Proof. Set

$$u_l(x) := \int_{-\infty}^{+\infty} q_l^{\Delta}(x,y) \mathrm{d}y f(y), \quad \text{for } l = 1, 2, \cdots, N \text{ and } x \in \mathbb{R}$$

which enables us to write $\mathbf{E}[\, f(X_T^N) | \mathcal{G}_l^N \,] = u_{N-l}(X_{t_l})$ for each $l = 1, 2, \cdots, N$.

By Theorem 2.2.1 and (3.9), we have

$$\| \mathrm{Err}_N \|_{L^2}^2 = \sum_{k=2}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^k}{k!} \mathbf{E}\Big[\mathbf{E}[\, \partial_l^k f(X_T^N) | \mathcal{G}_{l-1}^N \,]^2 \Big].$$

Since the operators $\mathbf{E}[\cdot | \mathcal{G}_l^N \,]$ and ∂_l commutes on $L^2(\mathcal{G}_N^N)$, we have

$$\mathbf{E}[\, \partial_l^k f(X_T^N) | \mathcal{G}_{l-1}^N \,] = \mathbf{E}[\, u_{N-l}^{(k)}(X_{t_l}^N) | \mathcal{G}_{l-1}^N \,] = v_{k,N-l}(X_{t_{l-1}}^N)$$

for each $k = 2, 3, \cdots, N$ and $l = 1, 2, \cdots, N$ where

$$v_{k,N-l}(x) := \int_{-\infty}^{+\infty} q_1^{\Delta}(x,y) \, \mathrm{d}y \, u_{N-l}^{(k)}(y), \quad x \in \mathbb{R}.$$

By Assumption 3.7.1 and Schwartz' inequality, we have

$$\begin{aligned}
&\mathbf{E}[\, v_{k,N-l}(X_{t_{l-1}}^N)^2 \,] \leq C \, \mathbf{E}[\, v_{k,N-l}(cW_{t_{l-1}}^{x_0})^2 \,] \\
&= C \, \mathbf{E}[\int_{-\infty}^{+\infty} p_{t_{l-1}}^c(x_0, x) \mathrm{d}x \Big\{ \int_{-\infty}^{+\infty} q_1^{\Delta}(x,y) \, \mathrm{d}y \, u_{N-l}^{(k)}(y) \Big\}^2 \,] \\
&\leq C \, \mathbf{E}[\int_{-\infty}^{+\infty} p_{t_{l-1}}^c(x_0, x) \, \mathrm{d}x \int_{-\infty}^{+\infty} q_1^{\Delta}(x,y) \, \mathrm{d}y \Big\{ u_{N-l}^{(k)}(y) \Big\}^2 \,] \\
&\leq C^2 \, \mathbf{E}[\int_{-\infty}^{+\infty} p_{t_l}^c(x_0, y) \mathrm{d}y \Big\{ u_{N-l}^{(k)}(y) \Big\}^2 \,] \\
&\leq C^2 c^k \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y \, \mathbf{E}[\, u_{N-l}^{(k)}(cW_{t_{l-1}}^{x_0} + cy)^2 \,]
\end{aligned}$$

where in the last inequality, we have used the assumption $c \geq 1$. Moreover we have

$$\mathbf{E}[\, u_{N-l}^{(k)}(cW_{t_{l-1}}^{x_0} + cy)^2 \,] = \sum_{n=0}^{\infty} \frac{c^n (t_{l-1})^n}{n!} \mathbf{E}[\, u_{N-l}^{(k+n)}(\, cW_{t_{l-1}}^{x_0} + cy \,) \,]^2.$$

Putting it all together, we have

$$\begin{aligned}\|\mathrm{Err}_N\|_{L^2}^2 &= C^2 \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y \sum_{l=1}^{N} \sum_{k=2}^{\infty} \sum_{n=0}^{\infty} \frac{(\Delta t)^k (t_{l-1})^n}{k!n!} \mathbf{E}[\, u_{N-l}^{(k+n)}(\, cW_{t_{l-1}}^{x_0} + cy\,)\,]^2 \\ &= C^2 \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y \sum_{n=2}^{\infty} \frac{1}{n!} \\ &\quad \times \sum_{l=1}^{N} \Big\{ (t_l)^n - (t_{l-1})^n - n(\Delta t)(t_{l-1})^{n-1} \Big\} c^n\, \mathbf{E}[\, u_{N-l}^{(n)}(cW_{t_{l-1}}^{x_0} + cy\,)\,]^2.\end{aligned}$$

By our assumption in (3.42), we obtain

$$\begin{aligned}&\int_{-\infty}^{+\infty} c^n\, \mathbf{E}[\, u_{N-l}^{(n)}(\, cW_{t_{l-1}}^{x_0} + cy\,)\,]^2 \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y \\ &= \frac{n!}{(t_{l-1})^n} \int_{-\infty}^{+\infty} \mathbf{E}[\, u_{N-l}(cW_{t_{l-1}}^{x_0} + cy) H_n\left(\frac{W_{t_{l-1}}}{\sqrt{t_{l-1}}}\right)]^2 \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y \\ &= \frac{n!}{(t_{l-1})^n} \int_{-\infty}^{+\infty} \mathbf{E}[\, f\Big(X_{T-t_l}^N(cW_{t_{l-1}}^{x_0} + cy)\Big) H_n\left(\frac{W_{t_{l-1}}}{\sqrt{t_{l-1}}}\right)]^2 \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y \\ &\leq \frac{Kn!}{T^n} \int_{-\infty}^{+\infty} \mathbf{E}[\, f(cW_T^{x_0} + cy) H_n\left(\frac{W_T}{\sqrt{T}}\right)]^2 \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y.\end{aligned}$$

Hence we have

$$\begin{aligned}&\|\mathrm{Err}_N\|_{L^2}^2 \\ &\leq KC^2 \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y \sum_{n=2}^{\infty} n \Big\{ \frac{1}{n} - \frac{1}{N} \sum_{l=0}^{N} \left(\frac{l}{N}\right)^{n-1} \Big\} \mathbf{E}[\, f(cW_T^{x_0} + cy) H_n\left(\frac{W_T}{\sqrt{T}}\right)]^2 \\ &\leq KC^2 N^{-s} \int_{-\infty}^{+\infty} \|f(cW_T^{x_0} + cy)\|_{\mathbb{D}_{2,s}}^2 \frac{\mathrm{e}^{-\frac{y^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}} \mathrm{d}y.\end{aligned}$$

□

CHAPTER 4

Discrete-Time Clark-Ocone Formula for Poisson Functionals

This part is based on the paper [**5**].

1. Introduction

Differently to the previous chapter, we use the symbol n (rather than N) in this chapter to denote the number of division points in the equidistant partition Δ:

$$\Delta = \Delta^{(n)} : 0 = t_0 < t_1 < \cdots < t_l = \frac{lT}{n} < \cdots < t_n = T$$

because we deal with Poisson processes $N = (N_t)_{0 \le t \le T}$ in the following.

When compared to the Brownian case, a Clark-Ocone formula for Poisson noises takes a bit different form (see e.g. [**15**]): for a Poisson functional F,

$$F = \mathbf{E}[F] + \int_0^{T+} \int_{\mathbf{X}} \mathbf{E}[\, D_{(t,x)} F \,|\, \mathcal{H}_t \,] \widetilde{N}(\mathrm{d}t\mathrm{d}x) \tag{4.1}$$

where $\widetilde{N}(\mathrm{d}t\mathrm{d}x)$ is a compensated Poisson random measure on $[0, T] \times \mathbf{X}$, $\mathbf{X}$ is a measurable space, and $\mathcal{H} = (\mathcal{H}_t)_{0 \le t \le T}$ is the natural filtration of $\widetilde{N}(\mathrm{d}t\mathrm{d}x)$. The expression (4.1), however, does not directly describes a martingale representation with respect to a process (martingale basis) unless the Lévy measure of $\widetilde{N}$ consists only of point masses.

To the best of the author's knowledge, the earliest work on the Poisson version of Clark-Ocone formula was due to Ševljakov [**53**], using a difference operator for $D_{(t,x)}$ (for its definition, see section 3.1, subsection 1.1 in [**59**] or Definition 6.4.1 and Proposition 6.4.7 in [**45**]). The Poisson version of Clark-Ocone formula has been established together with the development of the Malliavin calculus. It is known that the Malliavin calculus for Poisson processes has two different formulations: one is based on chaotic expansions which leads to the "difference calculus" on Poisson spaces (see e.g., Bichteler-Gravereaux-Jacod [**8**], Wu [**58**], Dermoune-Krée-Wu [**14**] or Nualart-Vives [**40**]), and the other is a differential calculus initiated by Carlen-Pardoux [**11**]. Independently to the results in Carlen-Pardoux [**11**], Elliott-Tsoi (Theorem 3.7 in [**16**]) obtained the formula (4.1) in a framework which is closely related to the one developed in Carlen-Pardoux [**11**] and which is in the case that $N(\mathrm{d}t\mathrm{d}x)$ is coming from a Poisson process and F is a functional of jump times of the Poisson process. Privault

(Theorem 1 in [**44**]) has also obtained the formula (4.1) for a Poisson functional F in L^2 within both of the two frameworks above See also Picard (Corollaire 6 in [**43**]) and Wu (Lemma 1.3 in [**59**]). The latter states (4.1) for stationary Poisson point processes within the former framework. In this article, we adopt the former framework, that is, the "difference calculus".

In this chapter, we are interested in a discretization of (4.1). In many applications the observations of the system are discrete; let say, $t_0, t_1, \cdots, t_N$, which is the case in the financial practice. In the financial context, a martingale representation of a functional F suggests a hedging strategy of the risk of F, but with a discrete observation the martingale representation fails, and instead one should work on a Riemann sum approximation of the integral with an error. We call this error "martingale representation error", which may correspond to a hedging error in the financial context.

In [**2**], the martingale representation error in the Brownian case is studied by introducing a discrete-time version of the Clark-Ocone formula. In this chapter, studied is that of Poissonian functionals when the Lévy measure is of finite point masses, the case where the Clark-Ocone formula (4.1) gives a martingale representation with respect to an explicit martingale basis.

The first main result is a discrete version of the Poissonian Clark-Ocone formula (see Theorem 2.4.1):

$$\begin{aligned} F &= \mathbf{E}[F] + \sum_{l=1}^{n} \mathbf{E}[\left(\vartheta_l^1 f\right)(\Delta N_1, \cdots, \Delta N_n) \,|\, \mathcal{H}_{l-1}^n] \Delta \widetilde{N}_l \\ &\quad + \sum_{m=2}^{\infty} \sum_{l=1}^{n} \mathbf{E}[\left(\vartheta_l^m f\right)(\Delta N_1, \cdots, \Delta N_n) \,|\, \mathcal{H}_{l-1}^n] \times \begin{pmatrix} m\text{-th order} \\ \text{chaos of } \Delta N_l \end{pmatrix}, \end{aligned} \tag{4.2}$$

where N is a (vector) Poisson process, $\widetilde{N}$ is the compensated one, $\mathcal{H}_l^n = \sigma(\Delta N_1, \cdots, \Delta N_l)$, and ϑ_l^m, $m = 1, 2, \cdots$, are some difference operators. We remark that the first order term can be also written as

$$\sum_{l=1}^{n} \int \mathbf{E}[\left(\vartheta_{l,x} f\right)(\Delta L_1, \cdots, \Delta L_n) \,|\, \mathcal{H}_{l-1}^n] \widetilde{N}\left((t_{l-1}, t_l], \mathrm{d}x\right),$$

where $\Delta L = \sum_i x_i \Delta N(\cdot, \{x_i\})$, and

$$\begin{aligned} &\left(\vartheta_{l,x} f\right)(\Delta L_1, \cdots, \Delta L_n) \\ &\quad = f(\Delta L_1, \cdots \Delta L_l + x, \cdots, \Delta L_n) - f(\Delta L_1, \cdots, \Delta L_l, \cdots, \Delta L_n). \end{aligned}$$

By this expression we may insist that (4.2) is a variant of Clark-Ocone formula.

Using the formula (4.2), we then investigate the asymptotic behavior(s) of the martingale representation error:

$$\mathrm{Err}_n := F_n - \mathbf{E}[F_n] - \sum_{l=0}^{n} \mathbf{E}[\left(\vartheta_l f_n\right)(\Delta N_1, \cdots, \Delta N_n) | \mathcal{H}_{l-1}^n] \Delta \widetilde{N}_l$$

for a sequence $(F_n)_{n=1}^{\infty}$ such that $F_n = f_n(\Delta N_1, \cdots, \Delta N_n)$ (for details, see section 4).

The study of such errors is a variant of the topic of discretization error of a stochastic integral, which has been intensively studied, especially for these fifteen years. Here is a brief review of the literature. For $X = (X_t)_{0 \le t \le T}$ being discontinuous processes, we have the following results in the current literature.

- Convergence in law of the normalized error:
$$\sqrt{n} \cdot \mathrm{Err}_n \overset{|\Delta| \to 0}{\to} 0 \quad \text{in probability (hence in law)}$$
when X is a Lévy-Itô process without diffusion component in [**54**].
- L^2-convergence of the error: In contrast to the previous result, it is claimed in [**10**] that
$$\| \mathrm{Err}_n \|_{L^2} = O(n^{-1/2}) \quad \text{as } |\Delta| = 1/n \to 0$$
for a pure-jump Lévy process X.

Along the same line of the previous chapter, we study the context with the following results:

(1) A central limit theorem for multi-level errors (Theorem 4.1.1), which generalizes (3.2) to Poisson functionals.
(2) A strong estimate under a smoothness condition (Theorem 4.2.1), which, though seemingly contradicts with the exisiting results, gives a view unifying Brownian and Poissonian functionals.
(3) A result corresponding to the irregular pay-off cases (Theorem 4.3.1). This result exhibits an advantage of the approach using discrete Clark-Ocone formula.

These results are obtained by an application of the discrete Clark-Ocone formula (Theorem 2.4.1), which is based on an action of a Heisenberg algebra on the discrete Poisson space (see Proposition 2.2.1). The calculus derived from the action might be called discrete Poisson Malliavin calculus (see section 2.3). In section 3.1, we describe how related is the discrete to the classical one.

2. A Discrete-Time Version of Poisson Clark-Ocone Formula

2.1. Notations. Let n be a natural number. We fix an interval $[0, T]$ and its division $\Delta = \Delta^{(n)} : 0 = t_0 < t_1 < \cdots < t_n = T$ where $t_k = \frac{kT}{n}$ and for $X : [0, T] \to \mathbb{R}$, we write

$$\Delta X_k := (\Delta X)_k := X_{t_k} - X_{t_{k-1}}$$

for $k = 1, 2, \cdots, n$. We also write $\Delta t \equiv t_k - t_{k-1} = \frac{T}{n}$.

Let $(N_t^\gamma)_{0 \le t \le T, \gamma \in \Gamma}$ be a family of independent Poisson processes indexed by a finite set Γ. Denote by λ_γ the parameter of the Poisson process $N^\gamma = (N_t^\gamma)_{0 \le t \le T}$. The $\underbrace{\mathbb{R}^\Gamma \times \cdots \times \mathbb{R}^\Gamma}_{n\text{-times}}$-valued random variable $(\Delta N_1^\gamma, \cdots, \Delta N_n^\gamma)_{\gamma \in \Gamma}$ is distributed as:

$$\begin{aligned}\nu\Big(\prod_{\gamma\in\Gamma}\{k_1^\gamma\} \times \cdots \times \prod_{\gamma\in\Gamma}\{k_n^\gamma\}\Big) &:= \mathbf{P}\Big(\Delta N_1 = \boldsymbol{k}_1; \cdots; \Delta N_n = \boldsymbol{k}_n\Big) \\ &= \prod_{\gamma\in\Gamma} \mathrm{e}^{-\lambda_\gamma T} \frac{(\lambda_\gamma \Delta t_1)^{k_1^\gamma} \cdots (\lambda_\gamma \Delta t_n)^{k_n^\gamma}}{k_1^\gamma! \cdots k_n^\gamma!} k_1^\gamma \cdots k_n^\gamma\end{aligned}$$

where $\boldsymbol{k}_l = (k_l^\gamma)_{\gamma\in\Gamma}$. We set a filtration $(\mathcal{H}_l^n)_{l=0}^n$ by $\mathcal{H}_0^n :=$ the trivial σ-algebra and

$$\mathcal{H}_l^n := \sigma(\Delta N_1^\gamma, \cdots, \Delta N_l^\gamma : \gamma \in \Gamma) \quad \text{for } l = 1, 2, \cdots, n.$$

2.2. A Heisenberg Algebra Acting on the Discrete Poisson Space. We denote by $\vartheta_{(l,\gamma)}$ the difference operator acting on functions

$$f : \underbrace{\mathbb{R}^\Gamma \times \cdots \times \mathbb{R}^\Gamma}_{n\text{-times}} \to \mathbb{R}$$

by

$$\begin{aligned}&\Big(\vartheta_{(l,\gamma)} f\Big)(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) \\ &\qquad := f(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_l + \boldsymbol{e}_\gamma, \cdots, \boldsymbol{k}_n) - f(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n),\end{aligned}$$

where $\boldsymbol{e}_\gamma \in \mathbb{R}^\Gamma = \{\text{map } \Gamma \to \mathbb{R}\}$ is defined by

$$\boldsymbol{e}_\gamma^\eta := \boldsymbol{e}_\gamma(\eta) := \begin{cases} 1 & \text{if } \gamma = \eta, \\ 0 & \text{if } \gamma \neq \eta. \end{cases}$$

We also define another family of difference operators by

$$\vartheta_{(l,\gamma)}^* f := \frac{1}{\lambda_\gamma \Delta t} \theta_{(l,\gamma)} f - f$$

where

$$\Big(\theta_{(l,\gamma)} f\Big)(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) := k_l^\gamma f(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_l - \boldsymbol{e}_\gamma, \cdots, \boldsymbol{k}_n).$$

We denote by $\mathcal{P}$ the space of all polynomials (over $\mathbb{R}$) in $\{k_l^\gamma : l = 1, 2, \cdots, n, \gamma \in \Gamma\}$. It is easy to see that the difference operators $\vartheta_{(l,\gamma)}$, $\vartheta_{(l,\gamma)}^*$ are linear transformations on $\mathcal{P}$. We set a family of polynomials $\{Q_m^{(l,\gamma)}\}$ by

$$Q_m^{(l,\gamma)}(k_l^\gamma) = \Big(\vartheta_{(l,\gamma)}^{*m} 1\Big)(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n)$$

where $\boldsymbol{k}_l = (k_l^\gamma)_{\gamma\in\Gamma} \in \mathbb{R}^\Gamma$ for $l = 1, 2, \cdots, n$. As an example, we see that

$$\vartheta^*_{(l,\gamma)} 1 = (\lambda_\gamma \Delta t)^{-1}(\Delta N_l^\gamma - \lambda\Delta t) =: (\lambda_\gamma\Delta t)^{-1}\Delta\widetilde{N}_l^\gamma. \tag{4.3}$$

Here (and henceforth) we identify the action of $\vartheta^*_{(l,\gamma)}$ on $\mathcal{P}[k_l^\gamma] \subset L^2(\nu)$ and that on $\mathcal{P}[\Delta N_l^\gamma] \subset L^2(\mathbf{P})$ induced by $\Omega \ni \omega \mapsto \{\Delta N_l^\gamma\}$.

We shall list up some algebraic properties which will play essential roles in the rest of this paper.

Proposition 2.2.1.

(i) *[Heisenberg commutation relation] Denoting by $(\delta_{\alpha,\beta})_{\alpha,\beta}$ the Kronecker delta, we have*

$$[\vartheta_{(l,\gamma)}, \vartheta^*_{(k,\eta)}] = \frac{\delta_{lk}\delta_{\gamma\eta}}{\lambda_\gamma\Delta t} \tag{4.4}$$

for each $l, k = 1, 2, \cdots, n$ and $\gamma, \eta \in \Gamma$.

(ii) *[Duality]*

$$\left\langle \vartheta_{(l,\gamma)} f, g\right\rangle_{L^2(\nu)} = \left\langle f, \vartheta^*_{(l,\gamma)} g\right\rangle_{L^2(\nu)} \tag{4.5}$$

for $l = 1, 2, \cdots, n$, $\gamma \in \Gamma$ and $f, g \in \mathcal{P}$.

(iii) *[Orthogonality] For $p, q = 0, 1, 2, \cdots$, $1 \le l, r \le n$ and $\gamma, \eta \in \Gamma$,*

$$\left\langle Q_p^{(l,\gamma)}(k_l^\gamma), Q_q^{(r,\eta)}(k_r^\eta)\right\rangle_{L^2(\nu)} = \frac{p!}{(\lambda_\gamma\Delta t)^p}\delta_{pq}\delta_{lr}\delta_{\gamma\eta}.$$

(iv) *[Completeness]*

$$\Big\{ \prod_{\gamma\in\Gamma}\prod_{l=1}^n \sqrt{\frac{(\lambda_\gamma\Delta t)^{m_{(l,\gamma)}}}{m_{(l,\gamma)}!}} Q^{(l,\gamma)}_{m_{(l,\gamma)}}(k_l^\gamma) : \sum_{\substack{1\le l\le n\\ \gamma\in\Gamma}} m_{(l,\gamma)} < \infty \Big\}$$

forms a complete orthonormal system (CONS, in short) of $L^2(\nu)$.

Proof. We prove only (iv). The others are straightforward or obvious. We fix a numbering and write $\Gamma = \{\gamma_j\}$. To prove (iv), it suffices to prove that $\mathcal{P}$ is dense in $L^2(\nu)$. Assume that $X \in L^2(\nu)$ is orthogonal to $\mathcal{P}$. Then since X is also orthogonal to its $L^2(\nu)$-closure $\overline{\mathcal{P}}$, we have

$$\begin{aligned} &E^\nu\Big[E^\nu\Big[X\Big|\sigma(k_1^j, \cdots, k_n^j : 1 \le j \le m)\Big]\exp\Big\{\sum_{l=1}^n\sum_{j=1}^m \sqrt{-1}\xi_{l,j}k_l^{\gamma_j}\Big\}\Big] \\ &= E^\nu\Big[X\exp\Big\{\sum_{l=1}^n\sum_{j=1}^m \sqrt{-1}\xi_{l,j}k_l^{\gamma_j}\Big\}\Big] = 0 \end{aligned}$$

for each $m = 1, 2, \cdots$ and $\xi_{l,j} \in \mathbb{R}$, where E^ν denotes the expectation with respect to ν. The injectivity of the Fourier transform implies $E^\nu[X|\sigma(k_1^j, \cdots, k_n^j : 1 \leq j \leq m)] = 0$ ν-a.s. for each $m = 1, 2, \cdots$. Therefore the martingale convergence theorem ensures that $X = 0$ ν-a.s. and hence $\mathcal{P}$ is dense in $L^2(\nu)$. □

Remark 2.1. Each of $Q_m^{(l,\gamma)}(k)$ is related to the so-called **Charlier polynomial** $C_m(k, \lambda_\gamma \Delta t)$ of order m and parameter $\lambda_\gamma \Delta t$ (See [**45**], p207, Chapter 6, Definition 6.2.7) which is defined by

$$\sum_{m=0}^{\infty} \frac{s^m}{m!} C_m(k, \lambda) = \exp\left\{k \log(1+s) - s\lambda\right\}$$

for $-1 < s < 1$. The relation between them is given by

$$C_m(k, \lambda_\gamma \Delta t) = (\lambda_\gamma \Delta t)^m Q_m^{(l,\gamma)}(k).$$

2.3. Generalized Poisson Functionals in Discrete Time and its Generalized Conditional Expectations. With the action of $\{\vartheta_{(l,\gamma)}, \vartheta^*_{(l,\gamma)}\}$, we can define a chaotic decomposition:

$$L^2(\nu) = \mathbb{R} \oplus C_1^{(n)} \oplus C_2^{(n)} \oplus \cdots,$$

with

$$C_k^{(n)} = \overline{\mathrm{span}\Big\{\prod_{\gamma\in\Gamma}\prod_{l=1}^{n} \sqrt{\frac{(\lambda_\gamma \Delta t)^{m_{(l,\gamma)}}}{m_{(l,\gamma)}!}} Q_{m_{(l,\gamma)}}^{(l,\gamma)}(k_l^\gamma) : \sum_{\substack{1\leq l\leq n\\ \gamma\in\Gamma}} m_{(l,\gamma)} = k\Big\}}^{\|\cdot\|_{L^2(\nu)}}$$

for $k = 1, \cdots$ and $C_0^{(n)} = \mathbf{R}$. Let $J_k^{(n)}$ denote the orthogonal projection onto $C_k^{(n)}$ for each $k = 0, 1, 2, \cdots$. Define $\|\cdot\|_{2,s,(n)}$ for $s \in \mathbb{R}$ by

$$\|F\|_{2,s,(n)}^2 = \sum_{k=1}^{\infty} (1+k)^s \|J_k^{(n)}(F)\|_{L^2(\nu)}^2.$$

We denote by $\mathbb{D}_{2,s}^{(n)}$ the completion of $\mathcal{P}$ with respect to $\|\cdot\|_{2,s,(n)}$.

The spaces $\mathbb{D}_{2,-\infty}^{(n)} := \cup_{s<0}\mathbb{D}_{2,s}^{(n)}$ and $\mathbb{D}_{2,\infty}^{(n)} := \cap_{s>0}\mathbb{D}_{2,s}^{(n)}$ are what one might call the spaces of *generalized Poisson functionals in discrete time* and *smooth Poisson functionals in discrete time*. By a standard argument, one can see that $\mathbb{D}_{2,-\infty}^{(n)}$ is the continuous dual of $\mathbb{D}_{2,\infty}^{(n)}$ (with respect to the projective topology on $\mathbb{D}_{2,\infty}^{(n)}$).

The operators $\vartheta_{(l,\gamma)}$ can be extended to $\mathbb{D}_{2,-\infty}^{(n)}$ by the pairing

$${}_{\mathbb{D}_{2,-\infty}^{(n)}}\left\langle \vartheta_{(l,\gamma)} F, G\right\rangle_{\mathbb{D}_{2,\infty}^{(n)}} = {}_{\mathbb{D}_{2,-\infty}^{(n)}}\left\langle F, \vartheta^*_{(l,\gamma)} G\right\rangle_{\mathbb{D}_{2,\infty}^{(n)}}$$

for each $F \in \mathbb{D}^{(n)}_{2,-\infty}$, $G \in \mathbb{D}^{(n)}_{2,\infty}$, $l = 1, 2, \cdots, n$ and $\gamma \in \Gamma$. Furthermore, for each $X \in \mathbb{D}^{(n)}_{2,-\infty}$, its conditional expectation $\mathbf{E}[X|\mathcal{A}]$ given a sub σ-algebra $\mathcal{A}$ of $\mathcal{H}^n_n$ is defined as the unique element

$$Y \in \bigcup_{s<0} \overline{\Big\{F \in L^2(\nu, \mathcal{A}) : \|F\|_{2,s,(n)} < \infty\Big\}}^{\|\cdot\|_{2,s,(n)}} =: \mathbb{D}^{(n)}_{2,-\infty}(\mathcal{A})$$

such that

$${}_{\mathbb{D}^{(n)}_{2,-\infty}}\langle X, Z\rangle_{\mathbb{D}^{(n)}_{2,\infty}} = {}_{\mathbb{D}^{(n)}_{2,-\infty}}\langle Y, Z\rangle_{\mathbb{D}^{(n)}_{2,\infty}}$$

for each

$$Z \in \bigcap_{s>0} \overline{\Big\{F \in L^2(\nu, \mathcal{A}) : \|F\|_{2,s,(n)} < \infty\Big\}}^{\|\cdot\|_{2,s,(n)}} =: \mathbb{D}^{(n)}_{2,\infty}(\mathcal{A}).$$

Since $\mathbf{E}[\cdot|\mathcal{A}]$ is a contraction operator on $L^2(\mathcal{H}^n_n)$, the existence of such a Y is ensured. The uniqueness follows from the non-singularity of the above pairing on $\mathbb{D}^{(n)}_{2,-\infty}(\mathcal{A}) \times \mathbb{D}^{(n)}_{2,\infty}(\mathcal{A})$.

The framework can be called *discrete Poisson-Malliavin calculus*.

2.4. Discrete-Time Clark-Ocone Formula. Now we present a discrete-time version of the Clark-Ocone formula for Poisson functionals.

Theorem 2.4.1 (A Discrete Version of Clark-Ocone Formula). *For each $F \in L^2(\mathcal{H}^n_n)$, we have the following L^2-convergent series expansion:*

$$\begin{aligned}&F - \mathbf{E}[F]\\ &= \sum_{m=1}^{\infty}\sum_{l=1}^{n}\sum_{\substack{\sum_{\gamma\in\Gamma} k_\gamma = m,\\ k_\gamma \geq 0 \text{ for all } \gamma}} \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma \Delta t)^{k_\gamma}}{k_\gamma!} \mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta^{k_\gamma}_{(l,\gamma)}\Big)F \Big| \mathcal{H}^n_{l-1}\Big]\Big(\prod_{\gamma\in\Gamma} \vartheta^{*k_\gamma}_{(l,\gamma)}\Big)1.\end{aligned} \tag{4.6}$$

Remark 2.2. In the above expression, the product of operators stands for the composition of operators, which is independent of the order how it composes because of (4.4) and is well-defined since they consists of actually a finite number of compositions. The conditional expectations are understood in the generalized sense.

Proof. We begin with the equation

$$F - \mathbf{E}[F] = \sum_{l=1}^{n} \Big\{\mathbf{E}[F|\mathcal{H}^n_l] - \mathbf{E}[F|\mathcal{H}^n_{l-1}]\Big\}.$$

We see that for $\mathbf{P}$-a.a. ω and under $\mathbf{P}(\cdot|\mathcal{H}^n_{l-1})(\omega)$, ΔN^γ_l is distributed in the Poisson law of the parameter $\lambda_\gamma \Delta t$ and $\mathbf{E}[F|\mathcal{H}^n_l]$ can be thought as a functional of ΔN^γ_l,

$\gamma \in \Gamma$. Moreover, from Proposition 2.2.1 we know that

$$\Big\{ \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma \Delta t)^{k_\gamma/2}}{\sqrt{k_\gamma!}} Q^{(l,\gamma)}_{k_\gamma}(\Delta N^\gamma_l) : \sum_{\gamma\in\Gamma} k_\gamma < \infty \Big\}$$

forms a CONS in $L^2(\sigma(\Delta N_l), \mathbf{P}(\cdot|\mathcal{H}^\Delta_l)(\omega))$.

From these observations and using the duality (4.5), we have

$$\begin{aligned}&\mathbf{E}[\,F\,|\,\mathcal{H}^n_l\,]\\ &= \sum_{m=0}^\infty \sum_{\sum_{\gamma\in\Gamma} k_\gamma = m} \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma \Delta t)^{k_\gamma}}{k_\gamma!} \mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta^{k_\gamma}_{(l,\gamma)}\Big) F \Big| \mathcal{H}^n_{l-1}\Big]\Big(\prod_{\gamma\in\Gamma} \vartheta^{*k_\gamma}_{(l,\gamma)}\Big) 1\end{aligned}$$

$\mathbf{P}$-a.s., and hence we obtain the result. □

Remark 2.3. For the one-dimensional case, we see that this discrete-version of Clark-Ocone formula takes a simpler form:

$$F - \mathbf{E}[F] = \sum_{m=1}^\infty \sum_{l=1}^n \frac{(\lambda \Delta t)^m}{m!} \mathbf{E}[\,\vartheta^m_l F\,|\,\mathcal{H}^n_{l-1}\,]\big(\vartheta^{*m}_l 1\big), \tag{4.7}$$

with an obvious reduction of the difference operators.

3. Consistency of the Discrete Poisson Malliavin Calculus with the Classical One

3.1. A Review of the Classical Poisson Malliavin Calculus. First, we review briefly the classical Malliavin calculus over the Poisson space to introduce notations which we will use in the following sections devoted to asymptotic analyses, and then will show how our framework, established in the previous sections is "embedded" to the classical Malliavin calculus (Proposition 3.2.1).

Let $(\mathbf{X}, \mathscr{B}_\mathbf{X}, \boldsymbol{n}(\mathrm{d}x))$ be a σ-finite Borel measure space. We denote by $\Pi_\mathbf{X}$ the space of all point functions (see **[24]**, Chapter I, subsection 9) $p = (p, \mathbf{D}_p)$ on $\mathbf{X}$. For each $p \in \Pi_\mathbf{X}$, a counting measure N_p on $[0,T] \times \mathbf{X}$ is defined by

$$N_p(\mathrm{d}t\mathrm{d}x) := \sharp\Big\{ s \in \mathbf{D}_p \cap \mathrm{d}t : p(s) \in \mathrm{d}x \Big\}.$$

The mapping $p \mapsto N_p$ induces a measurable structure on $\Pi_\mathbf{X}$. We denote by $(\mathcal{H}_t)_{0\le t\le T}$ the filtration defined by

$$\mathcal{H}_t := \sigma\Big(N_p((0,s] \times U) : 0 \le s \le t,\ U \in \mathscr{B}_\mathbf{X} \Big)$$

for $0 \le t \le T$. One can endow a probability measure $\mathbf{P}$ on $(\Pi_\mathbf{X}, \mathcal{H}_T)$ such that p is a stationary Poisson point process with the characteristic measure $\boldsymbol{n}$ (see **[24]**, Chapter I, subsection 9). Under the probability measure $\mathbf{P}$, N_p admits the Doob-Meyer decomposition

$$N_p(\mathrm{d}t\mathrm{d}x) = \widetilde{N}_p(\mathrm{d}t\mathrm{d}x) + \widehat{N}_p(\mathrm{d}t\mathrm{d}x)$$

where $\widehat{N}_p(\mathrm{d}t\mathrm{d}x) = \mathbf{E}[N_p(\mathrm{d}t\mathrm{d}x)] = \mathrm{d}t\,\boldsymbol{n}(\mathrm{d}x)$ and $\widetilde{N}_p(\mathrm{d}t\mathrm{d}x) = N_p(\mathrm{d}t\mathrm{d}x) - \widehat{N}_p(\mathrm{d}t\mathrm{d}x)$ is the martingale part of the decomposition.

Writing $K := L^2([0,T]\times\mathbf{X}, \mathrm{d}t\otimes\boldsymbol{n}(\mathrm{d}x))$, it is known that $L^2(\Pi_{\mathbf{X}})$ has the following orthogonal decomposition, which is called the *Wiener-Itô chaos expansion*:

$$L^2(\Pi_{\mathbf{X}}) = \mathbb{R} \oplus C_1 \oplus C_2 \oplus \cdots$$

where C_k is a closed linear subspace of $L^2(\Pi_{\mathbf{X}})$ which consists of all multiple Poisson integrals

$$\int_0^T\int_{\mathbf{X}}\cdots\int_0^{t_2-}\int_{\mathbf{X}} g\Big((t_1,x_1),\cdots,(t_k,x_k)\Big)\widetilde{N}_p(\mathrm{d}t_1\mathrm{d}x_1)\cdots\widetilde{N}_p(\mathrm{d}t_k\mathrm{d}x_k)$$

of k-th degree (see Theorem 10.2 in [**15**] or Theorem 2 in [**25**]), $g \in \underbrace{K\otimes\cdots\otimes K}_{k\text{-times}}$.

Each C_k is called the subspace of *k-th order chaos*. We denote by J_k the orthogonal projection onto C_k.

For each $s \in \mathbb{R}$, a Sobolev-type Hilbert space $\mathbb{D}_{2,s} = \mathbb{D}_{2,s}(\mathbb{R})$ is defined as the completion of $\{F \in L^2(\Pi_{\mathbf{X}}) : \|F\|_{2,s} < +\infty\}$ under the seminorm $\|\cdot\|_{2,s}$ on $L^2(\Pi_{\mathbf{X}})$ defined by

$$\|F\|_{2,s}^2 = \sum_{k=0}^{\infty}(1+k)^s\,\|J_kF\|_{L^2}^2, \quad F \in L^2(\Pi_{\mathbf{X}}) \tag{4.8}$$

which may be infinite in general.

It is also known that one can define a (continuous) linear operator $D : \mathbb{D}_{2,1} \to L^2(\Pi_{\mathbf{X}})\otimes K$ such that

$$\langle DF, g\rangle_K = \int_0^T\int_{\mathbf{X}} g(t,x)D_{(t,x)}F\,\mathrm{d}t\,\boldsymbol{n}(\mathrm{d}x) \in L^2(\Pi_{\mathbf{X}})$$

for every $g \in K$ and $F \in \mathbb{D}_{2,1}$ (see e.g., Definition 12.2 in [**15**], Definition 6.4.1 and Proposition 6.4.7 in [**45**]), where $D_{(t,x)}F$ is defined by

$$(D_{(t,x)}F)(p) = F(p^+_{(t,x)}) - F(p) \tag{4.9}$$

for a.a $(p,t,x) \in \Pi_{\mathbf{X}}\times[0,T]\times\mathbf{X}$ and $p^+_{(t,x)}$ is a point function defined by

$$p^+_{(t,x)}(s) := \begin{cases} p(s) & \text{if } s \neq t, \\ x & \text{if } s = t \end{cases} \qquad \text{for } s \in \mathbf{D}_{p^+_{(t,x)}} := \mathbf{D}_p \cup \{t\}.$$

The *difference operator* $D_{(t,x)}$ in (4.9) is well-defined for a.a. (p,t,x) because the image measure of $\mathbf{P}(\mathrm{d}p)\otimes\mathrm{d}t\otimes\boldsymbol{n}(\mathrm{d}x)$ under the map

$$\Pi_X\times[0,T]\times\mathbf{X} \ni (p,t,x) \mapsto (p^+_{(t,x)},t,x) \in \Pi_X\times[0,T]\times\mathbf{X}$$

is absolutely continuous with respect to $\mathbf{P}(\mathrm{d}p)\otimes\mathrm{d}t\otimes\boldsymbol{n}(\mathrm{d}x)$ (see e.g., Corollary 6.4.6 in [**45**] or subsection 1.1 in [**59**]). When we consider the case where $\mathbf{X}$ is a singleton $\{x\}$, we denote $D_t := D_{(t,x)}$ for a.e. $0 \le t \le T$.

3.2. Consistency of the Discrete Poisson Malliavin Calculus with the Classical Continuous One. Under these notations, we can state some relationships between our framework established in section 2 and that of the classical Poisson Malliavin calculus stated above, and prepare some additional framework to state our result.

We choose freely distinct points $x_\gamma \in X$ indexed by $\gamma \in \Gamma$, and construct a stationary Poisson point process p with characteristic measure $\sum_{\gamma\in\Gamma} \lambda_\gamma \delta_{\{\gamma\}}$ such that $N_t^\gamma = N_p((0,t] \times \{x_\gamma\})$.

The correspondences are summarized as follows. We omit the proof because it is immediate from the definition and the above known results.

Proposition 3.2.1.

(i) *For each $F \in \mathbb{D}_{2,1}^{(n)}$, we have*

$$(D_{(t,x_\gamma)}F)(p) = \sum_{l=1}^{n} 1_{[t_{l-1},t_l)}(t)\left(\vartheta_{(l,\gamma)}F\right)(p)$$

for any $\gamma \in \Gamma$ and a.a. $(p,t) \in \Pi_{\mathbf{X}} \times [0,T]$.

(ii) *For any n and s, $\mathbb{D}_{2,s}^{(n)} \subset \mathbb{D}_{2,s}$.*

(iii) *For each $F \in \mathbb{D}_{2,s}$ and $n = 1,2,\cdots$, we have $\mathbf{E}[F|\mathcal{H}_n^n] \in \mathbb{D}_{2,s}^{(n)}$ and*

$$\mathbf{E}[F|\mathcal{H}_n^n] \to F \quad \text{in } \mathbb{D}_{2,s}$$

as $n \to \infty$.

For each $F \in \mathbb{D}_{2,1}$, Proposition 3.2.1 implies that

$$(D_{(t,x_\gamma)}F)(p) = \lim_{n\to\infty} \sum_{l=1}^{n} 1_{\{t_{l-1}\le t<t_l\}}\vartheta_{(l,\gamma)}\mathbf{E}[F|\mathcal{H}_n^n](p) \tag{4.10}$$

for a.a. $(p,t,x) \in \Pi_{\mathbf{X}} \times [0,T] \times \mathbf{X}$. Note that in the Brownian motion case, the derivative D on the Wiener space is defined via such a relation (4.10) with $N = 2^n$ in [**31**]. Following this approach in [**31**], we define, for $F \in \mathbb{D}_{2,k}$,

$$D_{(\cdot,x_{\gamma_1},\cdots,x_{\gamma_k})}F \in L^2[0,T]$$

as the L^2-limit of the sequence

$$\sum_{l=1}^{n} 1_{\{t_{l-1}\le t<t_l\}}\vartheta_{(l,\gamma_k)}\cdots\vartheta_{(l,\gamma_1)}\mathbf{E}[F|\mathcal{H}_n^n]$$

if it exists (see [**31**], Theorem 1.10 for a sufficient condition for the existence of the limit). In the case where Γ, and hence $\{x_\gamma\}$, is a singleton, we denote

$D^k_t := D^k_{(t,\underbrace{x_\gamma, \cdots, x_\gamma)}_{k\text{-times}}}$ for a.a. $0 \le t \le T$ and therefore $D^k_t F$ is of the form

$$D^k_t F(p) = \lim_{n\to\infty} \sum_{l=1}^{n} 1_{\{t_{l-1}\le t<t_l\}} \Big(\vartheta^k_{(l,\gamma)} F\Big)(p)$$

for a.a. $(p,t) \in \Pi_{\mathbf{X}} \times [0,T]$.

4. Asymptotic Analysis of the Martingale Representation Errors

We start with the settings and notations in section 2. Instead of working on a fixed n, we deal with a sequence. To make this point clear, we write $(\mathcal{H}^n_l)^n_{l=0}$ and $\{\vartheta_{(l,\gamma,n)}, \vartheta^*_{(l,\gamma,n)}\}$, etc. Note that $\{N^\gamma : \gamma \in \Gamma\}$, and in particular Γ, are independent of n.

Throughout the section, we consider a sequence $(F_n)^\infty_{n=1}$, each F_n being $\mathcal{H}^n_n$-measurable, and for $m = 0, 1, 2, \cdots$, we put

$$\begin{aligned}&\mathrm{Err}_n(m) =: \mathrm{Err}(F_n)(m)\\ &:= F_n - \sum_{p=1}^{m}\sum_{l=1}^{n} \sum_{\substack{\sum_{\gamma\in\Gamma} k_\gamma = p,\\ k_\gamma \ge 0 \text{ for all } \gamma}} \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma \Delta t)^{k_\gamma}}{k_\gamma!} \mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta^{k_\gamma}_{(l,\gamma)}\Big)F \Big| \mathcal{H}^n_{l-1}\Big]\Big(\prod_{\gamma\in\Gamma} \vartheta^{*k_\gamma}_{(l,\gamma)}\Big)1,\end{aligned}$$

where we recall that the filtration $(\mathcal{H}^n_l)^n_{l=0}$ is given by $\mathcal{H}^n_0$ = the trivial σ-field and $\mathcal{H}^n_l = \sigma(\Delta N^\gamma_1, \cdots, \Delta N^\gamma_l : \gamma \in \Gamma)$ for $l = 1, 2, \cdots, n$.

4.1. A Central Limit Theorem for the Errors.

In this subsection, we treat only the case $\sharp\Gamma = 1$ for notational convenience, where we have

$$\mathrm{Err}_n(m) = F_n - \sum_{k=0}^{m}\sum_{l=1}^{n} \frac{(\lambda\Delta t)^m}{m!} \mathbf{E}[\,\vartheta^k_l F_n \,|\, \mathcal{H}^n_{l-1}\,]\Big(\vartheta^{*k}_l 1\Big).$$

Here are apparent notational reductions on the difference operators.

THEOREM 4.1.1. *Let $m \in \mathbb{N}$. Suppose that $F_n \in \mathbb{D}^{(n)}_{2,m+2}$ for each $n = 1, 2, \cdots$ and for some $F \in \mathbb{D}_{2,m+1}$, we have*

- ▷ *$F_n \to F$ in $L^2(\mathbf{P})$,*
- ▷ *$D^{k+1}_t F$ exists for a.a $t \in [0,T]$ and $\displaystyle\int_0^T \|D^{k+1}_t F_n - D^{k+1}_t F\|^2_{L^2} \mathrm{d}t \to 0$ as $n \to \infty$ for each $k = 0, 1, \cdots, m$ and*
- ▷ $\displaystyle\sup_n \int_0^T \|D^{m+2}_t F_n\|^2_{L^2} \mathrm{d}t < +\infty.$

Then we have

$$\begin{pmatrix} \mathrm{Err}_n(0) \\ (\Delta t)^{-1/2}\mathrm{Err}_n(1) \\ \vdots \\ (\Delta t)^{-m/2}\mathrm{Err}_n(m) \end{pmatrix} \to \begin{pmatrix} \int_0^T \mathbf{E}[D_t F \,|\, \mathcal{H}_{t-}]\mathrm{d}\widetilde{N}_t \\ \frac{\lambda^{1/2}}{\sqrt{2}} \int_0^T \mathbf{E}[D_t^2 F \,|\, \mathcal{H}_{t-}]\mathrm{d}B_t^1 \\ \vdots \\ \frac{\lambda^{m/2}}{\sqrt{(m+1)!}} \int_0^T \mathbf{E}[D_t^{m+1} F \,|\, \mathcal{H}_{t-}]\mathrm{d}B_t^m \end{pmatrix}$$

in probability on an extended probability space as $n \to \infty$, where $(B^1, \cdots, B^m)$ is an m-dimensional Brownian motion.

Remark 4.1. Although the Brownian motion $B = (B^1, \cdots, B^m)$ above is not adapted to the filtration $(\mathcal{H}_t)_{0 \le t \le T}$, the above stochastic integrals make sense because B is automatically independent of $\mathcal{H}_T$, so that it is an $\Big(\mathcal{H}_t \vee \sigma(B_s : 0 \le s \le t)\Big)_{0 \le t \le T}$-Brownian motion.

Proof. By Corollary 4.7, we have

$$\begin{aligned} (\lambda \Delta t)^{-k/2}\mathrm{Err}_n(k) &= \sum_{m'=k+1}^{\infty} \sum_{l=1}^{n} \frac{(\lambda \Delta t)^{m'-k/2}}{m'!} \mathbf{E}[\, \vartheta_{\Delta N_l}^{m'} F_n \,|\, \mathcal{H}_{l-1}^n \,]\Big(\vartheta_{\Delta N_l}^{*m'} 1\Big) \\ &=: A_{k+1} + \sum_{m'=k+2}^{\infty} A_{m'}. \end{aligned}$$

By using the integration by parts formula (4.5), change of variable, Schwartz' inequality and the consistency (Proposition 3.2.1), we see that

$$\begin{aligned} &\Big\| \sum_{m'=k+2}^{\infty} A_{m'} \Big\|_{L^2}^2 \\ &= (\lambda \Delta t)^2 \sum_{m'=0}^{\infty} \sum_{l=1}^{n} \frac{(\lambda \Delta t)^{m'}}{(m'+k+2)!} \Big\| \mathbf{E}\Big[\Big(\vartheta_{l,n}^{k+2} F_n\Big)\Big(\vartheta_{l,n}^{*m'} 1\Big) \Big| \mathcal{H}_{l-1}^n \Big] \Big\|_{L^2}^2 \\ &\le (\lambda \Delta t)^2 \sum_{m'=0}^{\infty} \sum_{l=1}^{n} \frac{m'!}{(m'+k+2)!} \| \vartheta_{l,n}^{k+2} F_n \|_{L^2}^2 \\ &\le (\Delta t) \sum_{m'=1}^{\infty} \frac{1}{(m')^{k+2}} \times \sum_{l=1}^{N} \| \vartheta_{l,n}^{k+2} F_n \|_{L^2}^2 \lambda^2 \Delta t \\ &= (\Delta t) \sum_{m'=1}^{\infty} \frac{1}{(m')^{k+2}} \times \int_0^T \| D_t^{k+2} F_n \|_{L^2}^2 \lambda^2 \mathrm{d}t \end{aligned}$$

which goes to zero as $n \to \infty$ for each $k = 0, 1, \cdots, m$ by the assumption.

Let us have a closer look at A_{k+1}. For each $k = 1, 2, \cdots, m$, we define a right-continuous process $L^{k,n} = (L^{k,n}_t)_{0 \le t \le T}$ with left-hand side limits by

$$L^{k,N}_t := \sum_{l=1}^{r} \frac{(\lambda \Delta t)^{(k+1)/2}}{\sqrt{(k+1)!}} \Big(\vartheta^{*(k+1)}_{l,n} 1 \Big) \quad \text{if } t_{r-1} \le t < t_r$$

for $r = 1, 2, \cdots, n$, and $L^{k,n}_T := L^{k,n}_{t_{n-1}}$. Since

$$\frac{(\lambda \Delta t)^{(k+1)/2}}{\sqrt{(k+1)!}} \Big(\vartheta^{*(k+1)}_{l,n} 1 \Big) = \frac{(\lambda \Delta t)^{-(k+1)/2}}{\sqrt{(k+1)!}} C_{k+1}(\Delta N_l, \lambda \Delta t),$$

are i.i.d. random variables as a family of $l = 1, 2, \cdots, n$ and they are orthogonal to each other as a family of $k = 1, 2, \cdots, m$ (Proposition 2.2.1), the central limit theorem of finite dimensional distributions of $(\Delta t)^{1/2} L^{k,n}$, $n = 1, 2, \cdots$ follows, as for each $0 \le s < t$, with taking $t_{q-1} \le s < t_q$ and $t_{r-1} \le t < t_r$,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\Big[\mathrm{e}^{i \sum_{k=1}^{m} \xi_k \left\{ (\Delta t)^{1/2} L^{k,n}_t - (\Delta t)^{1/2} L^{k,n}_s \right\}} \Big| \mathcal{F}^{L^{1,n}}_s \vee \cdots \vee \mathcal{F}^{L^{m,n}}_s \Big] \\
&= \lim_{n\to\infty} \prod_{l=q+1}^{r} \mathbf{E}\Big[\mathrm{e}^{i \sum_{k=1}^{m} \left(\xi_k \sqrt{t_r - t_q} \right) \cdot (r-q)^{-1/2} \frac{(\lambda \Delta t)^{(k+1)/2}}{\sqrt{(k+1)!}} \left(\vartheta^{*(k+1)}_{l,n} 1 \right)} \Big] \\
&= \lim_{n\to\infty} \prod_{l=q+1}^{r} \left\{ 1 - \frac{|\xi|^2}{2(r-q)} (t_r - t_q) + o\left(\frac{|\xi|^2}{r-q} \right) \right\} = \mathrm{e}^{-\frac{|\xi|^2}{2}(t-s)}.
\end{aligned}$$

for each $\xi = (\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^m$, where $(\mathcal{F}^Z_t)_{0 \le t \le T}$ denotes the filtration generated by a stochastic process $Z = (Z_t)_{0 \le t \le T}$ and the little-o-notation is with respect to the asymptotics when $n \to \infty$ (so that $r - q \to +\infty$). This implies that every finite dimensional distribution of m-dimensional process $((\Delta t)^{1/2} L^{k,n})_{k=1}^m$ converges to that of an m-dimensional Brownian motion $(B^1, B^2, \cdots, B^m) = (B^1_t, B^2_t, \cdots, B^m_t)_{0 \le t \le T}$.

Besides, using Kolmogorov's inequality, we have for each $k = 1, 2, \cdots, m$,

$$\begin{aligned}
&\lim_{K\to\infty} \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{0 \le t \le T} \left| (\Delta t)^{1/2} L^{k,n}_t \right| \ge K \Big) \\
&\le \lim_{K\to\infty} \frac{(\Delta t) \| L^{k,n}_T \|^2_{L^2}}{K^2} = 0
\end{aligned}$$

and for each $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}\Big(\inf_{\substack{\{s_j\}_j\subset[0,T]:\\ |s_j-s_{j+1}|>\delta}}\max_j\sup_{t,s\in[s_{j-1},s_j)}(\Delta t)^{1/2}|L^{k,n}_t-L^{k,n}_s|\geq\varepsilon\Big)\\
&\leq\limsup_{n\to\infty}\mathbf{P}\Big(\max_{j=1,2,\cdots,n}\sup_{t,s\in[s_{j-1},s_j)}(\Delta t)^{1/2}|L^{k,n}_t-L^{k,n}_s|\geq\varepsilon\Big)\\
&=\limsup_{n\to\infty}\mathbf{P}(0\geq\varepsilon)=0.
\end{aligned}$$

They imply the tightness of $\{(\Delta t)^{1/2}L^{k,n}\}_{n=1}^{\infty}$(see Billingsley [**9**], Theorem 13.2). Therefore,

$$\Big\{\Big((\Delta t)^{1/2}L^{1,n},(\Delta t)^{1/2}L^{2,n},\cdots,(\Delta t)^{1/2}L^{m,n}\Big)\Big\}_{n=1}^{\infty}$$

also forms a tight family. Hence we have

$$(\sqrt{\Delta t}L^{1,n},\sqrt{\Delta t}L^{2,n},\cdots,\sqrt{\Delta t}L^{m,n})\to(B^1,B^2,\cdots,B^m)$$

in law as $n\to\infty$. By the Skorohod representation theorem (see Ikeda-Watanabe [**24**], Theorem 2.7[1]), we may assume that the above convergence is realized as an almost sure convergence on an extended probability space.

Hence for $k=1,2,\cdots,m$, we have

$$\begin{aligned}
&\frac{(\lambda\Delta t)^{(k+1)-k/2}}{(k+1)!}\sum_{l=1}^{n}\mathbf{E}[\vartheta^{k+1}_{l,n}F_n|\mathcal{H}^n_{l-1}]\cdot\Big(\vartheta^{*(k+1)}_{l,n}1\Big)\\
&=\frac{1}{\sqrt{(k+1)!}}\sum_{l=1}^{n}\mathbf{E}[\vartheta^{k+1}_{l,n}F_n|\mathcal{H}_{t_{l-1}}]\Big\{(\Delta t)^{1/2}L^{k,n}_{t_l}-(\Delta t)^{1/2}L^{k,n}_{t_{l-1}}\Big\}\\
&\to\frac{1}{\sqrt{(k+1)!}}\int_0^T\mathbf{E}[D^{k+1}_tF|\mathcal{H}_t]\mathrm{d}B^k_t\quad\text{in probability as } n\to\infty.
\end{aligned}$$

Finally, for $k=0$, by using (4.3), we have

$$(\lambda\Delta t)\sum_{l=1}^{n}\mathbf{E}[\,\vartheta_{l,n}F_n\,|\,\mathcal{H}^n_{l-1}\,](\vartheta^*_{l,n}1)\to\int_0^T\mathbf{E}[\,D_tF\,|\,\mathcal{H}_{t-}\,]\mathrm{d}\widetilde{N}_t$$

in probability as $n\to\infty$. □

[1]On the space of all right-continuous functions with left-hand side limits, one can endow so-called the *Skorohod topology* which is metrizable and makes the space a complete separable metric space. For details, see Billingsley [**9**], Chapter 5.

4.2. Strong Convergence of the Error. The second error estimation result, where Γ is any finite set, is the following

THEOREM 4.2.1. *It holds that*

$$\|\mathrm{Err}_n\|_{L^2}^2 \leq \Delta t \sum_{l=1}^{n} \sum_{\gamma,\gamma'\in\Gamma} \| \vartheta_{(l,\gamma,n)} \vartheta_{(l,\gamma',n)} F_n \|_{L^2}^2 \lambda_\gamma \lambda_{\gamma'} \Delta t, \tag{4.11}$$

so that if one has

$$\sup_n \sum_{l=1}^{n} \sum_{\gamma,\gamma'\in\Gamma} \| \vartheta_{(l,\gamma,n)} \vartheta_{(l,\gamma',n)} F_n \|_{L^2}^2 \lambda_\gamma \lambda_{\gamma'} \Delta t < \infty,$$

then $\|\mathrm{Err}_n\|_{L^2} = O(n^{-1/2})$ *as* $n \to \infty$.

PROOF. By the equation (4.6), we have

$$\mathrm{Err}_n = E_{n,1} + E_{n,2}$$

where

$$E_{n,i} := \sum_{l=1}^{n} \sum_{k\in\mathcal{K}_{n,i}} \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma \Delta t)^{k_\gamma}}{k_\gamma!} \times \mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta_{(l,\gamma,n)}^{k_\gamma}\Big) F_n \Big| \mathcal{H}_{l-1}^n\Big] \cdot \Big(\prod_{\gamma\in\Gamma} \vartheta_{(l,\gamma,n)}^{*k_\gamma}\Big) 1$$

for $i = 1, 2$. Here we set $\mathcal{K}_{n,1}$ to be the set of all vector non-negative integers $\boldsymbol{k} = (k_\gamma)_{\gamma\in\Gamma}$ such that $k_\gamma \leq 1$ for all $\gamma \in \Gamma$ and $2 \leq \sum_{\gamma\in\Gamma} k_\gamma$, and $\mathcal{K}_{n,2}$ denotes the set of all $\boldsymbol{k} = (k_\gamma)_{\gamma\in\Gamma}$ such that $k_\gamma \geq 2$ for some $\gamma \in \Gamma$. Note that $\mathcal{K}_{n,1} = \emptyset$ when $\sharp\Gamma = 1$.

By Parseval's identity, we have

$$\|E_{n,1}\|_{L^2}^2 = \sum_{l=1}^{n} \sum_{k\in\mathcal{K}_{n,1}} \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma \Delta t)^{2k_\gamma}}{(k_\gamma!)^2} \times \mathbf{E}\Big[\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta_{(l,\gamma,n)}^{k_\gamma}\Big) F_n \Big| \mathcal{H}_{l-1}^n\Big]^2\Big] \Big\| \prod_{\gamma\in\Gamma} \vartheta_{(l,\gamma,n)}^{*k_\gamma} 1 \Big\|_{L^2}^2.$$

For each $\boldsymbol{k} \in \mathcal{K}_{n,1}$, there exists a pair (η, η') such that $\eta \neq \eta'$ and $k_\eta = k_{\eta'} = 1$. Then

$$\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta_{(l,\gamma,n)}^{k_\gamma}\Big) F_n \Big| \mathcal{H}_{l-1}^n\Big] \tag{4.12}$$

$$= \mathbf{E}\Big[\Big(\vartheta_{(l,\eta,n)} \vartheta_{(l,\eta',n)} F_n\Big)\Big(\prod_{\gamma\in\Gamma\setminus\{\eta,\eta'\}} \vartheta_{(l,\gamma,n)}^{*k_\gamma} 1\Big) \Big| \mathcal{H}_{l-1}^n\Big]. \tag{4.13}$$

By considering the Fourier expansion in $L^2(P|_{\mathcal{H}^n_{l-1}})$ of

$$A := \mathbf{E}\Big[\vartheta_{(l,\eta,n)}\vartheta_{(l,\eta',n)}F_n\Big|\mathcal{H}^n_{l-1}\vee\sigma\Big(\Delta N^{\gamma,n}_l:\gamma\in\Gamma\backslash\{\eta,\eta'\}\Big)\Big],$$

we have

$$\begin{aligned}
&\|A\|^2_{L^2}\\
&=\sum_{\substack{\boldsymbol{k}\in\mathbf{Z}^\Gamma_+\\ k_\eta=k_{\eta'}=1}}\Big\{\Big\|\prod_{\gamma\in\Gamma\backslash\{\eta,\eta'\}}\vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big\|^2_{L^2}\times\\
&\prod_{\gamma\in\Gamma\backslash\{\eta,\eta'\}}\frac{(\lambda_\gamma\Delta t)^{2k_\gamma}}{(k_\gamma!)^2}\Big\|\mathbf{E}\Big[\Big(\vartheta_{(l,\eta,n)}\vartheta_{(l,\eta',n)}F_n\Big)\Big(\prod_{\gamma\in\Gamma\backslash\{\eta,\eta'\}}\vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big)\Big|\mathcal{H}^n_{l-1}\Big]\Big\|^2_{L^2}\Big\}
\end{aligned}$$

(by (4.13), and by the mutual independence among $\Delta N^{\gamma,n}$,)

$$\begin{aligned}
&=\sum_{\substack{\boldsymbol{k}\in\mathbf{Z}^\Gamma_+\\ k_\eta=k_{\eta'}=1}}\Big\{\frac{\big\|\prod_{\gamma\in\Gamma}\vartheta^{*k_\gamma}_{(l,\gamma,n)}1\big\|^2_{L^2}}{\|\vartheta^*_{(l,\eta,n)}1\|^2_{L^2}\|\vartheta^*_{(l,\eta',n)}1\|^2_{L^2}}\\
&\qquad\times\lambda_\eta^{-2}\lambda_{\eta'}^{-2}(\Delta t)^{-4}\prod_{\gamma\in\Gamma}\frac{(\lambda_\gamma\Delta t)^{2k_\gamma}}{(k_\gamma!)^2}\Big\|\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta^{k_\gamma}_{(l,\gamma,n)}\Big)F_n\Big|\mathcal{H}^n_{l-1}\Big]\Big\|_{L^2}\Big\}.
\end{aligned}$$

Since

$$\|\vartheta^*_{(l,\eta,n)}1\|^2=\lambda_\eta^{-1}(\Delta t)^{-1},$$

we have

$$\begin{aligned}
&\|E_{n,1}\|^2_{L^2}\\
&\le\sum_{l=1}^n\sum_{\substack{\eta\neq\eta'\in\Gamma\\ k_\eta=k_{\eta'}=1}}\lambda_\eta\lambda_{\eta'}(\Delta t)^2\\
&\qquad\times\Big\|\mathbf{E}\Big[\vartheta_{(l,\eta,n)}\vartheta_{(l,\eta',n)}F_n\Big|\mathcal{H}^n_{l-1}\vee\sigma\Big(\Delta N^{\gamma,n}_l:\gamma\in\Gamma\backslash\{\eta,\eta'\}\Big)\Big]\Big\|^2_{L^2}\\
&\le\Delta t\sum_{l=1}^n\sum_{\substack{\eta,\eta'\in\Gamma\\ \eta\neq\eta'}}\|\vartheta_{(l,\eta,n)}\vartheta_{(l,\eta',n)}F_n\|^2_{L^2}\lambda_\eta\lambda_{\eta'}\Delta t.
\end{aligned}\tag{4.14}$$

On the other hand, for each $\boldsymbol{k}\in\mathcal{K}_{n,2}$, we have $\min_\gamma k_\gamma\ge 2$. We set

$$\mathcal{K}_{n,2}(m):=\{\boldsymbol{k}\in\mathcal{K}_{n,2}:\min_\gamma k_\gamma=m\}.$$

Then,

$$\mathcal{K}_{n,2}=\sum_{m\ge 2}\mathcal{K}_{n,2}(m).$$

For each $\boldsymbol{k} = (k_\gamma) \in \mathcal{K}_{n,2}(m)$ with $k_\eta = \min_\gamma k_\gamma$,

$$\begin{aligned}
&\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta^{k_\gamma}_{(l,\gamma,n)}\Big)F_n\Big|\mathcal{H}^n_{l-1}\Big]^2 \\
&= \mathbf{E}\Big[\mathbf{E}\Big[\Big(\vartheta^2_{(l,\eta,n)}F_n\Big)\Big(\prod_{\gamma\in\Gamma\setminus\{\eta\}} \vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big)\Big|\mathcal{H}^n_{l-1}\vee\sigma\Big(\Delta N^{\eta,n}_l\Big)\Big]\Big(\vartheta^{*(m-2)}_{(l,\eta,n)}1\Big)\Big|\mathcal{H}^n_{l-1}\Big]^2 \\
&\leq \frac{(m-2)!}{(\lambda_\eta\Delta t)^{m-2}}\mathbf{E}\Big[\mathbf{E}\Big[\Big(\vartheta^2_{(l,\eta,n)}F_n\Big)\Big(\prod_{\gamma\in\Gamma\setminus\{\eta\}} \vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big)\Big|\mathcal{H}^n_{l-1}\vee\sigma\Big(\Delta N^{\eta,n}_l\Big)\Big]^2\Big|\mathcal{H}^n_{l-1}\Big]
\end{aligned}$$

and

$$\Big\|\prod_{\gamma\in\Gamma}\vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big\|^2_{L^2} = \frac{m!}{(\lambda_\eta\Delta t)^m}\Big\|\prod_{\gamma\in\Gamma\setminus\{\eta\}}\vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big\|^2_{L^2}.$$

Therefore we have

$$\begin{aligned}
&\|E_{n,2}\|^2_{L^2} \\
&\leq \sum_{l=1}^n\sum_{m=2}^\infty\sum_{\boldsymbol{k}\in\mathcal{K}_{n,2}(m)}\prod_{\gamma\in\Gamma}\frac{(\lambda_{\eta(\boldsymbol{k})}\Delta t)^2}{m(m-1)} \\
&\qquad\times\mathbf{E}\Big[\mathbf{E}\Big[(\vartheta^2_{(l,\eta(\boldsymbol{k}),n)}F_n)\prod_{\gamma\in\Gamma\setminus\{\eta(\boldsymbol{k})\}}\vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big|\mathcal{H}^n_{l-1}\vee\sigma\Big(\Delta N^{\eta(\boldsymbol{k}),n}_l\Big)\Big]^2\Big] \\
&\qquad\qquad\times\Big\|\prod_{\gamma\in\Gamma\setminus\{\eta(\boldsymbol{k})\}}\vartheta^{*k_\gamma}_{(l,\gamma,n)}1\Big\|^2_{L^2} \\
&\leq \sum_{l=1}^n\sum_{\eta\in\Gamma}\sum_{m=2}^\infty\frac{(\lambda_\eta\Delta t)^2}{m(m-1)}\Big\|\mathbf{E}\Big[\vartheta^2_{(l,\eta,n)}F_n\Big|\mathcal{H}^n_{l-1}\vee\sigma\Big(\Delta N^{\eta,n}_l\Big)\Big]\Big\|^2_{L^2} \\
&\leq \Delta t\Big\{\sum_{m=2}^\infty\frac{1}{m(m-1)}\Big\}\sum_{l=1}^n\sum_{\eta\in\Gamma}\|\vartheta^2_{(l,\eta,n)}F_n\|^2_{L^2}\lambda^2_\eta\Delta t.
\end{aligned} \tag{4.15}$$

Combining the inequalities (4.14) and (4.15), we obtain

$$\begin{aligned}
\|\mathrm{Err}_n\|^2_{L^2} &= \|E_{n,1}\|^2_{L^2} + \|E_{n,2}\|^2_{L^2} \\
&\leq \Delta t\sum_{l=1}^n\sum_{\gamma,\gamma'\in\Gamma}\|\vartheta_{(l,\gamma,n)}\vartheta_{(l,\gamma',n)}F_n\|^2_{L^2}\lambda_\gamma\lambda_{\gamma'}\Delta t.
\end{aligned}$$

Thus we have completed our proof. □

4.3. The Cases with One Dimensional Functionals. In this subsection, we alternatively use notions related to point processes. We suppose that x_γ, $\gamma \in \Gamma$ are real numbers. Let p be as above, and N_p is the associated Poisson random measure. We define a real valued Lévy process

$$L_t := \sum_{\gamma\in\Gamma} x_\gamma N_p\Big((0,t]\times\{x_\gamma\}\Big)$$

for $0 \le t \le T$. We redefine difference operators

$$\Big(\vartheta_{x_\gamma} f\Big)(y) = f(y+x_\gamma) - f(y)$$

(which is actually independent of t) and

$$\Big(\vartheta^*_{(t,x_\gamma)} f\Big)(y) = \frac{1}{\lambda_\gamma t} f(y-x_\gamma) - f(y)$$

where $f : \mathbb{R} \to \mathbb{R}$ be a measurable map. We note that we still have the Heisenberg commutation relation:

$$[\vartheta_{x_\gamma}, \vartheta^*_{(t,x_\eta)}] = \frac{\delta_{\gamma\eta}}{\lambda_\gamma t}$$

and the duality in $L^2(\sigma(L_t - L_s)) =: H_{t,s}$:

$$\Big\langle \vartheta_{x_\gamma} f, g \Big\rangle_{H_{t,s}} = \Big\langle f, \vartheta^*_{(t-s,x_\gamma)} g \Big\rangle_{H_{t,s}}$$

for polynomials f and g in $L_t - L_s$. We set $F_n \equiv f(L_T)$.

Theorem 4.3.1. *We have that for each* $0 \le s \le 1$,

$$\|\mathrm{Err}_n\|^2_{L^2} \le n^{-s} \sum_{q=2}^\infty q^s \|J_q f(L_T)\|^2_{L^2} \le n^{-s} \|f(L_T)\|^2_{\mathbb{D}_{2,s}}.$$

Thus, if $\|f(L_T)\|_{\mathbb{D}_{2,s}} < +\infty$ *for some* $0 \le s \le 1$, *then we have* $\|\mathrm{Err}_n\|_{L^2} = O(n^{-s/2})$ *as* $n \to \infty$.

Proof. By the discrete Clark-Ocone formula (4.6) and Parseval's identity, we have

$$\begin{aligned}
&\|\mathrm{Err}_n\|^2_{L^2} \\
&= \sum_{m=2}^\infty \sum_{l=1}^n \sum_{\substack{k\in\mathbb{Z}_+^\Gamma \text{ with} \\ \sum_{\gamma\in\Gamma} k_\gamma = m}} \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma \Delta t)^{k_\gamma}}{k_\gamma!} \\
&\qquad\qquad \times \mathbf{E}\Big[\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma} \vartheta^{k_\gamma}_{(l,\gamma,n)}\Big) f(L^n_T) \Big| \mathcal{H}^n_{l-1}\Big]^2\Big].
\end{aligned}$$

The last factor $\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta^{k_\gamma}_{(l,\gamma,n)}\Big)f(L_T)\,\Big|\,\mathcal{H}^n_{l-1}\Big]$ has its L^2-expansion as

$$
\begin{aligned}
&\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta^{k_\gamma}_{(l,\gamma,n)}\Big)f(L_T)\,\Big|\,\mathcal{H}^n_{l-1}\Big]\\
&\quad=\sum_{m'=0}^{\infty}\sum_{\substack{k'\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma'\in\Gamma}k'_{\gamma'}=m'}}\prod_{\gamma'\in\Gamma}\frac{(\lambda_{\gamma'}t_{l-1})^{k'_{\gamma'}}}{k'_{\gamma'}!}\\
&\qquad\qquad\times\mathbf{E}\Big[\Big(\prod_{\gamma'\in\Gamma}\vartheta^{k'_{\gamma'}}_{x_{\gamma'}}\prod_{\gamma\in\Gamma}\vartheta^{k_\gamma}_{(l,\gamma,n)}\Big)f(L_T)\Big]\Big(\prod_{\gamma'\in\Gamma}\vartheta^{*k'_{\gamma'}}_{(t_{l-1},x_{\gamma'})}\Big)1.
\end{aligned}
$$

Moreover, we notice that

$$
\vartheta_{(l,\gamma,n)}f(L_T)=f(L_T+x_\gamma)-f(L_T)=\Big(\vartheta_{x_\gamma}f\Big)(L_T).
$$

From these observations, we have

$$
\begin{aligned}
&\mathbf{E}\Big[\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta^{k_\gamma}_{(l,\gamma,n)}\Big)f(L_T)\Big|\mathcal{H}^n_{l-1}\Big]^2\Big]\\
&\quad=\sum_{m'=0}^{\infty}\sum_{\substack{k'\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma'\in\Gamma}k'_{\gamma'}=m'}}\prod_{\gamma'\in\Gamma}\frac{(\lambda_{\gamma'}t_{l-1})^{k'_{\gamma'}}}{k'_{\gamma'}!}\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta^{k'_\gamma+k_\gamma}_{x_\gamma}\Big)f(L_T)\Big]^2.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\|\,\mathrm{Err}_n\,\|^2_{L^2}\\
&\quad=\sum_{l=1}^{n}\sum_{m=2}^{\infty}\sum_{\substack{k\in\mathbb{Z}_+^\Gamma \text{ with}\\ m_{\gamma\in\Gamma}k_\gamma=m}}\sum_{m'=0}^{\infty}\sum_{\substack{k'\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma'\in\Gamma}k'_{\gamma'}=m'}}\\
&\qquad\qquad\prod_{\gamma\in\Gamma}\frac{(\lambda_\gamma)^{k_\gamma+k'_\gamma}(\Delta t)^{k_\gamma}(t_{l-1})^{k'_\gamma}}{k_\gamma!k'_\gamma!}\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta^{k_\gamma+k'_\gamma}_{x_\gamma}\Big)f(L_T)\Big]^2.
\end{aligned}
$$

Here we apply the change of variables $m'\mapsto q:=m+m'$ for each $m=2,3,\cdots$ and then we again apply the change of variables $\boldsymbol{k}\mapsto\boldsymbol{j}:=\boldsymbol{k}+\boldsymbol{k}'$ for each

$\boldsymbol{k} = (k_\gamma)_{\gamma\in\Gamma}$. These procedures lead to

$$\begin{aligned}&\|\operatorname{Err}_n\|_{L^2}^2\\&= \sum_{l=1}^{n}\sum_{m=2}^{\infty}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma\in\Gamma}k_\gamma=m}}\sum_{q=m}^{\infty}\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}_+^\Gamma+\boldsymbol{k}\text{ with}\\ \sum_{\gamma\in\Gamma}j_\gamma=q}}\\&\qquad\prod_{\gamma\in\Gamma}\frac{(\lambda_\gamma)^{j_\gamma}(\Delta t)^{k_\gamma}(t_{l-1})^{j_\gamma-k_\gamma}}{k_\gamma!(j_\gamma-k_\gamma)!}\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta_{x_\gamma}^{j_\gamma}\Big)f(L_T)\Big]^2.\end{aligned}$$

We further change the order of the summation with respect to (m, q), after that, with respect to $(\boldsymbol{k}, \boldsymbol{j})$, and then the above takes the form of

$$\begin{aligned}&\|\operatorname{Err}_n\|_{L^2}^2\\&= \sum_{l=1}^{n}\sum_{q=2}^{\infty}\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma\in\Gamma}j_\gamma=q}}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma\in\Gamma}k_\gamma\geq 2,\\ k_\gamma\leq j_\gamma \text{ for all }\gamma}}\\&\qquad\prod_{\gamma\in\Gamma}\Big\{\binom{j_\gamma}{k_\gamma}(\Delta t)^{k_\gamma}(t_{l-1})^{j_\gamma-k_\gamma}\frac{(\lambda_\gamma)^{j_\gamma}}{j_\gamma!}\Big\}\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta_{x_\gamma}^{j_\gamma}\Big)f(L_T)\Big]^2.\end{aligned}$$

By iterative uses of the binomial theorem, we see that

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma\in\Gamma}k_\gamma\geq 2,\\ k_\gamma\leq j_\gamma \text{ for all }\gamma}}\prod_{\gamma\in\Gamma}\binom{j_\gamma}{k_\gamma}(\Delta t)^{k_\gamma}(t_{l-1})^{j_\gamma-k_\gamma} = (t_l)^q-(t_{l-1})^q-q(\Delta t)(t_{l-1})^{q-1},$$

where $q = \sum_{\gamma\in\Gamma} j_\gamma$. Thus we have

$$\begin{aligned}&\|\operatorname{Err}_n\|_{L^2}^2\\&= \sum_{q=2}^{\infty} qI_{n,q}\sum_{\substack{\boldsymbol{j}\in\mathbb{Z}_+^\Gamma \text{ with}\\ \sum_{\gamma\in\Gamma}j_\gamma=q}}\Big\{\prod_{\gamma\in\Gamma}\frac{(\lambda_\gamma T)^{j_\gamma}}{j_\gamma!}\Big\}\mathbf{E}\Big[\Big(\prod_{\gamma\in\Gamma}\vartheta_{x_\gamma}^{j_\gamma}\Big)f(L_T)\Big]^2,\end{aligned}$$

where we meant

$$\begin{aligned}I_{n,q} &:= \frac{1}{q}-\sum_{l=1}^{n}\Big(\frac{l-1}{n}\Big)^{q-1}\frac{1}{n}\\&= \sum_{l=1}^{n}\int_{(l-1)/n}^{l/n}\Big\{x^{q-1}-\Big(\frac{l-1}{n}\Big)^{q-1}\Big\}\mathrm{d}x > 0.\end{aligned}$$

We also notice that

$$\| J_r f(L_T) \|_{L^2}^2 = \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}_+^\Gamma \text{with} \\ \sum_{\gamma\in\Gamma} k_\gamma = r}} \Big\{ \prod_{\gamma\in\Gamma} \frac{(\lambda_\gamma T)^{k_\gamma}}{k_\gamma !} \Big\} \mathbf{E}[\, f(L_T) \prod_{\gamma} \vartheta^{*k_\gamma}_{T,x_\gamma} 1 \,]^2.$$

Finally, since $I_{n,q} \leq \dfrac{1}{q}$ and

$$I_{n,q} \leq \sum_{l=1}^{n} \Big\{ \Big(\frac{l}{n}\Big)^{q-1} - \Big(\frac{l-1}{n}\Big)^{q-1} \Big\} \frac{1}{n} = \frac{1}{n},$$

we have

$$I_{n,q} = (I_{n,q})^{1-s} (I_{n,q})^s \leq q^{s-1} n^{-s}.$$

Hence we have

$$\| \mathrm{Err}_n \|_{L^2}^2 \leq n^{-s} \sum_{q=2}^{\infty} q^s \| J_q f(L_T) \|_{L^2}^2 \leq n^{-s} \| f(L_T) \|_{\mathbb{D}_{2,s}}^2.$$

□

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