# A Study on Fermionic Wiener Functionals via Stochastic Areas and its Applications

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# Contents

**Abstract**

This thesis is organized into four sections.

In the first section, a probabilistic representation of the tau functions of KP (Kadomtsev-Petviashvili) solitons in terms of stochastic areas will be presented.

The second section is composed of a remark that Quadratic Gaussian term structures under suitable scale change are of multi-soliton type, (a collection of) solitary waves that are related to KdV solitons. In the real market, the term structure of forward rates exhibits some humps. The quadratic Gaussian term structure models or affine term structure models well explain this phenomena.

The third section presents a probabilistic "Bosonization" using stochastic areas. With the Bosonization, the "Fermions", which are anti-symmetric stochastic integrals generated by a representation of a Clifford algebra in Wiener space, are sent to the character polynomials. This Bosonization enables us to construct a probabilistic representation of tau functions of integrable systems.

In the last section, we present two equivalences in law among stochastic areas. The first one states that the law of a stochastic area at a fix time of a Gaussian process(stochastic integral of a deterministic $L^2$ function) is only dependent on its $L^2$-norm. The second one is on the pair of (generalized) stochastic areas. It says that the law of a pair related to Walsh system is again independent of the choice of the Walsh function.

# 1 Tau functions of KP solitons realized in Wiener space

## 1.1 Introduction

In the introduction, after giving a very short introduction to the theory of solitons following [24], we recall some existing results from probabilistic approaches.

### 1.1.1 Solitons, tau-functions, and Sato's Grassmannian

By *solitons*, we usually mean *solitary wave* solutions (behaving like a particle) to a class of non-linear wave equations including the KdV (Korteweg-de Vries) equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{2}u\frac{\partial u}{\partial x} \tag{1.1}$$

as the most notable example.

The first giant step in the study of solitons was made by Gardner, Greene, Kruskal and Miura [7], where they observed that (i) the eigenvalues of the Shorödinger operator

$$\frac{\partial^2}{\partial x^2} + u(t,x),$$

where $u$ is a solution to (2.16), are constant in time parameter $t$, and (ii) one can construct a soliton solution to (2.16) by applying the *inverse scattering method*, by which we mean the (mathematical) method to construct (unknown) potentials out of given *scattering data*, which had already been fully developed. The relation is most clearly seen when the potential is *reflectionless* as

$$u(t,x) = 2\frac{d^2}{dx^2}\log\det(I + G(x,t)), \tag{1.2}$$

where

$$G(x,t) := \left(\frac{\sqrt{m_i m_j}e^{(\eta_i+\eta_j)x+(\eta_i^3+\eta_j^3)t}}{\eta_i+\eta_j}\right)_{1\le i,j\le n}.$$

The constants $\eta_j, m_j$, $j = 1, \cdots, n$ are so-called *scattering data.*

The observation (i) together with the awareness of the existence of the infinite invariants in [7] motivated another seminal paper by P. Lax [20], where the KdV equation (2.16) is understood as the compatibility between the two equations:

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} + u(t,x)\right) w(=: Pw) = \kappa w, & (\kappa \text{ is an eigenvalue}) \\ \left(\frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + \frac{3}{4}\frac{\partial u}{\partial x}\right) w(=: Bw) = 0. & \end{cases}$$

This compatibility is rephrased as the celebrated "Lax equation":

$$\frac{\partial P}{\partial t} + [P, B] = 0, \tag{1.3}$$

where the bracket is the commutator; $[P, B] = PB - BP$.

By considering pseudo differential operators such as $\partial^{-n}$ for $n \in \mathbf{N}$ and their infinite series, we have in fact $B = (P^{3/2})_+$, where $(D)_+$ is the differential operator part of the pseudo differential operator $D$. In this Lax form, the existence of the infinite many invariants can be rephrased as

$$\frac{\partial P}{\partial x_k} + [P, (P^{k/2})_+] = 0, \quad k = 1, 3, 5, \cdots, 2n+1, \cdots,$$

where $u \equiv u(x_1, x_3, \cdots, x_{2n+1}, \cdots)$, a function of infinitely many variables. The KdV case (1.3) is retrieved by setting $x_1 = t, x_3 = x$. Each Lax equation generates a non-linear evolution equation with respect to $x_{2k+1}$ since $[P, (P^{k/2})_+]$'s are all multiplication operators. The totality of the generated equations is usually called *KdV hierarchy*.

If we instead start with the operator

$$L = \partial + \sum_{j=1}^{\infty} u_j \partial^{-j},$$

then we still have that $[L, (L^k)_+]$ are all multiplication operators, and hence we obtain infinitely many nonlinear differential equations with respect to $u_j$'s of infinitely many variables $x_1, x_2, \cdots, x_n, \cdots$ by the Lax equations:

$$\frac{\partial L}{\partial x_k} + [L, (L^k)_+] = 0, \quad k = 1, 2, \cdots.$$

The family is called *KP hierarchy* since the KP (Kadomtsev-Petviashvili) equation,

$$\frac{3}{4}\frac{\partial^2 u_1}{\partial x_2^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3} - \frac{3}{2}u_1\frac{\partial u_1}{\partial x_1} - \frac{1}{4}\frac{\partial^3 u_1}{\partial x_1^3}\right),$$

which is easily seen to be a generalization of the KdV to a two dimensional model, is deduced from the equations with $k = 2$ and $k = 3$. The KP hierarchy as a whole is also a generalization of the KdV hierarchy since the latter hierarchy is obtained by a reduction $(L^2)_- = 0$ from the former.

The equations in KP/KdV hierarchy are all "soliton equations" in the sense that they all have exact solutions of soliton type [1]. In fact, according to Sato's theory of infinite dimensional Grassmannian ([30], see also [24,29]), all the $u_j$'s of the hierarchy are simultaneously generated from a single function called *tau-function* $\tau$ in the following way: determine $w_1$, $w_2$, etc, by

$$\frac{\tau(x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, \cdots)}{\tau(x_1, x_2, \cdots)} = 1 + \frac{w_1}{k} + \frac{w_2}{k^2} + \cdots \tag{1.4}$$

by comparing the coefficients of $k^{-j}$, $j = 1, 2, \cdots$, and then $u_1, u_2$, etc by

$$L = \left(1 + \sum_{j=1}^{\infty} w_j \partial^{-j}\right) \circ \partial \circ \left(1 + \sum_{j=1}^{\infty} w_j \partial^{-j}\right)^{-1}. \tag{1.5}$$

For example, we have

$$u_1 = 2\frac{\partial^2}{\partial x_1^2} \log \tau. \tag{1.6}$$

In particular, we see that if $\tau$ is a polynomial of $e^{\sum c_{ij} x_j}$'s, then $u_j$'s are all "solitons" in that they are all rational functions of $e^{\sum c_{ij} x_j}$'s.

The tau functions are characterized as a solution to a family of quadratic differential equations called *Hirota equations*, which are nothing but Plücker relations that define Sato's infinite dimensional Grassmannian. That is to say, a tau function of the KP hierarchy is a point in the Sato's Grassmannian. It should be noted that in the Sato's theory, the KP hierarchy is the most universal one, out of which many well-known soliton equations are obtained by a reduction.

The following functions are known to be among the tau functions of the soliton solution of the KP equation:

$$\tau(x_1, x_2, \cdots) = \sum_{J \subset I} \left(\prod_{i \in J} m_i\right) \left(\prod_{i, i' \in J, i < i'} \frac{(p_i - p_{i'})(q_i - q_{i'})}{(p_i - q_{i'})(q_i - p_{i'})}\right) \exp\left(\sum_{i \in J} \sum_{l=1}^{\infty} (p_i^l - q_i^l) x_l\right)$$

[1] The solitons are basically rational functions of the exponential functions of the form $e^{\sum c_{ij} x_j}$ for some constants $c_{ij}$'s.

for $I = \{1, \cdots, n\}$, $n \in \mathbf{N}$, where $m_1, \cdots, m_n$, $p_1, \cdots, p_n$, and $q_1, \cdots, q_n$ are (indefinite) constants. This is alternatively written as

$$\tau(x_1, x_2, \cdots) = \det(I + G(x_1, x_2, \cdots)), \tag{1.7}$$

where

$$G(x_1, x_2, \cdots) = \left( \frac{\sqrt{m_i m_j}}{p_i - q_j} e^{\frac{1}{2}\sum_{l=1}^{\infty}\{(p_i^l - q_i^l) + (p_j^l - q_j^l)\}x_l} \right)_{1 \le i,j \le n}.$$

The formula (1.7) is a generalization of (1.2) since we retrieve it by (21) and the reductions of $q_j = -p_j$, $x_l = 0$ for $l \ge 4$.

*Remark* 1. It should be noted that, if $f$ is a solution to a Hirota equation then so is $Ce^{\sum_j c_j x_j} f$, for arbitrary constants $C, c_1, c_2, \cdots$. Therefore tau function is stable under the multiplication of the factor $Ce^{\sum_j c_j x_j}$. This property will be used in the proof of Theorem 7.

### 1.1.2 Probabilistic approach to solitons

As far as we know, the first attempt to represent solitons in terms of the expectation of Wiener functionals was made by S. Kotani [17] in 2000. According to [14], Kotani constructed the following correspondences. Let $\Sigma$ be the set of all pairs $(\sigma_+, \sigma_-) \equiv \sigma$ of non-negative measures each on $\mathbf{R}_-$ such that $\int_{\mathbf{R}_-} e^{\sqrt{-1}\lambda} \sigma_\pm(d\lambda) < \infty$ for any $\lambda > 0$. For $\sigma \in \Sigma$, associate a Gaussian process $X^\sigma$ with mean 0 whose covariance $C(u, v) = \mathbf{E}[X(u)X(v)]$ is given by

$$\begin{aligned} C(u, v; \sigma) = {} & \frac{1}{4} \int_{\mathbf{R}_-} (-z)^{-1/2} \left( e^{\sqrt{-z}(u+v)} - e^{\sqrt{-z}|u-v|} \right) \sigma_+(dz) \\ & + \frac{1}{4} \int_{\mathbf{R}_-} (-z)^{-1/2} \left( e^{-\sqrt{-z}|u-v|} - e^{-\sqrt{-z}(u+v)} \right) \sigma_-(dz). \end{aligned}$$

Let $\mathcal{Q}$ be the totality of the function $q^\sigma$ with $\sigma \in \Sigma$, where

$$q^\sigma(x) = -4 \frac{\partial^2}{\partial x^2} \log \mathbf{E} \left[ \exp\left( -\frac{1}{2} \int_0^x |X^\sigma(y)|^2 dy \right) \right]. \tag{1.8}$$

Then Kotani showed that $\mathcal{Q}$ is the closure (with respect to the topology of uniform convergence on compacts) of $\cup \mathcal{Q}_n$, where $\mathcal{Q}_n$ be totality of the reflectionless potentials of scattering data consisting of $2n$ constants.

In a somewhat different line, K. Hara and N. Ikeda [11] derived from the Fourier transform of a class of quadratic Wiener functionals a dynamics in the Grassmannian as a finite dimensional analogue to the Sato's framework (1.4)-(1.5) etc.

Soon after that N. Ikeda and S. Taniguchi [14] obtained a specific and more "stochastic analysis oriented" construction of the Gaussian process $X^\sigma$ in (1.8) than Kotani's method. They set

$$X_t^\sigma = \sqrt{a}\langle c, \xi_t^p \rangle, \tag{1.9}$$

where $a > 0$, $c \in \mathbf{R}_+^n$, $p \in \mathbf{R}^n$ and $\xi^p$ is an Ornstein-Uhlenbeck process in $\mathbf{R}^n$ starting at 0 defined as the solution to the following SDE:

$$d\xi_t = dW_t + \mathrm{diag}\{p_1, \cdots, p_n\}\xi_t dt. \tag{1.10}$$

The measure $\sigma$ in Kotani's correspondence is given as

$$\sigma_\pm(du) = 2a^2 \sum_{i:p_i \in \mathbf{R}_\pm} c_i^2 \delta_{-p_i^2}(du).$$

Related studies and surveys concerning the quadratic Wiener functionals can be found in [32–35], and more recently in [15].

Here we remark that all the probabilistic results cited here are on KdV solitons, and not extendable to KP. In this section, we will present a probabilistic representation of KP solitons using generalized *stochastic areas* (see Theorem 2 and Theorem 7).

### 1.1.3 Organization of the present paper

In section 1.2, we will introduce Lévy's stochastic area formula and present its generalization as Theorem 2 and its proof. Then in section 2.5, we will show that the generalized stochastic area formula is parameterized as a tau function of KP solitons. In section 2.6, we will give a probabilistic interpretation of the reduction from KP- to KdV-solitons.

## 1.2 A generalization of Lévy's stochastic area formula

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B \equiv (B^1, B^2)$ be a two-dimensional Brownian motion on it. The area enclosed by the curve $s \mapsto B_s$ and its chord up to time $t$, which is usually called stochastic area of $B$, is given by (1/2 of)

$$S_t := \left( \int_0^t B_s^2 \, dB_s^1 - \int_0^t B_s^1 \, dB_s^2 \right).$$

The characteristic function of $S_t$ is explicitly given as

$$\mathbf{E}[e^{\sqrt{-1}\xi S_t}] = (\cosh \xi t)^{-1} \quad (\xi \in \mathbf{R}), \tag{1.11}$$

and conditioned one is also given explicitly as

$$\mathbf{E}[e^{\sqrt{-1}\xi S_t} | B_t^1 = x, B_t^2 = y] = \frac{\xi t}{\sinh \xi t} e^{\frac{1}{2t}(x^2+y^2)(1-\xi t \coth \xi t)} \quad (\xi \in \mathbf{R}), \tag{1.12}$$

which were found by Paul Lévy [21] using Fourier series expansion of $W$. Either is often called Lévy's (stochastic area) formula(s). There have been plenty of studies related to the formulas. For example, the heat kernel of the Heisenberg group can be obtained by a slight modification of the formula ([3], see also [16]). Many alternative proofs and generalizations have been found ([36], [5], [9], [12], [13], etc).

In this section, we give the following generalization of (1.11). In its proof, the second Lévy formula (1.12) plays a crucial role.

**Theorem 2.** *Let $W^l \equiv (W^{l,1}, W^{l,2})$, $l = 1, \cdots, n$ be mutually independent two-dimensional Brownian motions starting at the origin, and stochastic areas of $W^l$ will be denoted by*

$$S^l := \int_0^1 \left( W_s^{l,2} dW_s^{l,1} - W_s^{l,1} dW_s^{l,2} \right).$$

*Let $\Lambda := \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$, where $\lambda_l, l = 1, 2, \cdots, n$ are positive numbers. Let $A \equiv (a_{i,j})_{1 \le i,j \le n}$ be a real $n \times n$ matrix, and $C^{\pm}$ be its symmetric and skew-symmetric part respectively, namely, $C^{\pm} = (A \pm A^*)/2$. Denote $\mathbf{W}_t^i = (W_t^{1,i}, \cdots, W_t^{n,i})$ for $i = 1, 2$, and define for $\sigma \in \mathbf{C}$*

$$\begin{aligned} \hat{S}(\sigma) &\equiv \hat{S}_{A,\Lambda}(\sigma) \\ &:= \sigma \sum_{l=1}^n \lambda_l S^l + \sigma \langle \Lambda^{\frac{1}{2}} C^- \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^2 \rangle_{\mathbf{R}^n} - \frac{\sigma^2}{2} \sum_{i=1,2} \langle \Lambda^{\frac{1}{2}} C^+ \Lambda^{\frac{1}{2}} \mathbf{W}_1^i, \mathbf{W}_1^i \rangle_{\mathbf{R}^n}. \end{aligned} \tag{1.13}$$

*Then, if both* $\max_l |\lambda_l|$ *and* $\|C^+\|$ *are sufficiently small, we have*

$$\mathbf{E}[e^{\hat{S}(\sqrt{-1})}]$$
$$= \begin{vmatrix} \cosh\lambda_1 + a_{1,1}\sinh\lambda_1 & a_{1,2}\sinh\lambda_2 & \cdots & a_{1,n}\sinh\lambda_n \\ a_{2,1}\sinh\lambda_1 & \cosh\lambda_2 + a_{2,2}\sinh\lambda_2 & \cdots & a_{2,n}\sinh\lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}\sinh\lambda_1 & a_{n,2}\sinh\lambda_2 & \cdots & \cosh\lambda_n + a_{n,n}\sinh\lambda_n \end{vmatrix}^{-1}.$$

*Proof.* We first calculate the conditional expectation of $e^{\hat{S}(\sigma)}$ conditioned by $\mathbf{W}_1 = (\mathbf{W}_1^1, \mathbf{W}_1^2)$ . By the Lévy's formula (1.12) with some analytic continuation, we have for sufficiently small $\sigma \in \mathbf{R}$ (such that the random variable $e^{\sigma \sum \lambda_l S^l}$ is integrable),

$$\begin{aligned} &E[e^{\sigma \sum_l \lambda_l S^l} | \mathbf{W}_1] \\ &= \prod_l \frac{\sigma\lambda_l}{\sin\sigma\lambda_l} \exp\Big( - \frac{(W_1^{l,1})^2 + (W_1^{l,2})^2}{2} (\sigma\lambda_l \cot\sigma\lambda_l - 1)\Big). \end{aligned} \tag{1.14}$$

Therefore we have

$$E[e^{\hat{S}(\sigma)} | \mathbf{W}_1] = \prod_l \frac{\sigma\lambda_l}{\sin\sigma\lambda_l} \exp\Big( - \frac{1}{2}\langle (M(\sigma) - \mathbf{I} + C(\sigma))\mathbf{W}_1, \mathbf{W}_1\rangle\Big)$$

where

$$M(\sigma) = \begin{pmatrix} \sigma\Lambda\cot\sigma\Lambda & 0 \\ 0 & \sigma\Lambda\cot\sigma\Lambda \end{pmatrix},$$

with

$$\cot\sigma\Lambda := \mathrm{diag}\{\cot\sigma\lambda_1, \cdots, \cot\sigma\lambda_n\}$$

as usual, and

$$C(\sigma) := \begin{pmatrix} \sigma^2\Lambda^{\frac{1}{2}}C^+\Lambda^{\frac{1}{2}} & \sigma\Lambda^{\frac{1}{2}}C^-\Lambda^{\frac{1}{2}} \\ -\sigma\Lambda^{\frac{1}{2}}C^-\Lambda^{\frac{1}{2}} & \sigma^2\Lambda^{\frac{1}{2}}C^+\Lambda^{\frac{1}{2}} \end{pmatrix}.$$

Since $\|M(\sigma) + C(\sigma) - I\| \to 0$ as $\sigma \to 0$, we can take $\sigma$ small enough to ensure that $M(\sigma) + C(\sigma)$ is positive definite. Then, applying quadratic Gaussian formula for such $\sigma$, we obtain

$$\mathbf{E}[e^{\hat{S}(\sigma)}] = \prod_{l=1}^n \frac{\sigma\lambda_l}{\sin\sigma\lambda_l} \ \det(M(\sigma) + C(\sigma))^{-\frac{1}{2}}. \tag{1.15}$$

We may go further as

$$
\begin{aligned}
&\det(M(\sigma)+C(\sigma)) \\
&= \det \begin{pmatrix} \sigma\Lambda^{\frac{1}{2}}(\Lambda^{\frac{1}{2}}\cot\sigma\Lambda+\sigma C^{+}\Lambda^{\frac{1}{2}}) & \sigma\Lambda^{\frac{1}{2}}C^{-}\Lambda^{\frac{1}{2}} \\ -\sigma\Lambda^{\frac{1}{2}}C^{-}\Lambda^{\frac{1}{2}} & \sigma\Lambda^{\frac{1}{2}}(\Lambda^{\frac{1}{2}}\cot\sigma\Lambda+\sigma C^{+}\Lambda^{\frac{1}{2}}) \end{pmatrix} \\
&= \det(\sigma\Lambda^{\frac{1}{2}}(\cot\sigma\Lambda+\sigma C^{+}+\sqrt{-1}C^{-})\Lambda^{\frac{1}{2}})\det(\sigma\Lambda^{\frac{1}{2}}(\cot\sigma\Lambda+\sigma C^{+}-\sqrt{-1}C^{-})\Lambda^{\frac{1}{2}}) \\
&\text{(Since } C^{-} \text{ is skew symmetric)} \\
&= \left\{\prod_{l}(\sigma\lambda_l)\det(\cot\sigma\Lambda+\sigma C^{+}+\sqrt{-1}C^{-})\right\}^2.
\end{aligned}
$$

Hence (1.15) is turned into the following equality:

$$
\mathbf{E}[e^{\hat{S}(\sigma)}] = \det\left(\cos\sigma\Lambda+(\sigma C^{+}+\sqrt{-1}C^{-})\sin\sigma\Lambda\right)^{-1} \tag{1.16}
$$

where $\sin\sigma\Lambda := \operatorname{diag}\{\sin\sigma\lambda_1,\cdots,\sin\sigma\lambda_n\}$.

The right-hand-side of (1.16) is meromorphic in $\sigma\in\mathbf{C}$. Now, we want to see if an analytic continuation to a domain including $\sigma=\sqrt{-1}$ is possible or not. To see this, it suffices to check the differentiability of the left-hand-side of (1.16) with respect to $\sigma$. Namely, we need to check the integrability of

$$
\begin{aligned}
&\mathbf{E}[\frac{d}{d\sigma}e^{\hat{S}(\sigma)}] \\
&= \mathbf{E}\left[e^{\hat{S}(\sigma)}\left(\sum_{l=1}^{n}\lambda_l S^l+\langle\Lambda^{\frac{1}{2}}C^{-}\Lambda^{\frac{1}{2}}\mathbf{W}_1^1,\mathbf{W}_1^2\rangle_{\mathbf{R}^n}-\sigma\sum_{i=1,2}\langle\Lambda^{\frac{1}{2}}C^{+}\Lambda^{\frac{1}{2}}\mathbf{W}_1^i,\mathbf{W}_1^i\rangle_{\mathbf{R}^n}\right)\right].
\end{aligned}
$$

Since $\hat{S}$ is quadratic Gaussian, the integrability is inherited from that of $e^{\hat{S}(\sigma)}$ itself, which is guaranteed if either $\max_l|\lambda_l|$ or $\|C^{+}\|$ is sufficiently small. □

## 1.3 Parameterization as a tau function of KP solitons

As we have stated, a tau function $\tau$ of the $n$-soliton solution of the Kadomtsev-Petviashvili equation (KP equation) is expressed by

$$
\tau(x_1,x_2,\cdots)=\det(I+G(x_1,x_2,\cdots)), \tag{1.17}
$$

with

$$
G(x_1,x_2,\cdots)=\left(\frac{\sqrt{m_i m_j}}{p_i-q_j}e^{\frac{1}{2}(\xi_i+\xi_j)}\right)_{1\le i,j\le n},
$$

where

$$\xi_i = (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + \cdots, \quad i = 1, \cdots, n,$$

and $m_i > 0$, $p_i$ and $q_i$ are parameters.

**Theorem 3.** *Let $P = \left(\frac{1}{p_i - q_j}\right)_{1 \le i,j \le n}$, and assume that $\min_{i,j} |p_i - q_j|$ is sufficiently large so that $I + P$ is invertible. Then, if we put $A = (I - P)(I + P)^{-1}$ and $\Lambda := \mathrm{diag}\{-\frac{1}{2}(\xi_1 + \log m_1), \cdots, -\frac{1}{2}(\xi_n + \log m_n)\}$, we have that $(\mathbf{E}[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}])^{-1}$, where $\hat{S}_{A,\Lambda}$ is defined by (1.13), defines a tau function of KP solitons.*

*Proof.* Since

$$G = e^{-\Lambda} P e^{-\Lambda},$$

we have

$$\begin{aligned} \tau &= \det(I + e^{-\Lambda} P e^{-\Lambda}) \\ &= \det e^{-\Lambda} \det(e^{\Lambda} + P e^{-\Lambda}) = \det(I + P e^{-2\Lambda}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\det(\cosh \Lambda + A \sinh \Lambda) \\ &= \det \left( \frac{e^{\Lambda} + e^{-\Lambda}}{2} + A \frac{e^{\Lambda} - e^{-\Lambda}}{2} \right) \\ &= 2^{-n} \det \left\{ (I + A) e^{\Lambda} + (I - A) e^{-\Lambda} \right\} \\ &= 2^{-n} \det\{(I + A) e^{\Lambda}\} \det \left( I + (I + A)^{-1} (I - A) e^{-2\Lambda} \right) \\ &= 2^{-n} \det(I + A) e^{-\frac{1}{2} \sum (\xi_i + \log m_i)} \det \left( I + P e^{-2\Lambda} \right). \end{aligned}$$

The last equality follows since

$$A = (I - P)(I + P)^{-1} \iff P = (I + A)^{-1}(I - A).$$

As we have stated in Remark 1, $2^{-n} \det(I + A) e^{-\frac{1}{2}(\xi_i + \log m_i)}$ is a trivial factor and thus by Theorem 2 we have the assertion. $\square$

## 1.4 Reduction to Ikeda-Taniguchi's construction

As we have discussed in subsection 1.1.1, we have (1.2) by the reduction of $q_j = -p_j$ in (1.7). In this subsection, we review this from the perspective of stochastic analysis. We will show that when $C^- = 0$, the expectation of the exponential of the generalized stochastic area is reduced to that of the exponential of the time integral of an Ornstein-Uhlenbeck process, which corresponds to the Taniguchi-Ikeda's construction (1.8), (1.9) and (1.10) of reflectionless potentials/tau functions of KdV solitons.

Precisely speaking, we have the following

**Proposition 4.** *Suppose that $A$ in Theorem 2 is symmetric. Then*

$$\mathbf{E}[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}] = \left(\mathbf{E}[e^{-\int_0^1 X_s^{A,\Lambda} ds}]\right)^2 e^{\mathrm{tr}\Lambda A},$$

*where $X^{A,\Lambda} = \langle(\Lambda - A\Lambda A)\xi, \xi\rangle$ and $\xi$ is an Ornstein-Uhlenbeck process on $\mathbf{R}^d$ starting at $0$ and satisfying*

$$d\xi_t = \Lambda^{\frac{1}{2}} dB_t + \Lambda A \xi_t \, dt, \tag{1.18}$$

*with $B$ being an $n$-dimensional standard Brownian motion.*

*Proof.* We first note the following identity since its right-hand-side also equals to that of (1.14) with $\sigma$ replaced by $\sqrt{-1}$ (see e.g. [22]):

$$\mathbf{E}[e^{\sqrt{-1}\sum_l \lambda_l S^l} | \mathbf{W}_1] = \mathbf{E}[e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 \{(W_s^{l,1})^2 + (W_s^{l,2})^2\} ds} | \mathbf{W}_1].$$

Then since $C^+ = A$ and $C^- = 0$, we have

$$\begin{aligned}
\mathbf{E}[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}] &= \prod_{i=1,2} \mathbf{E}[e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 (W_s^{l,i})^2 ds + \frac{1}{2}\langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_1^i, \mathbf{W}_1^i\rangle}] \\
&= \left(\mathbf{E}[e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 (W_s^{l,1})^2 ds + \frac{1}{2}\langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^1\rangle}]\right)^2.
\end{aligned}$$

By applying Itô's formula,

$$\begin{aligned}
&e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 (W_s^{l,1})^2 ds + \frac{1}{2}\langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^1\rangle} \\
&= e^{\frac{1}{2}\mathrm{tr}\Lambda A} e^{\int_0^1 \langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1, d\mathbf{W}_s^1\rangle - \frac{1}{2}\int_0^1 |\Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1|^2 ds} e^{-\frac{1}{2}\int_0^1 \langle (\Lambda - A\Lambda A)\Lambda^{\frac{1}{2}} \mathbf{W}_s^1, \Lambda^{\frac{1}{2}} \mathbf{W}_s^1\rangle ds}.
\end{aligned}$$

Define $Q$ by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_1} = e^{\int_0^1 \langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1, d\mathbf{W}_s^1 \rangle - \frac{1}{2}\int_0^1 |\Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1|^2 \, ds}.$$

Then by the Maruyama-Girsanov theorem, we see that $\mathbf{W}$ under $Q$ has the same law as $\xi$ of (2.18). This completes the proof. □

**Remark 1.** *Note that the variable $x$ appearing in (1.8) is suppressed in the integral over $[0,1]$ thanks to the scaling property of Brownian motion.*

**Remark 2.** *We note that the $2n$-dimensional Brownian motion used to represent $n$-solitons in Theorem 2 can be replaced by a 2-dimensional one irrespective of $n$. Let $W \equiv (W^1, W^2)$ be a 2-dimensional Brownian motion starting at the origin, and set*

$$f_i(t) := \sqrt{n} \sum_{l=1}^{n} \delta_l^i 1_{[\frac{l-1}{n}, \frac{l}{n})}(t), \quad i = 1, 2, \cdots, n,$$

*where*

$$\delta_j^i = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise} \end{cases}.$$

*Define*

$$S_{i,j}^+ := \sum_{a=1,2} \left( \int_0^1 f_i(t) \, dW_t^a \right) \left( \int_0^1 f_j(t) \, dW_t^a \right),$$

*and*

$$S_{i,j}^- := \int_0^1 \left( \int_0^t f_j(s) \, dW_s^2 \right) f_i(t) \, dW_t^1 - \int_0^1 \left( \int_0^t f_i(s) \, dW_s^1 \right) f_j(t) \, dW_t^2.$$

*We assume that $\lambda_i > 0$ for all $i$. We shall denote the $(i,j)$ entry of the matrices $\Lambda^{\frac{1}{2}} C^+ \Lambda^{\frac{1}{2}}$ and $\Lambda^{\frac{1}{2}}(I + C^-)\Lambda^{\frac{1}{2}}$ by $\lambda_{i,j}^+$, and $\lambda_{i,j}^-$, respectively. Note that $\lambda_{ii}^- = \lambda_i$. We also assume that either $\max_l |\lambda_l|$ or $\|C^+\|$ is sufficiently small to ensure the integrability. Then we have that*

$$\begin{aligned} \mathbf{E}[e^{\sum_{i,j}\left(\sqrt{-1}\lambda_{i,j}^- S_{i,j}^- + \frac{1}{2}\lambda_{i,j}^+ S_{i,j}^+\right)}] (= \mathbf{E}[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}]) \\ = \det(\cosh \Lambda + A \sinh \Lambda)^{-1}. \end{aligned}$$

*With this identification, it would be possible to obtain another class of $\tau$-function to KP hierarchy by letting $n \to \infty$ as in [22].*

# 2 Affine term structure as multi-soliton

## 2.1 Introduction

The spot interest rate $r(t,T)$ is the rate per unit of time (normally it is one year) at which one can (in practice, the rate can vary depending on who they are and how it is agreed but we ignore such credit risks/counter party risks here) borrow (lend) cash at time $t$ and repay (be repaid) at time $T$. Theoretically it is related to the price $P(t,T)$ of the zero-coupon bond maturing at $T$ as

$$r(t,T) = -\frac{1}{T-t}\log P(t,T).$$

In practice, the rate so defined is called *zero rate*. The function

$$T \mapsto r(t,T)$$

is what we call *term structure of spot rates*, or in practice it is rather function in $x = T - t$;

$$x \mapsto r(t, t+x),$$

which is often referred to as *yield curve*.

In theoretical finance, one rather work on the term structure of (the instantaneous) forward rates, which is given by

$$T \mapsto f(t,T) = -\partial_T \log P(t,T),$$

or

$$x \mapsto f(t,t+x) = -\partial_T \log P(t,T)\Big|_{T=t+x}.$$

This is because the forward rate is easier to handle *mathematically*. In particular to impose arbitrage-free property to the term structure.

In real market, however, the term structure of spot rates behaves nicer. According to the series of studies by N.L.Liu and her collaborators [25–27], from the term structure of spot rates only two or three factors up to almost 99% are detected when applied a principal component analysis (or its variants), while that of forward rates exhibits more than 10, sometimes 15, or even more factors. Much more straightforward peculiarity is that the samples of the term structure of forward rates often have more *humps* than those of spot rates.

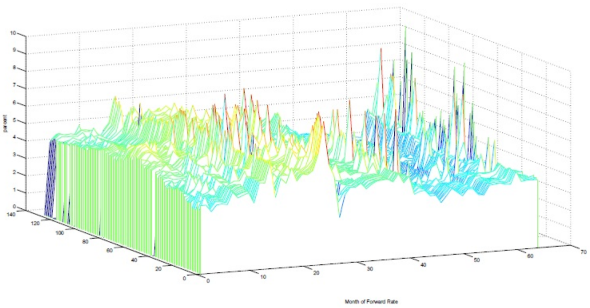



Figure 1: Typical forward rate movement: EU zero rate

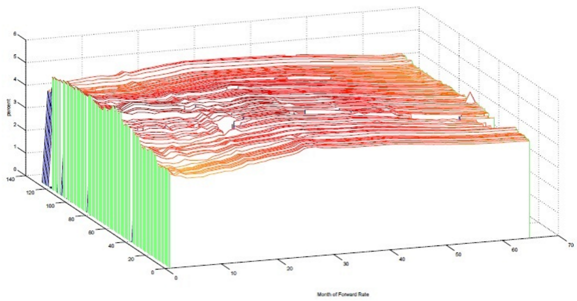



Figure 2: Spot rate movement of the same data as Fig. 1

The main aim of the present paper is to propose a new point of view where the humps are understood as a kind of *solitons*.

The rest of the paper is organized as follows. In section 2.2, we illustrate our idea by a primitive one dimensional example. In section 2.2.1, we present a brief introduction to *solitons*. In section 2.3, we give a multi-dimensional version of the observation made in section 2.2. We emphasize that a class of affine (quadratic Gaussian) models exhibits multi-soliton shape term structures. Finally in section 2.4, we remark that the solitons appearing in the term structure models are related to a non-linear partial differential equation called *KdV equations*.

## 2.2 A primitive example

To explain the idea, we start with a primitive example. Let

$$P(t,T) = Ex[\exp\{-\frac{1}{2}\int_t^T c^2|W_s|^2\,ds\}|W_t], \quad 0 \le t \le T, \tag{2.1}$$

where $W$ is a 1-dimensional Brownian motion. This formula defines an arbitrage-free bond market, which is a simplest example of the quadratic Gaussian model, and at the same time, an affine term structure model (see e.g.[8]) where we consider $|W|^2$ to be a state variable. In fact, we have an explicit expression as

$$\begin{aligned} P(t,T) &= \{\cosh(c(T-t))\}^{-1/2} \\ &\quad \times \exp\{-\frac{c}{2}\tanh(c(T-t))|W_t|^2\}, \end{aligned}$$

and the (instantaneous) forward rate $f(t,T) = -\partial_T \log P(t,T)$ is then expressed as

$$\begin{aligned} f(t,T) &= \frac{c}{2}\tanh(c(T-t)) \\ &\quad + \frac{c^2|W_t|^2}{2}\operatorname{sech}^2(c(T-t)), \end{aligned} \tag{2.2}$$

which is an affine function in the state variable.

By (2.1), we know that

$$T \mapsto -\log P(t,T)$$

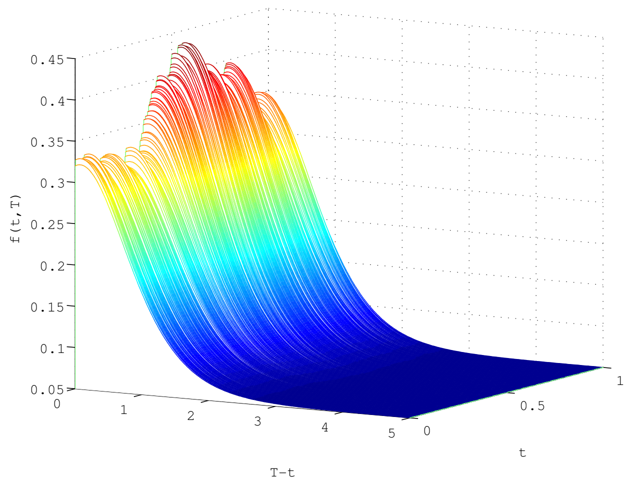



Figure 3: A sample path of the forward rate given by (2.2) with $W_0 = 8$, $c = 0.1$.

is increasing, and therefore the term structure of spot rates under this model behaves nicely, while one notices that

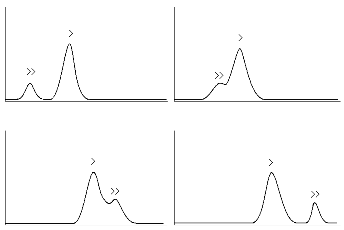
$$T \mapsto f(t, T)$$

is a rational function of $e^{c(T-t)}$ and $e^{-c(T-t)}$, which is, what we will call in local terminology, a *soliton*.

Fig. 4 exemplifies a sample path of the affine forward rate.

### 2.2.1 Solitons

In general, a traveling wave solution to a non-linear (evolution-type) differential equation is not stable; it collapses from the top. The soliton solutions are exceptions. They have (sometimes more than two) solitary waves=humps, and the humps are quite stable even after the "collisions". Somehow they behave like particles, and that is why they are called "solitons".



Mathematically, solitons can be defined as some rational functions of exponentials (see [10]). More precisely, it is something like

$$u(t,x) = \frac{f}{g} = \frac{\sum_i K_i e^{A_i t - B_i x}}{\sum_i L_i e^{C_i t - D_i x}}, \tag{2.3}$$

where $A_i, B_i, C_i, D_i, K_i$ and $L_i$ are constants, and the summations are finite ones. Here we assume $\max_i C_i \geq \max_i A_i$ and $\min_i C_i \leq \min_i A_i$ to ensure the existence of the limits at $x \to \pm\infty$. If we require the inequality to be strict, then the graph $x \mapsto u(t,x)$ is hump-shaped. Note that solitons of this definition are stable under summation, multiplication, and differentiations. Note that the forward rate (2.2) in the previous section is a soliton in $T$ or $x = T - t$ in this sense.

## 2.3 Affine Term Structure as Multi-Soliton

We generalize the observation made in section 2.2. Let $W = (W^1, \cdots, W^n)$ be an $n$-dimensional Brownian motion starting at $x = (x^1, \cdots x^n) \in \mathbf{R}^n$, defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with for each $\lambda_i \in \mathbf{R}$ $(i = 1, 2, \cdots, n)$, and $C \in M(n)$ be a positive definite matrix.

Let

$$\begin{aligned} P(t,T) &:= e^{\langle CW_t, W_t \rangle} \\ &\times E_x \left[ \exp\left\{ -\frac{1}{2} \int_t^T |\Lambda W_s|^2 \, ds - \langle CW_T, W_T \rangle \right\} |W_t \right]. \end{aligned} \tag{2.4}$$

Then $\{P(\cdot, T)\}$ defines an arbitrage-free bond market with

$$\pi_t = \exp\left\{ -\frac{1}{2} \int_0^t |\Lambda W_s|^2 \, ds - \langle CW_t, W_t \rangle \right\}.$$

being a state price density.

**Proposition 5.** *Under the model (2.4), the forward rate is an $n$-soliton; a rational function in $e^{\pm(T-t)\lambda_i}$, $i = 1, \cdots, n$, of degree at most $2n$ for any state $W_t$.*

*Proof.* Let

$$\begin{aligned} K(t) &= -\cosh(t\Lambda)C - \frac{1}{2}\Lambda\sinh(t\Lambda), \\ L(t) &= 2\sinh(t\Lambda)\Lambda^{-1}C + \cosh(t\Lambda), \end{aligned} \tag{2.5}$$

and

$$H(t) = K(t) \cdot L(t)^{-1}. \tag{2.6}$$

Note that

$$K'(t) = -\frac{1}{2}\Lambda^2 L(t) \tag{2.7}$$

and

$$L'(t) = -2K(t). \tag{2.8}$$

We will show that

$$\begin{aligned} &P(t, T) \\ &= \{\det(L(T-t))\}^{-1/2}\exp\langle(H(T-t) + C)W_t, W_t\rangle\}. \end{aligned} \tag{2.9}$$

By the Feynman-Kac formula,

$$\begin{aligned} &u(t, x) := \\ &E\left[\exp\left\{-\frac{1}{2}\int_0^t |\Lambda W_s|^2\, ds - \langle CW_t, W_t\rangle\right\} |W_0 = x\right], \end{aligned}$$

where $x = (x^1, \cdots x^n)$, satisfies the following differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u - \dfrac{1}{2}\langle\Lambda^2 x, x\rangle u, \\ u(0, x) = e^{-\langle Cx, x\rangle}, \end{cases} \tag{2.10}$$

where $\Delta$ is the Laplacian. Note that

$$P(t, T) = e^{\langle CW_t, W_t\rangle} u(T-t, W_t). \tag{2.11}$$

It is well-recognized that the solution $u$ to (2.10) is expressed by

$$\exp\Big(H_0(t) + \langle H(t)x, x\rangle\Big), \tag{2.12}$$

where $H$ is a symmetric-matrix valued differentiable function satisfying

$$\frac{dH}{dt}(t) = 2H(t)^2 - \frac{1}{2}\Lambda^2, \quad H(0) = -C, \tag{2.13}$$

and $H_0$ is given by

$$\frac{dH_0}{dt}(t) = \mathrm{tr} H(t), \quad H_0(0) = 0. \tag{2.14}$$

Now we see that $H$ given by (2.5) and (2.6) is the unique solution to (2.13). In fact, by (2.7) and (2.8), we have

$$\begin{aligned} H' &= (KL^{-1})' = -KL^{-1}L'L^{-1} + K'L^{-1} \\ &= 2(KL^{-1})^2 - \frac{1}{2}\Lambda^2 \\ &= 2H^2 - \frac{1}{2}\Lambda^2, \end{aligned}$$

and also $L(0) = I$ and $K(0) = -C$, which imply $H(0) = -C$.

Further, by (2.14),

$$\begin{aligned} e^{H_0(t)} &= e^{\mathrm{tr}\{-\frac{1}{2}\int_0^t L'(s)L(s)^{-1}\,ds\}} \\ &= \det\{e^{-\frac{1}{2}\int_0^t L'(s)L(s)^{-1}\,ds}\} \\ &= (\det\{e^{\int_0^t L'(s)L(s)^{-1}\,ds}\})^{-1/2}. \end{aligned}$$

Since

$$(e^{\int_0^t L'(s)L(s)^{-1}\,ds})' = L'(t)L(t)^{-1}e^{\int_0^t L'(s)L(s)^{-1}\,ds},$$

we see, by the uniqueness of the matrix-valued first order linear differential equation, that

$$L(t) = e^{\int_0^t L'(s)L(s)^{-1}\,ds}.$$

Thus we have confirmed (2.9), at the same time (2.11) with (2.12), by which we have

$$\begin{aligned} f(t,T) = &-\frac{\partial}{\partial T}H_0(T-t) \\ &+ \frac{\partial}{\partial T}\langle H(T-t)W_t, W_t\rangle. \end{aligned}$$

Then, by substituting (2.13) and (2.14), we get

$$\begin{aligned} f(t,T) &= -\mathrm{tr}H(T-t) \\ &\quad - \frac{1}{2}\langle (4H(T-t)^2 - \Lambda^2)W_t, W_t\rangle. \end{aligned} \tag{2.15}$$

We note that the $(i,j)$-th entries $k_{ij}$ and $l_{ij}$ of $K(t)$ and $L(t)$ are given by

$$k_{ij} = -\cosh(t\lambda_i)c_{ij} - \frac{1}{2}\delta_{ij}\sinh(t\lambda_i),$$

and

$$l_{ij} = 2\sinh(t\lambda_i)\lambda_i^{-1}c_{ij} + \delta_{ij}\cosh(t\lambda_i),$$

and thus they are polynomials in $e^{\pm t\lambda_i}$. Since

$$H(t) = K(t)L(t)^{-1} = K(t)\tilde{L}(t)(\det(L(t)))^{-1},$$

where $\tilde{L}(t)$ is the cofactor matrix of $L(t)$, we see that each entry of $H(t)$ is a rational function in $e^{\pm t\lambda_i}$, $i = 1, \cdots, n$, with degree $n$. Hence, by the expression (2.15), we have the assertion. □

*Remark* 6. It is known that the forward rates stay positive if $\pi$ is a strict supermartingale. In fact, for $T_1 > T_2$ we have

$$Ex[\pi_{T_1}|\mathcal{F}_t] < Ex[\pi_{T_2}|\mathcal{F}_t]$$

by the supermartingale property of $\pi$, and the formula reads

$$P(t,T_1) = \frac{Ex[\pi_{T_1}|\mathcal{F}_t]}{\pi_t} \leq \frac{Ex[\pi_{T_2}|\mathcal{F}_t]}{\pi_t} = P(t,T_2),$$

meaning that $P(t,\cdot)$ and hence $\log P(t,\cdot)$ is decreasing. This in turn implies that $f(t,T) = -\partial_T \log P(t,T)$ is positive.

We give a sufficient condition that ensures the positivity. Since

$$\begin{aligned} &d\pi_t \\ &= \pi_t(-d\langle CW_t, W_t\rangle - \frac{1}{2}|\Lambda W_t|^2\,dt + \frac{1}{2}d[\langle CW_t, W_t\rangle]_t) \\ &= -2\langle CW_t, dW_t\rangle \\ &\qquad - \mathrm{tr}C dt - \frac{1}{2}|\Lambda W_t|^2\,dt + \frac{2^2}{2}|CW_t|^2\,dt, \end{aligned}$$

we see that $\pi$ is a supermartingale, and hence the forward rates stay positive, if

$$\Lambda^2 - 4C^2 > 0$$

since $C > 0$ is already assumed.

## 2.4 Remarks on a relation with KdV equation

Let $\tilde{f}(t,T) := f(\frac{c^2}{2^4}t, \frac{1}{2^2}T)$. Then, we have

$$\begin{aligned}\tilde{f}(t,T) &= \frac{c}{2^3}\tanh\Big(\frac{1}{2}(\frac{c}{2}T - \frac{c^3}{2^3}t)\Big) \\ &\quad + \frac{c^2|W_t|^2}{2^3}\mathrm{sech}^2\Big(\frac{1}{2}(\frac{c}{2}T - \frac{c^3}{2^3}t)\Big) \\ &=: v(t,T) + |W_t|^2 u(t,T).\end{aligned}$$

By this scale change, the functions $u$ and $v$ satisfy $4\frac{\partial v}{\partial T} = u$ and

$$\frac{\partial u}{\partial t} = -6u\frac{\partial u}{\partial T} - \frac{\partial^3 u}{\partial T^3}. \tag{2.16}$$

The equation (2.16) is known as the Korteweg-de Vries equation (KdV equation for short), which describes waves on shallow water surfaces. The KdV equation is mathematically as well as physically quite important in that there are many infinite dimensional symmetries which allow it to have great many explicit solutions including elliptic ones, rational ones, and most importantly in our context, *soliton* ones.

The relation has been extensively studied, especially by N. Ikeda and S. Taniguchi [14,15,33–35]. An extended relation to KP solitons using stochastic areas is given in [1].

## 2.5 Parameterization as a tau function of KP solitons

As we have stated, a tau function $\tau$ of the $n$-soliton solution of the Kadomtsev-Petviashvili equation (KP equation) is expressed by

$$\tau(x_1, x_2, \cdots) = \det(I + G(x_1, x_2, \cdots)), \tag{2.17}$$

with

$$G(x_1, x_2, \cdots) = \left(\frac{\sqrt{m_i m_j}}{p_i - q_j} e^{\frac{1}{2}(\xi_i + \xi_j)}\right)_{1 \le i,j \le n},$$

where

$$\xi_i = (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + \cdots, \quad i = 1, \cdots, n,$$

and $m_i > 0$, $p_i$ and $q_i$ are parameters.

**Theorem 7.** *Let* $P = \left(\frac{1}{p_i - q_j}\right)_{1\le i,j\le n}$*, and assume that* $\min_{i,j}|p_i - q_j|$ *is sufficiently large so that* $I+P$ *is invertible. Then, if we put* $A = (I-P)(I+P)^{-1}$ *and* $\Lambda := \mathrm{diag}\{-\frac{1}{2}(\xi_1 + \log m_1), \cdots, -\frac{1}{2}(\xi_n + \log m_n)\}$*, we have that* $(\mathbf{E}[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}])^{-1}$*, where* $\hat{S}_{A,\Lambda}$ *is defined by (1.13), defines a tau function of KP solitons.*

*Proof.* Since

$$G = e^{-\Lambda} P e^{-\Lambda},$$

we have

$$\begin{aligned}\tau &= \det(I + e^{-\Lambda} P e^{-\Lambda}) \\ &= \det e^{-\Lambda} \det(e^{\Lambda} + P e^{-\Lambda}) = \det(I + P e^{-2\Lambda}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\det(\cosh\Lambda + A\sinh\Lambda) \\ &= \det\left(\frac{e^{\Lambda}+e^{-\Lambda}}{2} + A\frac{e^{\Lambda}-e^{-\Lambda}}{2}\right) \\ &= 2^{-n}\det\left\{(I+A)e^{\Lambda} + (I-A)e^{-\Lambda}\right\} \\ &= 2^{-n}\det\{(I+A)e^{\Lambda}\}\det\left(I + (I+A)^{-1}(I-A)e^{-2\Lambda})\right) \\ &= 2^{-n}\det(I+A)e^{-\frac{1}{2}\sum(\xi_i + \log m_i)}\det\left(I + Pe^{-2\Lambda}\right).\end{aligned}$$

The last equality follows since

$$A = (I-P)(I+P)^{-1} \iff P = (I+A)^{-1}(I-A).$$

As we have stated in Remark 1, $2^{-n}\det(I+A)e^{-\frac{1}{2}(\xi_i + \log m_i)}$ is a trivial factor and thus by Theorem 2 we have the assertion. □

## 2.6 Reduction to Ikeda-Taniguchi's construction

As we have discussed in section 1.1.1, we have (1.2) by the reduction of $q_j = -p_j$ in (1.7). In this section, we review this from the perspective of stochastic analysis. We will show that when $C^- = 0$, the expectation of the exponential of the generalized stochastic area is reduced to that of the exponential of the time integral of an Ornstein-Uhlenbeck process, which corresponds to the Taniguchi-Ikeda's construction (1.8), (1.9) and (1.10) of reflectionless potentials/tau functions of KdV solitons.

Precisely speaking, we have the following

**Proposition 8.** *Suppose that $A$ in Theorem 2 is symmetric. Then*

$$\mathbf{E}[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}] = \left(\mathbf{E}[e^{-\int_0^1 X_s^{A,\Lambda} ds}]\right)^2 e^{\mathrm{tr}\Lambda A},$$

*where $X^{A,\Lambda} = \langle(\Lambda - A\Lambda A)\xi, \xi\rangle$ and $\xi$ is an Ornstein-Uhlenbeck process on $\mathbf{R}^d$ starting at $0$ and satisfying*

$$d\xi_t = \Lambda^{\frac{1}{2}} dB_t + \Lambda A \xi_t \, dt, \tag{2.18}$$

*with $B$ being an $n$-dimensional standard Brownian motion.*

*Proof.* We first note the following identity since its right-hand-side also equals to that of (1.14) with $\sigma$ replaced by $\sqrt{-1}$ (see e.g. [22]):

$$\mathbf{E}[e^{\sqrt{-1}\sum_l \lambda_l S^l} | \mathbf{W}_1] = \mathbf{E}[e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 \{(W_s^{l,1})^2 + (W_s^{l,2})^2\} ds} | \mathbf{W}_1].$$

Then since $C^+ = A$ and $C^- = 0$, we have

$$\begin{aligned}\mathbf{E}[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}] &= \prod_{i=1,2} \mathbf{E}[e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 (W_s^{l,i})^2 ds + \frac{1}{2}\langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_1^i, \mathbf{W}_1^i\rangle}] \\ &= \left(\mathbf{E}[e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 (W_s^{l,1})^2 ds + \frac{1}{2}\langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^1\rangle}]\right)^2.\end{aligned}$$

By applying Itô's formula,

$$\begin{aligned}&e^{-\sum_l \frac{\lambda_l^2}{2}\int_0^1 (W_s^{l,1})^2 ds + \frac{1}{2}\langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^1\rangle} \\ &= e^{\frac{1}{2}\mathrm{tr}\Lambda A} e^{\int_0^1 \langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1, d\mathbf{W}_s^1\rangle - \frac{1}{2}\int_0^1 |\Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1|^2 ds} e^{-\frac{1}{2}\int_0^1 \langle (\Lambda - A\Lambda A)\Lambda^{\frac{1}{2}} \mathbf{W}_s^1, \Lambda^{\frac{1}{2}} \mathbf{W}_s^1\rangle ds}.\end{aligned}$$

Define $Q$ by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_1} = e^{\int_0^1 \langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1, d\mathbf{W}_s^1\rangle - \frac{1}{2}\int_0^1 |\Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1|^2 ds}.$$

Then by the Maruyama-Girsanov theorem, we see that $\mathbf{W}$ under $Q$ has the same law as $\xi$ of (2.18). This completes the proof. $\square$

**Remark 3.** *Note that the variable $x$ appearing in (1.8) is suppressed in the integral over $[0, 1]$ thanks to the scaling property of Brownian motion.*

**Remark 4.** *We note that the $2n$-dimensional Brownian motion used to represent $n$-solitons in Theorem 2 can be replaced by a 2-dimensional one irrespective of $n$. Let $W \equiv (W^1, W^2)$ be a 2-dimensional Brownian motion starting at the origin, and set*

$$f_i(t) := \sqrt{n} \sum_{l=1}^{n} \delta_l^i 1_{[\frac{l-1}{n}, \frac{l}{n})}(t), \quad i = 1, 2, \cdots, n,$$

*where*

$$\delta_j^i = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise} \end{cases}.$$

*Define*

$$S_{i,j}^+ := \sum_{a=1,2} \left( \int_0^1 f_i(t)\, dW_t^a \right) \left( \int_0^1 f_j(t)\, dW_t^a \right),$$

*and*

$$S_{i,j}^- := \int_0^1 \left( \int_0^t f_j(s)\, dW_s^2 \right) f_i(t)\, dW_t^1 - \int_0^1 \left( \int_0^t f_i(s)\, dW_s^1 \right) f_j(t)\, dW_t^2.$$

*We assume that $\lambda_i > 0$ for all $i$. We shall denote the $(i, j)$ entry of the matrices $\Lambda^{\frac{1}{2}} C^+ \Lambda^{\frac{1}{2}}$ and $\Lambda^{\frac{1}{2}} (I + C^-) \Lambda^{\frac{1}{2}}$ by $\lambda_{i,j}^+$, and $\lambda_{i,j}^-$, respectively. Note that $\lambda_{ii}^- = \lambda_i$. We also assume that either $\max_l |\lambda_l|$ or $\|C^+\|$ is sufficiently small to ensure the integrability. Then we have that*

$$\begin{aligned} E_x[e^{\sum_{i,j} \left( \sqrt{-1} \lambda_{i,j}^- S_{i,j}^- + \frac{1}{2} \lambda_{i,j}^+ S_{i,j}^+ \right)}] (= E_x[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}]) \\ = \det(\cosh \Lambda + A \sinh \Lambda)^{-1}. \end{aligned}$$

*With this identification, it would be possible to obtain another class of $\tau$-function to KP hierarchy by letting $n \to \infty$ as in* [22].

# 3 Wiener Functionals as Fermions and their Bosonization via Stochastic Areas

## 3.1 Introduction

### 3.1.1 What is done in this section?

As is well-known, the Wiener chaos expansion induces a representation of the Heisenberg algebra, which fact is a keynote of the Malliavin calculus. The fact that the expansion also induces a representation of Clifford algebra is, as P. A. Mayer pointed out in his book [23], *a fact whose significance is not generally appreciated* by probabilists.

In [2], some of the properties of the representation are studied. This section is a continuation of [2], concentrating on *Bosonization*(s) of the representation. As a main result, Theorem 17 presents a Bosonization that is *probabilistic* in that the map is given by an "integral operator" whose kernel is given in terms of stochastic areas: roughly, the result is illustrated as

$$\begin{aligned}&\int e^{\sum x_i \times \text{stochastic areas}}(\text{a } \textit{Fermion} \text{ in Wiener space}) d\mu \\ &= (\text{the corresponding Boson; a character polynomial in } (x_1, \cdots)),\end{aligned}$$

where $\mu$ is the two-dimensional Wiener measure and the "stochastic areas" are namely the areas drawn by transformed paths.

The following three observations are the keys to Theorem 17.

1. The representation is unitary (Theorem 12 and Theorem 15) and therefore the "vacuum expectation value" becomes the standard expectation in Wiener space (Theorem 14).

2. A Fermionic stochastic integral (multi-order stochastic area) decomposes into "Pfaffian" of (second order) stochastic areas (Lemma 18, a result in [2]), and among the Pfaffian expression, the charge-zero part reduces to "determinant" (Lemma 19).

3. In the representation, the products of second order fermions again become orthogonal to each other (Lemma 20).

#### 3.1.2 Why a probabilistic Bosonization is important?

A motivation of the series of studies [1], [2] (and this paper) lies in a probabilistic representation of "tau-functions"[2]. There has been a strong belief among (a part of) probabilists that there are (hidden) beautiful probabilistic interpretations to special functions such as zeta-, theta-, and tau- functions, and our motivation is in line with these.

[2]For a detailed introduction to tau-functions, see [1].

According to the results by Sato's school (see [24] and the references therein), we have roughly

$$\{\text{tau functions}\} = \text{Bosonization of } \exp\{\text{ quadratic forms of Fermions that form an } \infty\text{-dimensional Lie algebra}\}.$$

Since we have already at hand fermioninc Wiener functionals, our Bosonization gives a totally probabilistic representation of tau functions.

### 3.1.3 The Organization of the Present Section

Section 3.2 is devoted to a survey on an abstract theory of *Fermions* and *Bosons*, or (the) *Clifford algebra* and (the) *Heisenberg algebra*, following mainly [24]. Section 3.3 is the main part. After introducing a representation of Clifford algebra in Wiener space in section 3.3.1, we will show that it is unitary, and then the vacuum expectation value is realized as the standard expectation with respect to the Wiener measure in section 3.3.2. Then in section 3.3.3 we give a first Bosonization, which is rather algebraic than probabilistic. Finally in section 3.3.4 we shall present our main result and its proof, based on several lemmas.

## 3.2 Fermions and Bosons

### 3.2.1 Heisenberg Algebra

Let $C \equiv C[x_1, x_2, \cdots]$ be the space of all polynomials of infinite variables $x_1, x_2, \cdots, x_n, \cdots$. Define $a_n, a_n^* \in \mathrm{End}(C)$, $n \in \mathbf{N}$ by

$$a_n f = \frac{\partial f}{\partial x_n}, \quad a_n^* f = x_n f. \tag{3.1}$$

Then, they satisfy the canonical commutation relations: for $n, m \in \mathbf{N}$,

$$[a_n, a_m] = a_n a_m - a_m a_n = 0,\ [a_n^*, a_m^*] = 0,\ \text{and } [a_n, a_m^*] = \delta_{nm}, \tag{3.2}$$

where $\delta_{nm}$ is Kronecker's delta. Clearly,

$$C = \mathrm{span}\{a_{i_1}^* \cdots a_{i_n}^* 1 : i_1, \cdots, i_n \in \mathbf{N}, n \in \mathbf{Z}_+\}.$$

In general, abstract symbols endowed with a multiplication satisfying the relations (3.2) are called *Bosons*, and the algebra generated by the symbols with (3.2) defining relations is called the *Heisenberg algebra*. The above (3.1) can be understood to be constructing a representation of the Heisenberg algebra, where $C$ is the representation space. Namely the algebra is realized as a subalgebra of $\mathrm{End}(C)$. We may alternatively call the algebra an $\mathcal{H}$-module if we denote the Heisenberg algebra by $\mathcal{H}$.

If there is an element $v$ in a representation space $V$ such that

$$V = (\text{a closure of}) \ \mathrm{span}\{a_{i_1}^* \cdots a_{i_n}^* v : i_1, \cdots, i_n \in \mathbf{N}, n \in \mathbf{Z}_+\},$$

then the space is called a Bosonic Fock space, and in this representation $a_n^*$'s are called *creations* and $a_n$'s are *annihilations*. It is obvious that $C$ is a Bosonic Fock space. A *symmetric Fock space*, which is usually constructed from a separable infinite dimensional Hilbert space $H$ by

$$\overline{\bigoplus_{n=0}^{\infty} H^{n\otimes_{\mathrm{sym}}}}$$

is also a Bosonic Fock space in the above sense.

### 3.2.2 Clifford Algebra

The Clifford algebra $Cl$ is the algebra generated by the symbols $\varphi_n$, $\varphi_n^*$, indexed by half integers $n \in \mathbf{Z} + 1/2$, with defining relations

$$[\varphi_m, \varphi_n]_+ = \varphi_m \varphi_n + \varphi_n \varphi_m = 0, \ [\varphi_m^*, \varphi_n^*]_+ = 0, \ \text{and} \ [\varphi_m^*, \varphi_n]_+ = \delta_{m+n,0}.$$

The generators are called *fermions*, and those with negative index are called *creations* and the others are called *annihilations*. It can be easily checked that

$$\begin{aligned} Cl = \mathrm{span}\{&\varphi_{-i_1} \cdots \varphi_{-i_r} \varphi_{-j_1}^* \cdots \varphi_{-j_m}^* \\ &: 0 < i_r < \cdots < i_1, 0 < j_m < \cdots < j_1 \text{are half integers, and } r, m \in \mathbf{Z}_+\} \end{aligned}$$

as a vector space, and so an irreducible left $Cl$-module, which is called a *Fermionic Fock space*, always takes the form of (a closure of)

$$\begin{aligned} \mathrm{span}\{&\varphi_{-i_1} \cdots \varphi_{-i_r} \varphi_{-j_1}^* \cdots \varphi_{-j_m}^* v \\ &: 0 < i_r < \cdots < i_1, 0 < j_m < \cdots < j_1 \text{are half integers, and } r, n \in \mathbf{Z}_+\} \end{aligned} \tag{3.3}$$

for an element $v$ in the representation space, which is called a *vacuum state*. Similarly, an irreducible right *Cl*-module, which is referred to as a dual Fermionic Fock space, always takes the form of (a closure of)

$$\begin{aligned}&\mathrm{span}\{v\varphi_{j_m}\cdots\varphi_{j_1}\varphi^*_{i_r}\cdots\varphi^*_{i_1}:\\&0<i_r<\cdots<i_1, 0<j_m<\cdots<j_1 \text{are half integers, and } r,m\in\mathbf{Z}_+\}.\end{aligned}$$

An *anti-symmetric Fock space*, which is usually constructed from a separable infinite dimensional Hilbert space $H$ by

$$\overline{\oplus_{n=0}^{\infty}H^{n\wedge}}$$

is also a Fermionic Fock space in the above sense.

### 3.2.3 Vacuum Expectation Value

Let $\mathcal{F}$ be a Fermionic Fock space and $\mathcal{F}^*$ be a dual Fermionic Fock space. For a fixed pair of vacuum states $v^*\in\mathcal{F}^*$ and $v\in\mathcal{F}$, define a bilinear form $\langle\cdot|\cdot\rangle:\mathcal{F}^*\times\mathcal{F}\to\mathbf{C}$ by $\langle v^*|v\rangle=1$ and

$$\begin{aligned}&\langle v^*\varphi_{j'_{m'}}\cdots\varphi_{j'_1}\varphi^*_{i'_{r'}}\cdots\varphi^*_{i'_1}|\varphi_{-i_1}\cdots\varphi_{-i_r}\varphi^*_{-j_1}\cdots\varphi^*_{-j_m}v\rangle\\&=\begin{cases}\delta_{i'_1-i_1,0}\cdots\delta_{i'_r-i_r,0}\delta_{j'_1-j_1,0}\cdots\delta_{j'_m-j_m,0} & r'=r \text{ and } m'=m\\0 & \text{otherwise.}\end{cases}\end{aligned}$$

The map is called a *vacuum expectation value*. We note that

$$\langle v^*a|bv\rangle=\langle v^*|abv\rangle=\langle v^*ab|v\rangle. \tag{3.4}$$

The vacuum states are often denoted by vac and an element of $\mathcal{F}$ is by $u|\mathrm{vac}\rangle$ or simply by $|u\rangle$ for $u\in Cl$. We also note that by (3.4), the expectation can be denoted by (and understood as) $\langle\mathrm{vac}|a|\mathrm{vac}\rangle$ or simply $\langle a\rangle$ for $a\in Cl$.

### 3.2.4 Bosonization

A representation of the Heisenberg algebra in a Fermionic Fock space can be constructed as follows. Let

$$H_n:=\sum_{j\in\mathbf{Z}_++1/2}:\varphi_{-j}\varphi^*_{j+n}:,\quad n\in\mathbf{Z},$$

where $:\ \cdot\ :$ is the operation called *normally ordered product*: the axioms are:

1. $: a :$ is linear in $a$, and all the Fermions within the colons anticommute,

2. $: 1 := 1 \in \mathbf{C}$, and

$$\begin{cases} : a\varphi :=: a : \varphi & \text{for } \varphi \text{ an annihilation,} \\ : \varphi a := \varphi : a : & \text{for } \varphi \text{ a creation.} \end{cases}$$

Note that for each of the basis in the expression (3.3) of the Fermionic Fock space only finite terms are acting.

One can prove (see e.g. [24] for details) that

$$[H_m, H_n] = m\delta_{m+n,0}, \quad m, n \in \mathbf{Z}$$

so that

$$a_n := H_n, \quad a_n^* := \frac{1}{n} H_{-n}, \quad n \in \mathbf{N}$$

satisfies the canonical commutation relations (3.2). With these Bosons, we can construct an isomorphism between the Fermionic Fock space and the Bosonic Fock space $C$ introduced in section 3.2.1. Let

$$H(\mathbf{x}) \equiv H(x_1, \cdots, x_n, \cdots) := \sum_{n=1}^{\infty} x_n H_n,$$

and

$$C[z, z^{-1}] := \oplus_{l \in \mathbf{Z}} z^l C.$$

For an integer $l$, define $\langle l| \in \mathcal{F}^*$ by

$$\langle l| = \begin{cases} \langle \mathrm{vac}|\varphi_{1/2} \cdots \varphi_{-l-1/2} & l < 0 \\ \langle \mathrm{vac}| & l = 0 \\ \langle \mathrm{vac}|\varphi_{1/2}^* \cdots \varphi_{l-1/2}^* & l > 0. \end{cases}$$

Then,

**Fact 9** (Bosonization, see Theorem 5.1 in [24]). *The map* $\Phi : \mathcal{F} \to C[z, z^{-1}]$ *defined by*

$$\Phi(|u\rangle) = \sum_{l \in \mathbf{Z}} z^l \langle l| e^{H(\mathbf{x})} |u\rangle, \quad u \in Cl,$$

*is an isomorphism of vector spaces. Moreover, we have*

$$\Phi(H_n|u\rangle) = \begin{cases} \frac{\partial}{\partial x_n} \Phi(|u\rangle) & n > 0 \\ -n x_{-n} \Phi(|u\rangle) & n < 0. \end{cases}$$

### 3.2.5 Young Diagrams

A *Young diagram* is a non-increasing sequence of positive integers $(f_1, f_2, \cdots)$ all but finite members of which is zero. In a pictorial form, a Young diagram is viewed as a figure in the fourth quadrant of the plane, made up of a number of rows of congruent square tiles, with the rows aligned along their left sides, the first row having $f_1$ tiles, the second row $f_2$ tiles, and so on. The only requirement is that the number of tiles in a row does not increase when we move down from one row to the next.

Young diagrams have an alternative description. Suppose that $Y = (f_1, f_2, \cdots, f_r)$ for $r$ in $\mathbf{N}$ is a Young diagram, and let $s$ be the diagonal width of $Y$ when viewed from the top left-hand corner. We write $m_1 > m_2 > \cdots > m_s$ for the number of tiles lying above the NW-SE diagonal line (excluding those straddling the line) in each horizontal row, and $n_1 n_2, \cdots, n_s$ for the number of tiles lying below the diagonal line (excluding those straddling the line) in each vertical column. Then we write $Y = (m_1, m_2, \cdots, m_s | n_1, n_2, \cdots, n_s)$ for the Young diagram.

Using this notation, we define the *character polynomial* of $Y$ to be

$$F_Y(\mathbf{x}) = \det(h_{m_i,n_j}(\mathbf{x})).$$

Here $\mathbf{x} = x_1, x_2, \cdots$, and

$$h_{m,n}(\mathbf{x}) = (-1)^n \sum_{l \geq 0} p_{l+m+1}(\mathbf{x}) p_{n-l}(-\mathbf{x}),$$

where the $p_i(\mathbf{x})$ are defined in $e^{\sum_{j=1}^{\infty} k^j x_j} = \sum_{j=0}^{\infty} p_j(\mathbf{x}) k^j$ with $p_j(\mathbf{x}) = 0$ if $j < 0$. Here $h_{m,n}(\mathbf{x}) = F_{m+1,1^n}(\mathbf{x})$ is the character polynomial corresponding to the hook shaped Young diagram $(m+1, 1^n)$, where $1^n$ stands for the series of $n$ terms $(1, 1, \cdots, 1)$.

The following fact plays an important role in our results.

**Fact 10** (A characterization of bozonization, see Theorem 9.4 in [24])**.** *The bosonization $\Phi$ is characterized by the following property: the basis vector*

$$\varphi_{m_1} \cdots \varphi_{m_r} \varphi^*_{n_1} \cdots \varphi^*_{n_r} |vac\rangle,$$

*for $m_1 < m_2 < \cdots < m_r < 0$ and $n_1 < n_2 < \cdots < n_r < 0$ of the Fock space of charge 0 goes over into the character polynomial of the Young diagram of the form*

$$Y = \Big( -m_1 - \frac{1}{2}, \cdots, -m_r - \frac{1}{2} | -n_1 - \frac{1}{2}, \cdots, -n_r - \frac{1}{2} \Big)$$

*multiplied by the sign* $(-1)^{\sum_{i=1}^{r}(n_i+1/2)+r(r-1)/2}$.

Here, the meaning of *charge* will be clarified in section 3.3.1.

## 3.3 A Probabilistic Bosonization

### 3.3.1 A Realization of Fermions in Wiener Space

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $W \equiv (W^1, W^2)$ be a two-dimensional real Brownian motion on it starting at the origin, and we set a one-dimensional complex Brownian motion $Z = W^1 + iW^2$, and $\mathcal{F}_1^Z = \sigma(\{Z_t; t \leq 1\})$.

We decompose $L^2[0,1] = L_+ \oplus L_-$, where $L_+$ and $L_-$ are mutually isomorphic and orthogonal. Let $\{f_i\}_{i\geq 1}$ and $\{g_i\}_{i\geq 1}$ be orthonormal bases of $L_+$ and $L_-$, respectively. Let $\mathcal{H}_l$ be the closure of the subspace of $L^2(\mathcal{F}_1^Z)$ spanned by

$$\int_0^1 \int \cdots \int f_{m_1} dZ \curlywedge \cdots \curlywedge f_{m_{k_1}} dZ \curlywedge g_{n_1} d\overline{Z} \curlywedge \cdots \curlywedge g_{n_{k_2}} d\overline{Z};$$

$$\begin{aligned} &m_1, \cdots, m_{k_1} \text{ are distinct natural numbers} \\ &\text{and so are } n_1, \cdots, n_{k_2}, \\ &\text{with the constraint that } k_1 - k_2 = l, \end{aligned}$$

where the operation $\curlywedge$ is defined as follows; for complex semi-martingales $X^1, \cdots, X^n$,

$$\begin{aligned} &\int_0^1 \int \cdots \int dX^1 \curlywedge \cdots \curlywedge dX^n \\ &= \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}\sigma \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} dX_{t_n}^{\sigma(n)} \cdots dX_{t_1}^{\sigma(1)}, \end{aligned}$$

where $\mathfrak{S}_n$ is the $n$-th symmetric group of permutations. An element of $\mathcal{H}_l$ is said to be of *charge* $l$. We set $\mathcal{H} = \oplus_{l\geq 0} \mathcal{H}_l$.

We define bounded operators $\psi_i$ and $\psi_i^*$ acting on $\mathcal{H}$, indexed by half-integers, as follows; for $i > 0$,

$$\begin{aligned} &\psi_i \Big( \int_0^1 \int \cdots \int f_{m_1} dZ \curlywedge \cdots \curlywedge f_{m_{k_1}} dZ \curlywedge g_{n_1} d\overline{Z} \curlywedge \cdots \curlywedge g_{n_{k_2}} d\overline{Z} \Big) \\ &= \begin{cases} (-1)^{k_1+j-1} \int_0^1 \int \cdots \int \cdots \curlywedge g_{n_{j-1}} d\overline{Z} \curlywedge g_{n_{j+1}} d\overline{Z} \curlywedge \cdots \\ \qquad\qquad\qquad\qquad \exists j \text{ such that } i = n_j + 1/2, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases} \end{aligned}$$

$$\psi_{-i}\left(\int_0^1\int\cdots\int f_{m_1}dZ\curlywedge\cdots\curlywedge f_{m_{k_1}}dZ\curlywedge g_{n_1}d\overline{Z}\curlywedge\cdots\curlywedge g_{n_{k_2}}d\overline{Z}\right)$$
$$=\begin{cases}0 & \exists j \text{ such that } i=m_j,\\ \int_0^1\int\cdots\int f_i dZ\curlywedge f_{m_1}dZ\curlywedge\cdots & \text{otherwise,}\end{cases}$$

$$\psi_i^*\left(\int_0^1\int\cdots\int f_{m_1}dZ\curlywedge\cdots\curlywedge f_{m_{k_1}}dZ\curlywedge g_{n_1}d\overline{Z}\curlywedge\cdots\curlywedge g_{n_{k_2}}d\overline{Z}\right)$$
$$=\begin{cases}(-1)^{j-1}\int_0^1\int\cdots\int\cdots\curlywedge f_{m_{j-1}-1/2}dZ\curlywedge f_{m_{j+1}-1/2}dZ\curlywedge\cdots \\ \qquad\qquad\qquad\qquad \exists j \text{ such that } i=m_j+1/2,\\ 0 \qquad\qquad\qquad\qquad \text{otherwise,}\end{cases}$$

and

$$\psi_{-i}^*\left(\int_0^1\int\cdots\int f_{m_1}dZ\curlywedge\cdots\curlywedge f_{m_{k_1}}dZ\curlywedge g_{n_1}d\overline{Z}\curlywedge\cdots\curlywedge g_{n_{k_2}}d\overline{Z}\right)$$
$$=\begin{cases}0 \qquad\qquad \exists j \text{ such that } i=n_j+1/2,\\ (-1)^{k_1}\int_0^1\int\cdots\int\cdots\curlywedge f_{m_{k_1}}dZ\curlywedge g_i d\overline{Z}\curlywedge g_{n_1}d\overline{Z}\curlywedge\cdots \\ \qquad\qquad\qquad\qquad \text{otherwise.}\end{cases}$$

The following is obtained as a special case of the result in [2].

**Proposition 11** ([2])**.** *Let $\mathcal{C}$ be the algebra generated by $\{\psi_m,\psi_n^*; m,n\in\mathbf{Z}+\frac{1}{2}\}$. (i) Then $\mathcal{C}$ is a Clifford algebra ; i.e.*

$$[\psi_m,\psi_n]_+=0,\ [\psi_m^*,\psi_n^*]_+=0, \text{and } [\psi_m,\psi_n^*]_+=\delta_{m+n,0}.$$

*(ii) The set $\psi_{-m_1}\cdots\psi_{-m_k}\psi_{-n_1}^*\cdots\psi_{-n_l}^*(1)$ is an orthogonal basis of $\mathcal{H}$.*

Proposition 11 above states that $\mathcal{H}$ is an irreducible *Cl*-module. We may identify

$$\psi_{-m_1}\cdots\psi_{-m_k}\psi_{-n_1}^*\cdots\psi_{-n_l}^*|\text{vac}\rangle=\psi_{-m_1}\cdots\psi_{-m_k}\psi_{-n_1}^*\cdots\psi_{-n_l}^*(1),$$

for $0<m_i,n_j\in\mathbf{Z}+\frac{1}{2}, i,j\in\mathbf{N}$, and so on.

### 3.3.2 The Vacuum Expectation in Wiener Space

We begin with the following

**Lemma 12.** *For $k,l \in \mathbf{N}$ with $l = k+1$, if a half-integer $i$ satisfies any one of the following conditions (a),(b); a) $m_{s+1} < i < m_s$ with $1 \leq s < s+1 \leq k$; in this case we rename the subscripts, $n_{s'} = m'_{s'}$ if $1 \leq s' \leq s$, $m_{s'} = m'_{s'+1}$ if $s+1 \leq s' \leq k$ and $i = m'_{s+1}$ otherwise; b) $i = m_s$, then*

$$\begin{aligned}
&\big(\psi_{-i}, \psi_{-m_1} \cdots \psi_{-m_k} \psi^*_{-n_1} \cdots \psi^*{}_{-n_l}(1), \\
&\qquad\qquad \psi_{-m'_1} \cdots \psi_{-m'_{k+1}} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1)\big)_{\mathcal{H}} \\
&= \big(\psi_{-m_1}, \cdots \psi_{-m_k} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1), \\
&\qquad\qquad \psi^*_i \psi_{-m'_1} \cdots \psi^*_{-m'_{k+1}} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1)\big)_{\mathcal{H}},
\end{aligned} \tag{3.5}$$

*where* $0 < m_1, \cdots, m_k, m'_1, \cdots, m'_{k+1}, n_1, \cdots, n_l$ *runs the half-integers* $\mathbf{Z} + \frac{1}{2}$.

*Proof. case a)*. Suppose $i = m'_u$ with $1 \leq u \leq k+1$.

$$\begin{aligned}
&\text{Left hand side of the equation (3.5)} \\
&= \big(\psi_{-i}\psi_{-m_1} \cdots \psi_{-m_k} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1), \\
&\qquad\qquad \psi_{-m'_1} \cdots \psi_{-m'_{u-1}} \psi_{-i} \psi_{-m'_{u+1}} \cdots, \psi_{-m'_{k+1}} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1)\big)_{\mathcal{H}} \\
&= (-1)^{u-1} \big(\psi_{-i}\psi_{-m_1} \cdots \psi_{-m_k} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1) \\
&\qquad\qquad \psi_{-i}\psi_{-m'_1} \cdots \psi_{-m'_{u-1}} \psi_{-m'_{u+1}} \cdots \psi_{-m'_{k+1}} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1)\big)_{\mathcal{H}} \\
&= 1 \times (-1)^{u-1},
\end{aligned}$$

and

$$\begin{aligned}
&\text{Right hand side of the equation (3.5)} \\
&= \big(\psi_{-m_1} \cdots \psi_{-m_k} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1), \\
&\qquad\qquad \psi^*_i \psi_{-m'_1} \cdots, \psi_{-m'_{u-1}} \psi_{-m'_u} \psi_{-m'_{u+1}} \cdots \psi_{-m'_{k+1}} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1), \big)_{\mathcal{H}} \\
&= (-1)^{u-1} \big(\psi_{-m_1} \cdots \psi_{-m_k} \psi^*_{-n_1} \cdots \psi_{-n_l}(1), \\
&\qquad\qquad \psi_{-m'_1} \cdots \psi_{-m'_{u-1}} \psi^*_i \psi_{-m'_u} \psi_{-m'_{u+1}} \cdots \psi_{-m'_{k+1}} \psi^*_{-n_1} \cdots \psi^*_{-n_l}(1)\rangle\big)_{\mathcal{H}} \\
&= (-1)^{u-1} \times 1.
\end{aligned}$$

Thus (3.5) holds.

*case b*). Suppose $i = n_s$ with $1 \le s \le l+1$.

$$
\begin{aligned}
&\text{Left hand side of the equation (3.5)} \\
&= \big(\psi_{-i}\psi_{-m_1}\cdots\psi_{-m_k}\psi^*_{-n_1}\cdots\psi^*_{-n_l}(1), \\
&\qquad\qquad \psi_{-m'_1}\cdots\psi_{-m'_{k+1}}\psi^*_{-n_1}\cdots\psi^*_{-n_l}(1),\big)_{\mathcal{H}} \\
&= (-1)^{s-1}\big(\psi_{-m_1}\cdots\psi_{-m_{s-1}}\psi_{-i}\psi_{-m_s}\psi_{-m_{s+1}}\cdots\psi_{-m_k}\psi^*_{-n_1}\cdots\psi^*_{-n_l}(1), \\
&\qquad\qquad\qquad\qquad \psi_{-m'_1}\cdots\psi_{-m'_{k+1}}\psi^*_{-n_1}\cdots\psi^*_{-n_l}(1)\big)_{\mathcal{H}} \\
&= (-1)^{s-1} \times 0 = 0.
\end{aligned}
$$

$\square$

The above lemma describes adjoint operators of Fermions $(\psi_i)^{adj.}$ and $(\psi_i^*)^{adj.}$ ;

**Theorem 13.** *We have that*

$$(\psi_i)^{adj.} = \psi^*_{-i}, \text{ and } (\psi^*_i)^{adj.} = \psi_{-i} \tag{3.6}$$

*for* $i \in \mathbf{Z} + \frac{1}{2}$.

As a corollary, we show that the vacuum expectation introduced in section 3.2.3 is expressed by the usual expectation in Wiener space:

**Theorem 14.** *We have*

$$\langle u|v\rangle = E[\overline{u^{adj.}(1)}v(1)], \tag{3.7}$$

*for* $u, v \in Cl$.

*Proof.* We put

$$u = \psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_s}, \quad v = \psi_{m'_1}\cdots\psi_{m'_{r'}}\psi^*_{n'_1}\cdots\psi^*_{n'_{s'}},$$

where $m_1 > \cdots > m_r > 0$, $n_1 > \cdots > n_s > 0$, $0 < m'_1 < \cdots < m'_{r'}$, and

$0 < n'_1 < \cdots < n'_{s'}$, then

$$
\begin{aligned}
& E[\overline{u^{adj.}(1)}v(1)] \\
= \ & E[uv(1)] \\
= \ & E\Big[\psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_s} \\
& \quad \int_0^1 \int\cdots\int f_{m'_1-1/2}dZ \curlywedge \cdots \curlywedge f_{m'_{r'}-1/2}dZ \curlywedge g_{n'_1-1/2}d\overline{Z} \curlywedge \cdots \curlywedge g_{n'_{s'}-1/2}d\overline{Z}\Big] \\
= \ & \begin{cases} (-1)^{j_s-1}E\Big[\psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_{s-1}} \\ \qquad \int_0^1\int\cdots\int\cdots\curlywedge f_{m'_{j_s-1}-1/2}dZ \curlywedge f_{m'_{j_s+1}-1/2}dZ \curlywedge\cdots\Big] \\ \qquad\qquad \exists j_s\in\{1,\cdots,r'\} \text{ such that } m'_{j_s}=n_s \\ 0 \qquad\qquad \text{otherwise} \end{cases} \\
= \ & \cdots \\
= \ & \begin{cases} (-1)^{\sum_{k=1}^s k-s-\sum_{l_3=2}^s\sum_{l_2}^{s-1}\sum_{l_1=1}^{l_2}\delta_{\{j_{l_1}<j_{l_3}\}}} \\ \qquad \times\prod_{\alpha=1,j_\alpha\in\{1,\cdots,r'\}}^s\delta_{-m'_{j_\alpha}+n_\alpha}\prod_{\beta=1,j_\beta\in\{1,\cdots,s'\}}^r\delta_{m_{j_\beta}-n'_\beta} \\ \qquad\qquad r=s',\ r'=s \\ 0 \qquad\qquad \text{otherwise} \end{cases} \\
= \ & \begin{cases} \prod_{\alpha=1,j_\alpha\in\{1,\cdots,r'\}}^s\delta_{-m'_{j_\alpha}+n_\alpha}\prod_{\beta=1,j_\beta\in\{1,\cdots,s'\}}^r\delta_{m_{j_\beta}-n'_\beta} & r=s',\ r'=s \\ 0 & \text{otherwise} \end{cases} \\
= \ & \langle u|v\rangle.
\end{aligned}
$$

□

### 3.3.3 A Bosonization

Let

$$H_n := \sum_{j\in\mathbf{Z}_++1/2} :\psi_{-j}\psi^*_{j+n}:, \quad n\in\mathbf{Z}.$$

We also have the following as a corollary to Theorem 14;

**Theorem 15.** *We have*

$$H_k^{adj.} = H_{-k}. \tag{3.8}$$

*Proof.* We proof only when $k > 0$.

$$\begin{aligned}
H_k^{adj.} &= \left( \sum_{j \in \mathbf{Z}_+ + 1/2} : \psi_{-j} \psi_{j+k}^* : \right)^{adj.} \\
&= \left( \sum_{j \in \mathbf{Z}_+ + 1/2} \Big( \psi_{-j} \psi_{j+k}^* - \delta_{k,0} \theta(j < -k) \Big) \right)^{adj.} \\
&= \sum_{j \in \mathbf{Z}_+ + 1/2} \Big( \psi_{-j} \psi_{j+k}^* \Big)^{adj.} - \sum_{j \in \mathbf{Z}_+ + 1/2, j<0} \delta_{k,0} \theta(j < -k) \\
&= \sum_{j \in \mathbf{Z}_+ + 1/2} \psi_{-j-k} \psi_j^* - \sum_{j \in \mathbf{Z}_+ + 1/2, j<0} \delta_{k,0} \theta(j < -k) \\
&= H_{-k}.
\end{aligned}$$

□

The following is a direct consequence of Fact 9, Fact 10, Lemma 12, and Theorem 15.

**Corollary 16.** *The map*

$$\Psi : \mathcal{H} \to \mathbf{C}[x_1, x_2, \cdots],$$

*given by*

$$\Psi(X) = \sum_{l \in \mathbf{Z}} z^l E[\overline{e^{H^{adj.}(x)} l^{adj.}(1)} X], \quad X \in \mathcal{H},$$

*or equivalently* $\Psi(X) = \sum_{l \in \mathbf{Z}} z^l E[l e^H X]$ *or* $\Psi(X) = \sum_{l \in \mathbf{Z}} z^l E[l^{adj.}(1) e^H X]$, *where*

$$l(1) = \begin{cases} \psi_{l+1/2}^* \cdots \psi_{-1/2}^*(1) & \text{for } l < 0, \\ 1 & l = 0, \\ \psi_{-l+1/2} \cdots \psi_{-1/2}(1) & \text{for } l > 0, \end{cases}$$

*is a Bosonization;*

$$\Psi\Big(H_n u(1)\Big) = \begin{cases} \frac{\partial}{\partial x_n} \Psi\Big(u(1)\Big) & n > 0 \\ -n x_{-n} \Psi\Big(u(1)\Big) & n < 0, \end{cases}$$

*and a basis vector*

$$\psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_r}(1),$$

*for* $m_1 < m_2 < \cdots < m_r < 0$ *and* $n_1 < n_2 < \cdots < n_r < 0$ *of charge* $0$ *goes over into the character polynomial of the Young diagram of the form*

$$Y = \Big(-m_1-\frac{1}{2},\cdots,-m_r-\frac{1}{2}|-n_1-\frac{1}{2},\cdots,-n_r-\frac{1}{2}\Big)$$

*multiplied by the sign* $(-1)^{\sum_{i=1}^r (n_i+1/2)+r(r-1)/2}$.

#### 3.3.4 A Probabilistic Bosonization in terms of Stochastic Areas

**Theorem 17.** *We have that*

$$\Phi(X) = E[\overline{e^{\sum_{i,j\geq 1} F_{i,j}A_t^{i,j}}X}], \quad X\in\mathcal{H}_0 \tag{3.9}$$

*where* $F_{i,j}$ *is the character polynomial of* $(i|j)$ *for* $i,j\geq 1$, *and* $A^{i,j}$ *is stochastic area of* $f_i dZ$ *and* $g_j d\overline{Z}$*;*

$$\begin{aligned} A_t^{i,j} \quad &:= \quad \int_0^t\int f_i dZ \curlywedge g_j d\overline{Z} \\ \Bigg( &= \quad \int_0^t\int_0^s f_i(s)g_j(u)d\overline{Z}_u dZ_s - \int_0^t\int_0^s g_j(s)f_i(u)dZ_u d\overline{Z}_s\Bigg). \end{aligned}$$

For the proof, we prepare the following lemmas.

**Lemma 18** ([2])**.** *An order* $2n$ *curlywedge stochastic integral is expressed as the Pfaffian of the ones of order* $2$*:*

$$\begin{aligned} &\int_0^1\int\cdots\int h_{i_1}dZ^{i_1}\curlywedge\cdots\curlywedge h_{i_{2n}}dZ^{i_{2n}} \\ = \quad &\frac{(2n-1)!!}{(2n)!}\sum_{\sigma\in\mathfrak{S}_{2n}}\Bigg(\int_0^1\int h_{i_{\sigma(1)}}dZ^{i_{\sigma(1)}}\curlywedge h_{i_{\sigma(2)}}dZ^{i_{\sigma(2)}}\Bigg) \\ &\qquad\cdots\Bigg(\int_0^1\int h_{i_{\sigma(2n-1)}}dZ^{i_{\sigma(2n-1)}}\curlywedge h_{i_{\sigma(2n)}}dZ^{i_{\sigma(2n)}}\Bigg). \end{aligned}$$

If the curlywedge stochastic integral is a charge-zero fermion, then the Pfaffian is decomposed into "determinant" part and others.

**Lemma 19.** *We have*

$$\int_0^1 \int \cdots \int f_{i_1} dZ^{i_1} \curlywedge \cdots \curlywedge f_{i_n} dZ \curlywedge g_{j_1} d\overline{Z} \curlywedge \cdots \curlywedge g_{j_n} d\overline{Z}$$
$$= \det\left( \int_0^1 \int f_{i_k} dZ \curlywedge g_{j_l} d\overline{Z} \right)_{k,l=1,2,\cdots,n}$$
$$+ \left\{ \sum \left( \int \int f dZ \curlywedge f dZ \right) \cdots \left( \int \int g d\overline{Z} \curlywedge g d\overline{Z} \right), \text{ etc} \ldots \right\}$$

*Proof.* It is a direct consequence of Lemma 18. ∎

**Lemma 20.** *The products of stochastic areas of $f_i dZ$ and $g_j d\overline{Z}$; $A_t^{i,j}$ are mutually orthogonal;*

$$E[\overline{A_t^{i_1,j_1} \cdots A_t^{i_r,j_r}} A_t^{i'_1,j'_1} \cdots A_t^{i'_r,j'_r}] = \sum_{\sigma \in \mathfrak{S}_r} \delta_{i_1,i'_{\sigma(1)}} \delta_{j_1,j'_{\sigma(1)}} \cdots \delta_{i_r,i'_{\sigma(r)}} \delta_{i_r,i'_{\sigma(r)}}, \tag{3.10}$$

*where $\delta$ is Kronecker delta and $i_1, \cdots, i_r$, $i'_1, \cdots, i'_r$, $j_1, \cdots, j_r$, $j'_1, \cdots, j'_r \in \mathbf{N}$.*

*Proof.* By Itô's formula, we have for $n \in \mathbf{N}$,

$$\begin{aligned}
& A_t^{i_1,j_1} \cdots A_t^{i_{2n},j_{2n}} \\
= & \sum_{l=1,2,\cdots,2n} \int_0^t \prod_{k \neq l} A_s^{i_k,j_k} dA_s^{i_l,j_l} \\
& + \sum_{\substack{l_1, l_2 = 1,2,\cdots,2n, \\ l_1 < l_2}} \sum_{l \neq l_1, l_2} \int_0^t \int_0^s \prod_{k \neq l, l_1, l_2} A_u^{i_k,j_k} dA_u^{i_l,j_l} d\langle A_\bullet^{i_{l_1},j_{l_1}}, A_\bullet^{i_{l_2},j_{l_2}} \rangle_s \\
& + \cdots \\
& + \sum_{\substack{l_\alpha < l_{\alpha+1}, \\ \alpha = 1,3,\cdots,2n-3, \\ l_1 < l_3 < \cdots < l_{2n-3}}} \sum_{l \neq l_1, \cdots, l_{2n-2}} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \prod_{k \neq l_1, \cdots, l_{2n-2}} \\
& \quad A_{t_{n-1}}^{i_k,j_k} dA_{t_{n-1}}^{i_l,j_l} d\langle A_\bullet^{i_{l_{2n-3}},j_{l_{2n-3}}}, A_\bullet^{i_{l_{2n-2}},j_{l_{2n-2}}} \rangle_{t_{n-2}} \cdots d\langle A_\bullet^{i_{l_1},j_{l_1}}, A_\bullet^{i_2,j_2} \rangle_{t_1} \\
& + \sum_{\substack{l_\alpha < l_{\alpha+1}, \\ \alpha = 1,3,\cdots,2n-2, \\ l_1 < l_3 < \cdots < l_{2n-1}}} \langle A_\bullet^{i_{l_1},j_{l_1}}, A_\bullet^{i_{l_2},j_{l_2}} \rangle_t \cdots \langle A_\bullet^{i_{l_{2n-1}},j_{l_{2n-1}}}, A_\bullet^{i_{l_{2n}},j_{l_{2n}}} \rangle_t
\end{aligned}$$

.

Therefore,

$$
\begin{aligned}
&E[\overline{A_t^{i_1,j_1}\cdots A_t^{i_r,j_r}}A_t^{i'_1,j'_1}\cdots A_t^{i'_r,j'_r}] \\
&= E[\sum_{\sigma\in\mathfrak{S}_r}\langle\overline{A_\bullet^{i_1,j_1}}, A_\bullet^{i'_{\sigma(1)},j'_{\sigma(1)}}\rangle_t\cdots\langle\overline{A_\bullet^{i_r,j_r}}, A_\bullet^{i'_{\sigma(r)},j'_{\sigma(r)}}\rangle_t].
\end{aligned}
$$

Thus (3.10) holds. □

Now we are ready to give a proof of Theorem 17.

*Proof of Theorem 17.* We put $\tilde{\Psi}(X) = E\left[\overline{e^{\sum_{i,j\geq 1}F_{i,j}A_t^{i,j}}}X\right]$. It suffices to show $\Psi(X) = \tilde{\Psi}(X)$ when $X = \psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_r}(1)$. We first notice that

$$
\begin{aligned}
\tilde{\Psi}(X) &= E\left[\sum_{n\geq 0}\frac{1}{n!}\left(\sum_{i,j\in\mathbf{Z}+1/2}F_{i,j}A_t^{i,j}\right)^n\psi_{m_1}\cdots\psi_{m_r}\psi^*_{n_1}\cdots\psi^*_{n_r}(1)\right] \\
&= E\Big[\frac{1}{r!}\sum_{i_1,\cdots,i_r,j_1,\cdots,j_r}F_{i_1,j_1}\cdots F_{i_r,j_r}\overline{A_t^{i_1,j_1}\cdots A_t^{i_r,j_r}} \\
&\qquad\qquad \sum_{\sigma\in\mathfrak{S}_r}\mathrm{sgn}\sigma A_t^{m_1,n_{\sigma(1)}}\cdots A_t^{m_r,n_{\sigma(r)}}\Big].
\end{aligned}
$$

By Lemma 20,

$$
\begin{aligned}
&= \sum_{i_1,\cdots,i_r,j_1,\cdots,j_r}F_{i_1,j_1}\cdots F_{i_r,j_r}\sum_{\sigma\in\mathfrak{S}_r}\mathrm{sgn}\sigma\prod_{k,l=1,2,\cdots,r}\delta_{i_k-m_l,0}\delta_{j_k-n_{\sigma(l)},0} \\
&= \sum_{\sigma\in\mathfrak{S}_r}\mathrm{sgn}\sigma F_{m_1,n_{\sigma(1)}}\cdots F_{m_r,n_{\sigma(r)}} \\
&= \det\left(F_{m_i,n_j}\right)_{i,j=1,2,\cdots,r}.
\end{aligned}
$$

By the definition of character expectation,

$$
= F_{(m_1,\cdots,m_r|n_1,\cdots,n_r)}.
$$

Thus we have completed the proof. □

# 4 Some Universal Properties of Stochastic Areas

## 4.1 Introduction

Let $W = (W^1, W^2)$ be a two dimensional Wiener process starting at 0. The classical stochastic area is defined by

$$\int_0^1 \int_0^t dW_s^2 dW_t^1 - \int_0^1 \int_0^t dW_s^1 dW_t^2 =: A_0,$$

which is actually twice the area drawn by the two dimensional Wiener process. The quantity is often called *Lévy's stochastic area* or simply *Lévy area* since it was Paul Lévy who first proved the celebrated formula

$$\mathbf{E}[e^{\sqrt{-1}\lambda A_0}] = (\cosh \lambda)^{-1} \tag{4.1}$$

in [21].

In the present section, we are interested in the (joint) law of

$$\begin{aligned} A(f,g) := \int_0^1 f(t) \int_0^t g(s) dW_s^2 dW_t^1 \\ - \int_0^1 g(t) \int_0^t f(s) dW_s^1 dW_t^2, \end{aligned}$$

for $f, g \in L^2([0,1] \to \mathbf{R})$. The results are twofold. The first one is

**Theorem 21.** *For any $f \in L^2[0,1]$ with $\|f\|_{L^2} = 1$, $A(f,f)$ is identically distributed as $A_0$.*

The second one is concerned with the so-called *Walsh system* $\{\tau_k\}_{k=0}^\infty$, which is given as follows: $\tau_0 \equiv 1$,

$$\tau_{2^k}(x) = \begin{cases} 1 & x \in \bigcup_{j=1}^{2^k} [2(j-1)/2^{k+1}, (2j-1)/2^{k+1}) \\ -1 & \text{otherwise}, \end{cases}$$

for $k = 0, 1, \cdots$, and for $n = 2^{k_1} + \cdots + 2^{k_l}$,

$$\tau_n = \tau_{2^{k_1}} \cdots \tau_{2^{k_l}}.$$

The second result is

**Theorem 22.** *For $n \neq m$, $A(\tau_n, \tau_m) + A(\tau_m, \tau_n)$ is identically distributed as $A_0 + A_0'$, where $A_0'$ is an independent copy of $A_0$.*

Both of the theorem are abtained almost as a corollary to Proposition 23, which is actually obtained by a restriction of a result in [1].

This paper is organized as folliws, In section 4.2, we recall (a spccial case of) the generlalized Lévy-Area Formula of [1]. We state the result as Proposition 23 and give a bried proof. In sections 4.3 and 4.4, we give a proof of Theorem 21 and Theorem 22, respectively.

## 4.2 A Generalization of the Lévy-Area Formula

In this section we give a generalization of the Lévy-area Formula (4.1). It is actually a corollaty to Theorem 2 in [1], where the formula is used to obatain a probabistic representation of *tau functions*.

**Proposition 23.** *Let*

$$f_i(x) = \sum_{j=1}^{n} a_j^i 1_{[(j-1)/n, j/n)}(x), \quad x \in [0,1] \tag{4.2}$$

*where $a_j^i$, $j = 1, \cdots, n$, $i = 1, 2, \cdots, m$ are real numbers. For real numbers $c_{i,i'}$, $1 \leq i, i' \leq m$, we put*

$$\varphi(\lambda) := E[e^{\sqrt{-1}\lambda \sum c_{i,i'} A(f_i, f_{i'})}], \quad \lambda \in \mathbf{R}.$$

*If $c_{i,i'} = c_{i',i}$, then*

$$\varphi(\lambda) = (\det D)\{\det(D \cosh \lambda D - B \sinh \lambda D)\}^{-1}, \tag{4.3}$$

*where*

$$B = (b_{j,k}), \quad D := \operatorname{diag}[d_j]$$

*with*

$$b_{j,k} = \begin{cases} n \sum_{i,i'} c_{i,i'} a_j^i a_k^{i'} & j > k \\ -n \sum_{i,i'} c_{i,i'} a_j^i a_k^{i'} & j < k \\ 0 & j = k, \end{cases}$$

*and*

$$d_j = n \sum_{i,i'} c_{i,i'} a_j^i a_j^{i'}.$$

*Proof.* By a direct calculation, we have

$$
\begin{aligned}
&A(f_i, f_{i'}) \\
&= \sum_{j>k} (a^i_j a^{i'}_k \Delta W^{n,1}_j \Delta W^{n,2}_k - a^{i'}_j a^i_k \Delta W^{n,2}_j \Delta W^{n,1}_k) \\
&+ \sum_j a^i_j a^{i'}_j \Big( \int_{(j-1)/n}^{j/n} \int_{(j-1)/n}^{t} dW^2_s dW^1_t \\
&\qquad\qquad - \int_{(j-1)/n}^{j/n} \int_{(j-1)/n}^{t} dW^1_s dW^2_t \Big),
\end{aligned}
$$

where

$$\Delta W^{n,\cdot}_k = W^{\cdot}_{k/n} - W^{\cdot}_{(k-1)/n}.$$

Then we notice that, by the scaling property of the Wiener process, $\sum c_{i,i'} A(f_i, f_{i'})$ is identically distributed as

$$
\begin{aligned}
&\sum_{j,k} b_{j,k} W^{1,j}_1 W^{2,k}_1 \\
&+ \sum_j d_j \left( \int_0^1 \int_0^t dW^{1,j}_t dW^{2,j}_t - \int_0^1 \int_0^t dW^{2,j}_s dW^{1,j}_t \right),
\end{aligned}
$$

where

$$(W^{1,1}, \cdots, W^{1,n}, W^{2,1}, \cdots, W^{2,n})$$

is a $2n$-dimensional Wiener process starting at 0. Now we can apply Theorem 2 in [1] to obtain (4.3), since $B$ is skew-symmetric by the assumption that $c_{i,i'} = c_{i',i}$. $\square$

## 4.3 Proof of Theorem 21

We first show the property for a step function $f$ as (4.2);

$$f(x) = \sum_{j=1}^{n} a_j 1_{[(j-1)/n, j/n)}(x), \quad x \in [0,1].$$

We first note that, for $A(f,f)$, we have

$$D = n\mathrm{diag}[a_1^2, \cdots, a_n^2]$$

and

$$B = D^{1/2} P D^{1/2}$$

with

$$P = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ -1 & \cdots & -1 & 0 \end{pmatrix}.$$

Then, we notice that

$$\begin{aligned} &\varphi(\lambda) \\ &= \det D \{\det(D \cosh \lambda D - D^{1/2} P D^{1/2} \sinh \lambda D)\}^{-1} \\ &= \{\det(\cosh \lambda D - P \sinh \lambda D)\}^{-1}. \end{aligned}$$

We put

$$\begin{aligned} \phi(\lambda) &= \det(\cosh \lambda D - P \sinh \lambda D) \\ &= \begin{vmatrix} \cosh \lambda d_1 & \sinh \lambda d_2 & \cdots & \sinh \lambda d_n \\ -\sinh \lambda d_1 & \cosh \lambda d_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -\sinh \lambda d_1 & -\sinh \lambda d_2 & \cdots & \cosh \lambda d_n \end{vmatrix}, \end{aligned}$$

which is a polynomial in

$$x_j = \sinh \lambda d_j,\ y_j = \cosh \lambda d_j,$$

$j = 1, \cdots, n$. One further sees that each monomial is of the form,

$$c_\nu x_{\nu(1)} \cdots x_{\nu(l)} y_{\nu(l+1)} \cdots y_{\nu(l)},$$

indexed by a permutation $\nu \in \mathfrak{S}_n$ with the property that

$$\nu(1) < \cdots < \nu(l), \nu(l+1) < \cdots < \nu(n),$$

where $c_\nu$ is an integer. We claim that

$$c_\nu = \begin{cases} 1 & l \text{ is even} \\ 0 & l \text{ is odd.} \end{cases} \tag{4.4}$$

If it is the case, we will have that

$$\phi(\lambda) = \cosh(\lambda \|f\|^2)$$

since we can prove by induction that

$$\cosh(\lambda(d_1 + \cdots + d_n)) = \sum_{l \text{ is even}} x_{j_1} \cdots x_{j_l} y_{j'_1} \cdots y_{j'_{n-l}}$$

and

$$\sinh(\lambda(d_1 + \cdots + d_n)) = \sum_{l \text{ is odd}} x_{j_1} \cdots x_{j_l} y_{j'_1} \cdots y_{j'_{n-l}}.$$

To prove (4.4), we first define, as a function of matrices,

$$C_\nu((a_{ij})) = \sum_{\sigma \in \mathfrak{S}_n(\nu)} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

where

$$\begin{aligned} \mathfrak{S}_n(\nu) := \{\sigma \in \mathfrak{S}_n : \sigma\nu(1) \neq \nu(1), \cdots, \sigma\nu(l) \neq \nu(l), \\ \sigma\nu(l+1) = \nu(1+1), \cdots, \sigma\nu(n) = \nu(n)\} \end{aligned}$$

Then,

$$c_\nu = C_\nu(I + P).$$

Since

$$(I+P)_{ij} = \begin{cases} -1 & i > j \\ +1 & i \leq j \end{cases},$$

and since sgn restricted to $\mathfrak{S}_n(\nu)$ coincides with the signature on $\mathfrak{S}_l$, we have

$$c_\nu = \sum_{\sigma \in \mathcal{S}_l} \operatorname{sgn}(\sigma)(-1)^{\sharp\{i : \sigma(i) < j\}},$$

which is known to satisfy (4.4).

Finally, let $\{f_n\}$ be a sequence in $L^2$ defined by

$$f_n = \sum_{k=0}^{2^n - 1} \langle f, \tau_k \rangle \tau_k,$$

which, as a matter of course, converge in $L^2$ to $f$. Since we have established

$$\mathbf{E}[e^{\sqrt{-1}\lambda A(f_n, f_n)}] = (\cosh \lambda \|f\|^2)^{-1},$$

we the desired result since

$$\begin{aligned}&|\mathbf{E}[e^{\sqrt{-1}\lambda A(f_n,f_n)}] - \mathbf{E}[e^{\sqrt{-1}\lambda A(f,f)}]| \\ &\leq \mathbf{E}[|A(f_n,f_n) - A(f,f)|^2] \leq 4\|f_n - f\|_{L^2}.\end{aligned}$$

□

## 4.4 Proof of Theorem 22

Our proof is based on the following two lemmas.

**Lemma 24.** *For any $l$, $m$, $n \in \mathbb{Z}_0$, $A(\tau_l\tau_m, \tau_l\tau_n) + A(\tau_l\tau_n, \tau_l\tau_m)$ is identically distributed as $A(\tau_m, \tau_n) + A(\tau_n, \tau_m)$.*

This lemma suggests that our target is reduced to showing that

$$A(\tau_0, \tau_m) + A(\tau_m, \tau_0) \stackrel{\mathrm{d}}{=} A(\tau_0, \tau_1) + A(\tau_1, \tau_0). \tag{4.5}$$

The right-hand-side of (4.5) is easily seen to be equal to

$$\begin{aligned}&2\int_0^{1/2} (W_s^2 dW_s^1 - W_s^1 dW_s^2) \\ &+ 2\int_{1/2}^1 \{(W_s^2 - W_{1/2}^2)dW_s^1 - (W_s^1 - W_{1/2}^1)dW_s^2\},\end{aligned}$$

hence identically distributed as $A_0 + A_0'$, as desired.

The equivalence in law (4.5) is obtained by the following

**Lemma 25.** *Let $\theta_{k,N} : [0,1] \to [0,1]$, $1 \leq j < k \leq 2^N$, be defined by*

$$\theta_{j,k,N}(x) = \begin{cases} x + (k-j)/2^N & x \in [(j-1)/2^N, j/2^N) \\ x - (k-j)/2^N & x \in [(k-1)/2^N, k/2^N) \\ x \ \textit{otherwise}. \end{cases}$$

*Then, for any pair $(f_1, f_2)$ of the form (4.2) with $n = 2^N$, and for any $(j,k)$,*

$$\begin{aligned}&A(f_1, f_2) + A(f_2, f_1) \\ &\stackrel{\mathrm{d}}{=} A(f_1 \circ \theta_{j,k,N}, f_2 \circ \theta_{j,k,N}) + A(f_2 \circ \theta_{j,k,N}, f_1 \circ \theta_{j,k,N}).\end{aligned}$$

In fact, since we have $\tau_0 \circ \theta_{j,k,N} = \tau_0$ and $\tau_1 = \tau_m \circ \theta_{j_1,k_1,N} \circ \cdots \circ \theta_{j_{m'},k_{m'},N}$ for some $j_1, k_1, \cdots, j_{m'}, k_{m'}$, we have the equivalence in law (4.5).

Both of the lemmas are proven by looking at the generalized Lévy-area formula (4.3). Actually, the determinants in (4.3) is invariant under both of the operations $(tau_m, \tau_n) \mapsto (\tau_l\tau_m, \tau_l\tau_n)$ and $f \mapsto f \circ \theta_{j,k}$.

□

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