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# Seiberg-Witten Geometry
# and
# the Nekrasov-type Partition Function
# in
# E-string Theory

（E弦理論におけるサイバーグ-ウィッテン幾何学と

ネクラソフ型分配関数）

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Doctoral Program in Advanced Mathematics and Physics
Graduate School of Science and Engineering
Ritsumeikan University
立命館大学大学院理工学研究科
基礎理工学専攻博士課程後期課程

ISHII Takenori
石井　健准

Supervisor: Professor SUGAWARA Yuji
研究指導教員：菅原　祐二　教授

**Abstract**

In this thesis, the E-string theory, that is the interacting, non-gravitational local quantum field theory with (1,0) supersymmetry and the $E_8$ global symmetry in six dimensions, is surveyed. By the toroidal compactification, we can obtain the $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions. This theory is allowed to have its Seiberg-Witten description. In 2012, the Nekrasov-type partition function for E-string theory appeared. As the original Nekrasov partition function was required the proof of the correctness, the Nekrasov-type partition function also was. “The proof” is given by extracting the Seiberg-Witten description in the thermodynamic limit from the Nekrasov-type partition function, following the idea by Nekrasov and Okounkov. Due to the toroidal compactification, in E-string theory we obtain an elliptic function on the way to prove. The elliptic function gives the Seiberg-Witten description. The Nekrasov-type partition function is also valid in the cases with the general Wilson lines. Moreover, the Nekrasov-type partition function clarifies the dependence of the Seiberg-Witten curve on the Wilson lines.

# Contents

# 1 Introduction

String theory is the most reliable model for the study of the fundamental structure of an object. Superstring theory, or simply superstring, is the string theory with supersymmetry. The study of superstring has given not only a lot of physical predictions but also a lot of mathematical predictions. However, there are five different types of superstrings, so we have a big problem: which can describe our physics? However, E. Witten showed that there exists the theory in eleven dimensions which contains all the superstrings[1]. This eleven-dimensional theory is called *M-theory* and is expected to be the more fundamental theory. M-theory has membranes as the ingredients which are called an *M2*- and an *M5*-brane. One of methods of study of M-theory is to study the worldvolume theory on the M5-brane. The worldvolume theory is the six-dimensional theory and has supersymmetry. The number of supersymmetry depends on the configuration of the M2- and M5-branes. The six-dimensional theory can have three types of the number of supersymmetry and they are called the (2,0), (1,1), and (1,0) theories respectively[1] . The six-dimensional (2,0) theory is the maximal supersymmetric theory. Recent years, this theory has been intensively studied and has given a lot of interesting results not only in physics but also in mathematics. One of these is the *AGT correspondence*[2, 3]. The AGT correspondence relates the Nekrasov partition function of an $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions to the conformal block(roughly speaking, the correlation function) of a conformal field theory in two dimensions.

The six-dimensional (1,0) theory is the minimal supersymmetric theory, on the one hand. The six-dimensional (1,0) theory with the least field content, more explicitly just one tensor multiplet, and the $E_8$ global symmetry is called the *E-string theory* and this theory is the main subject in this thesis.

The history of E-string theory was started by P. Horava-E. Witten and O. J. Ganor-A. Hanany[4, 5, 6]. Briefly speaking, Horava and Witten showed that in M-theory on $S^1/\mathbb{Z}_2$ each $E_8$ gauge field must live on the end-of-the-world brane(M9-brane) to reproduce the $E_8 \times E_8$ heterotic superstring theory in the small $S^1/\mathbb{Z}_2$ limit[2]. Ganor and Hanany showed that, in such a case, in the limit where an M5-brane between the M9-branes approaches one of the M9-branes, an M2-brane between the M5-brane and the M9-brane becomes a *tensionless* and *non-critical*

[1]We make some comments on the study of the (2,0) and (1,0) theories in Introduction. However we make no mention of the (1,1) theory here.

[2]Before the discussion of a small instanton associated to the $E_8 \times E_8$ heterotic superstring theory, Witten discussed that assciated to the *SO(32)* heterotic superstring[7].

*string*. This string is called the *E-string* and the theory which describes the behaviour of the E-string on the M5-brane is called the *E-string theory*. We will review this theory in more detail in section 3.

E-string theory is known as the simplest, six-dimensional (1,0) theory. The theory consists of just one tensor multiplet, namely has no gravity. In addition, the theory has the $E_8$ global symmetry. However, the theory does not have its Lagrangian description at present.

When it comes to the four-dimensional theories, in particular with eight supercharges, we know a lot. More explicitly, the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories have two major descriptions: the Seiberg-Witten theory[8] and the Nekrasov partition function[11]. Thus in that sense, we can say that the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are the best understood theories. So we would like to move on from the world in six dimensions to that in four dimensions. By taking two dimensions to be very small in the six-dimensional theories, we can obtain the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and study them. This procedure is called *dimensional reduction*. Then to keep supersymmetry, the two dimensions must be a compact manifold, i.e. $\mathbb{R}^4 \times M^2$ where $M^2$ denotes the two-dimensional compact manifold. This procedure is called *compactification*. In E-string theory, the two dimensions must be a torus $T^2$ because we have to keep the eight supercharges. This is called the *toroidal compactification*. By the toroidal compactification, we can obtain the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory, but the nature of it is so different from that of the ordinary four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories[9]. For instance, the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory obtained from E-string theory is asymptotically non-free. More details will be mentioned in section 3. Nevertheless, it was shown that it has the Seiberg-Witten description[9, 10]. Hence, following the history of the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, namely the discovery of the Nekrasov partition function several years later since the appearance of the Seiberg-Witten description, it is natural to expect that there also exists the Nekrasov-type partition function in E-string theory. The Nekrasov partition function includes the Seiberg-Witten description as the special limit[11, 12]. Hence it is the very important problem to ask whether the Nekrasov-type partition function in E-string theory exists or not for the development of E-string theory.

From such an expectation, in 2012 the Nekrasov-type partition function was given by K. Sakai[13, 14]. The correctness or the validity of the Nekrasov-type partition function was checked order by order, by comparing the values given by the Nekrasov-type partiton function with the ones given by the partition function of the $\mathcal{N} = 4$ topological super Yang-Mills theory on $\frac{1}{2}$K3[16] and the ones given by the prepotential. In addition it was also checked that the Nekrasov-type partition function satisfies the holomorphic anomaly equation[35]. However, “ order by order ” means “ not all orders. ” Namely, in order for the Nekrasov-type partition function to be correct, i.e. it is the *formula*, that it is correct for all orders has to be shown. The Nekrasov partition function and some statements associated to it were given in [11] and then their correctness was given in the subsequent paper[12]. Hence it is natural to expect that we can show the correctness of the Nekrasov-type partition function for all orders by following the idea of [12]. K. Sakai and the author showed it in [15]. More details will be discussed in section 3.

Briefly speaking, our goal is to show that we can extract the known Seiberg-Witten description in E-string theory from the Nekrasov-type partition function. Following the idea of Nekrasov and Okounkov[12], the Nekrasov-type partition function is represented by some Young diagrams and functions associated to them which are called the *profile functions*. In the semiclassical limit which is called the *thermodynamic limit*, we can fix them explicitly. Then in E-string theory, due to the toroidal compactification, the profile functions are fixed by an elliptic function. This elliptic function plays a crucial role in our discussion. Namely, we showed that we can extract the Seiberg-Witten description if we give the elliptic function depending on a case. Here “ case ” means that for the special values of the Wilson lines, the global symmetry would be partially broken.

In the subsequent paper[29], the author generalised the result of [15] to more general cases. Namely, whilst in [15] we studied some cases with the Wilson lines given the concrete values, in [29] the cases with three and four general Wilson lines were discussed. By the study[29], it was shown that the Nekrasov-type partition function is also valid for the general case and was shown the dependence of the Seiberg-Witten curve on the Wilson lines *explicitly*. In particular, the latter is interesting for us. The Seiberg-Witten curve in the case with three general Wilson lines is already known[26]. It was obtained by the so-called *geometric engineering* approach. However, in that case, the dependence of the Seiberg-Witten curve on the Wilson lines was not clear. We will comment about this point in Conclusion again.

This thesis is organised as follows. In the next section, we briefly review the Seiberg-Witten description and the Nekrasov partition function. In particular, we will focus on the ideas which we will need for the later discussions. In section 3, we will review the formulation of E-string theory, its Seiberg-Witten description, and the Nekrasov-type partition function. In addition, we will see briefly the validity, or in other words the interpretation of the Nekrasov-type partition function. In section 4, we will prove the correctness of the Nekrasov-type partition function, i.e. we will show that we can extract the Seiberg-Witten description from the Nekrasov-type partition function. In section 5, the result of section 4 is generalised to the cases with the Wilson lines. In section 6, the examples of the cases with the broken symmetries are summarised. Sections 4, 5, and 6 are the original parts based on [15, 29]. Finally, in section 7 we will summarise the stories. In appendix A, the function $\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)$ which will be used in sections 2 and 4 will be summarised. In appendix B, the definitions and the useful relations of the elliptic functions which will be used in sections 4, 5, and 6 will be summarised.

# 2 Two Prescriptions in $4d$ $\mathcal{N}=2$ Gauge Theory

In this section, we review the ordinary $\mathcal{N}=2$ supersymmetric gauge theory in four dimensions before we move on to our main discussion, that is, the Nekrasov-type partition function for E-string theory. Firstly, we very briefly recall the Seiberg-Witten description for the ordinary theory and E-string theory. And secondly, we briefly review the discussion of the so-called Nekrasov partition function to move on to that for E-string theory.

## 2.1 Seiberg-Witten description

In 1994, N. Seiberg and E. Witten determined the low-energy effective action in the $\mathcal{N}=2$ supersymmetric gauge theory in four dimensions[3][8]. More precisely speaking, they gave the prescription to determine the prepotential of the theory which gives the action(we say Lagrangian alternatively henceforth). Here the *low-energy $effective$* means so-called *$Wilsonian$*, that is, by integrating out the massive modes, we make the theory consist only of the almost massless modes.

The low-energy effective Lagrangian of the theory with $\mathcal{N}=2$ supersymmetry is given by

$$L_{eff}=\frac{1}{8\pi i}\int d^2\theta\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W^\alpha W_\alpha+\frac{1}{4\pi i}\int d^2\theta d^2\bar{\theta}\Phi^\dagger\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi}+h.c. \tag{2.1}$$

where $\mathcal{F}$ denotes the prepotential, $\Phi(z)$ the complex scalar superfield, and $W_\alpha$ the chiral superfield including the vector field. Note, here, that from (2.1) we have the

[3]For the good review, see e.g. [40, 41].

gauge coupling $\tau$:

$$\tau = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^2}. \tag{2.2}$$

The prepotential $\mathcal{F}(\Phi)$ is a holomorphic function. Thus determining the holomorphic function is to give the Lagrangian. The prepotential can be written, by $\mathcal{N} = 2$ supersymmetry, as

$$\text{Prepotential} = \text{classical} + \text{1-loop} + \text{non-perturbative}. \tag{2.3}$$

In the r.h.s., the first two terms can be determined by the classical discussion as

$$\frac{1}{2}\tau_{cl} a^2 + \frac{i}{2\pi} a^2 \log\Big(\frac{a^2}{\Lambda^2}\Big), \tag{2.4}$$

where $a$ denotes the Higgs vev and $\Lambda$ the so-called QCD dynamical scale.

The long-standing problem was to determine the non-perturbative part. The form was known as

$$\sum_{k=1}^{\infty} \mathcal{F}_k \Big(\frac{\Lambda}{a}\Big)^{4k} a^2. \tag{2.5}$$

This is known as the so-called *instanton expansion*, which forms the series by the instanton number $k$. Seiberg and Witten showed that this can be determined by using the period integrals in algebraic geometry which is known as the moduli theory of torus.

The key ingredients of their discussion are a one-form, two one-cycles, and an elliptic curve, which are called the Seiberg-Witten *differential*, the $\alpha$- and $\beta$-cycles, and the Seiberg-Witten *curve* respectively. For instance, for $SU(2)$, we give just the result:

$$\begin{aligned}
a_D &:= \frac{\partial \mathcal{F}(a)}{\partial a}, \\
\omega &:= \oint_\alpha \frac{dz}{y}, \quad \omega_D := \oint_\beta \frac{dz}{y}, \\
\tau(a) &= \frac{\partial a_D}{\partial a} = \frac{\partial a_D}{\partial u}\frac{\partial u}{\partial a}, \quad 2\pi i \frac{\partial a_D}{\partial u} = \omega, \quad 2\pi i \frac{\partial a}{\partial u} = \omega_D, \\
a(u) &= \oint_\alpha ds, \quad a_D(u) = \oint_\beta ds, \quad ds = \frac{1}{\pi i}\frac{z^2 dz}{y}, \\
y^2 &= (z^2 - u)^2 - 4\Lambda^2 = \prod_{i=1}^{2} (z - \alpha_i^-)(z - \alpha_i^+).
\end{aligned} \tag{2.6}$$

Treating the Higgs vev $a$ and its dual $a_D$ as functions with respect to the modulus $u := \langle \mathrm{Tr}\phi^2 \rangle = a^2/2$ is the basic idea. The prepotential is fixed by the beta-cycle integral of the Seiberg-Witten differential $ds$. The two cycles are taken by two cuts over $[\alpha_1^-, \alpha_1^+]$ and $[\alpha_2^-, \alpha_2^+]$. The period integral or the Seiberg-Witten differential is fixed by the Seiberg-Witten curve $y^2$. The discussion by Seiberg and Witten states that we are able to determine the prepotential by going up the river.

## 2.2 Nekrasov partition function

As seen in the last subsection, we can obtain the prepotential by using the *acrobatic* algebraic geometrically method. However, it is hard to determine the prepotential practically. A few years later, a new approach appeared. It can combinatorially determine the prepotential and what we need is just the field content of the theory. In other words, if we even know the field content, we can directly give the prepotential. More precisely, it gives directly not the prepotential but the *partition function*. More explicitly, Nekrasov proposed the expression[4][11]

$$Z(\vec{a}, \epsilon_1, \epsilon_2, \Lambda) = \exp\Big( - \frac{\mathcal{F}^{inst}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda)}{\epsilon_1 \epsilon_2} \Big). \tag{2.7}$$

The detail including the notation is given later. The important thing is that the partition function $Z(\vec{a}, \epsilon_1, \epsilon_2, \Lambda)$ and the instanton part of the prepotential $\mathcal{F}^{inst}$ are both generic, that is, they are generalised by the two parameters $\epsilon_{1,2}$. So Nekrasov proposed one more strong relation[11]: in the limit $\epsilon_{1,2} \to 0$, (2.7) would reproduce the Seiberg-Witten description, i.e.[5]

$$\mathcal{F}^{inst}(\vec{a}, \epsilon_1, \epsilon_2, \Lambda)|_{\epsilon_{1,2}=0} \text{ is the instanton part of the prepotential of the low-energy theory.} \tag{2.8}$$

In the rest of this subsection we review the so-called Nekrasov partition function more detail and pursue the idea of the proof of it, that is, extracting the Seiberg-Witten description.

[4]The prepotential consists of the perturbative part and the non-perturbative part. Similarly, the Nekrasov partition function also consists of the perturbative part and the non-perturbative part. In this thesis, we restrict ourselves to the non-perturbative part, i.e. the instanton part. In the sense, by the Nekrasov partition function $Z$ we mean the instanton partition function of the full partition function.

[5]The full prepotential including the perturbative part is given by the full partition function.

From now on, we also consider the Nekrasov partition function in the field theoretic limit, i.e. $\epsilon_1 = -\epsilon_2 = \hbar$ and we take the gauge group to be $SU(N)$. Then the Nekrasov partition function can be expressed as the sum over the partitions[11]:

$$Z(a, \hbar, -\hbar, q) = \sum_{\vec{\mathbf{k}}} q^{|\mathbf{k}|} \prod_{(l,i)\neq(n,j)} \frac{a_{ln} + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_{ln} + \hbar(j-i)}. \tag{2.9}$$

Here $a_{ln} = a_l - a_n$, $a_i$ denotes the diagonal components of the Higgs vev, $q \sim \Lambda^{2N}$ is the dynamical scale, and $\epsilon_{1,2}$ are the deformation parameters which appear in the Omega-background[11, 12]. Note that the dyanamical scale can be expressed as

$$\Lambda^{2N} \sim \mu^{2N} e^{-\frac{8\pi^2}{g_0^2} + 2\pi i \vartheta_0}, \tag{2.10}$$

where $\mu$ is an UV cutoff[6] and $g_0, \vartheta_0$ are the bare couplings. The partition is defined as follows: $\vec{\mathbf{k}} = (\mathbf{k}_1, \cdots, \mathbf{k}_N)$, $\mathbf{k}_l = \{k_{l,1} \geqslant k_{l,2} \geqslant \cdots \geqslant k_{l,n_l} \geqslant k_{l,n_l+1} = k_{l,n_l+2} = \cdots = 0\}$, $|\vec{\mathbf{k}}| = \sum_{l,i} k_{l,i}$. The indexes $l, n$ run from 1 to $N$ and $i, j$ is 1 or above.

Including the perturbative part, the Nekrasov partition function can be totally written as

$$\begin{aligned} Z(a, \hbar, \Lambda) &= \sum_{\vec{\mathbf{k}}} \Lambda^{2N|\vec{\mathbf{k}}|} Z_{\vec{\mathbf{k}}}(a, \hbar), \\ Z_{\vec{\mathbf{k}}}(a, \hbar) &= Z^{pert}(a, \hbar) \mu^2_{\vec{\mathbf{k}}}(a, \hbar), \\ \mu^2_{\vec{\mathbf{k}}}(a, \hbar) &= \prod_{(l,i)\neq(n,j)} \frac{a_{ln} + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_{ln} + \hbar(j-i)}. \end{aligned} \tag{2.11}$$

In the semiclassical limit $\hbar \to 0$, which is called the *thermodynamic limit*, this partition function can be expressed as the genus expansion

$$Z(a, \hbar, \Lambda) = \exp\Big(\sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(a, \Lambda)\Big). \tag{2.12}$$

$\mathcal{F}_0$ is the prepotential of the low-energy effective theory:

$$\mathcal{F}_0(\vec{a}, \Lambda) = -\frac{1}{2} \sum_{l,n} (a_l - a_n)^2 \Big(\log\Big(\frac{a_l - a_n}{\Lambda}\Big) - \frac{3}{2}\Big) + \sum_{k=1}^{\infty} \Lambda^{2kN} f_k(\vec{a}). \tag{2.13}$$

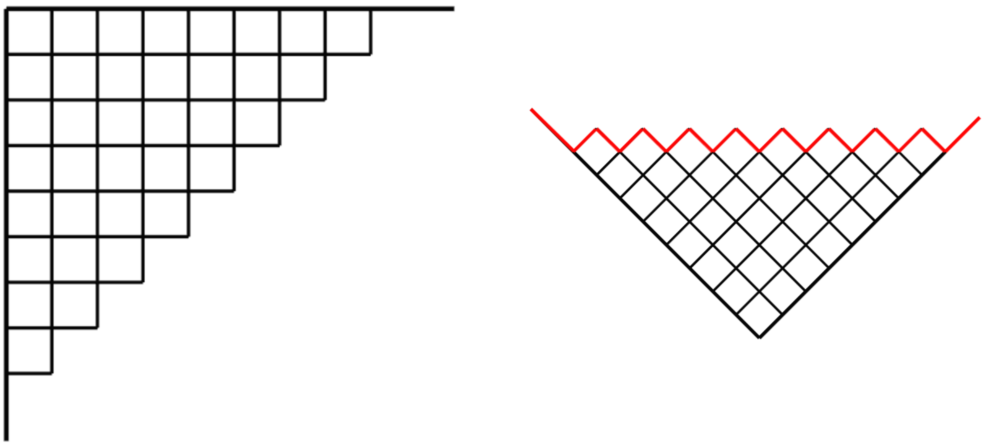

Figure 1: A Young diagram and the profile function(red line). left: A set of partitions(i.e. a Young diagram). right: A Russian style of the Young diagram and the profile function. The piecewise-linear function becomes the continuous function in the limit $\epsilon_{1,2} \to 0$.

There are three different approaches to prove the correctness of the Nekrasov partition function[12, 28, 32, 33]. Since we discuss the Nekrasov and Okounkov approach for E-string theory in the next section, we review the approach only here.

Their idea is to represent the sum of the partitions by that of the Young diagrams(see the figure 1). The shape(the red line in the figure) is expressed by a piecewise-linear function $f_{\mathbf{k}}(x)$

$$f_{\mathbf{k}}(x) = |x| + \sum_{i=1}^{\infty} \Big[ |x - k_i + i - 1| - |x - k_i + i| + |x + i| - |x + i - 1| \Big]. \tag{2.14}$$

We call this function the *profile function*. In general, the profile function with $\epsilon_{1,2}$ is expressed as

$$\begin{aligned} f_{\mathbf{k}}(x|\epsilon_1, \epsilon_2) &= |x| + \sum_{i=1}^{\infty} \Big[ |x + \epsilon_1 - \epsilon_2 k_i - \epsilon_1 i| - |x - \epsilon_2 k_i - \epsilon_1 i| - |x + \epsilon_1 - \epsilon_1 i| + |x - \epsilon_1 i| \Big], \\ &= |x| + \sum_{j=1}^{\infty} \Big[ |x + \epsilon_2 - \epsilon_1 \tilde{k}_j - \epsilon_2 j| - |x - \epsilon_1 \tilde{k}_j - \epsilon_2 j| - |x + \epsilon_2 - \epsilon_2 j| + |x - \epsilon_2 j| \Big]. \end{aligned} \tag{2.15}$$

[6] We use a letter $\mu$ just below again but it does not mean an UV cutoff.

This general profile function satisfies the following conditions

$$\begin{aligned} f'_{\mathbf{k}}(x|\epsilon_1,\epsilon_2) &= \pm 1, \\ f_{\mathbf{k}}(x|\epsilon_1,\epsilon_2) &\geqslant |x|, \\ f_{\mathbf{k}}(x|\epsilon_1,\epsilon_2) &= |x|, \quad \text{for } |x| \gg 0. \end{aligned} \tag{2.16}$$

Moreover, the shifted profile function which is called a charged partition is defined by

$$f_{a;\mathbf{k}}(x|\epsilon_1,\epsilon_2) = f_{\mathbf{k}}(x-a|\epsilon_1,\epsilon_2). \tag{2.17}$$

Then, the instanton charge $a$ and the size of the partitions $|\mathbf{k}|$, i.e. the size of the Young diagrams, are recovered from the charged profile function as

$$\begin{aligned} a &= \frac{1}{2}\int_{\mathbb{R}} dx\, x f''_{a;\mathbf{k}}(x|\epsilon_1,\epsilon_2) = -\frac{1}{2}\int_{\mathbb{R}} dx\ f'_{a;\mathbf{k}}(x|\epsilon_1,\epsilon_2), \\ |\mathbf{k}| &= \frac{a^2}{2\epsilon_1\epsilon_2} - \frac{1}{4\epsilon_1\epsilon_2}\int dx\, x^2 f''_{a;\mathbf{k}}(x|\epsilon_1,\epsilon_2) = \frac{1}{2\epsilon_1\epsilon_2}\Big(a^2 - \int dx\,(f_{a;\mathbf{k}}(x|\epsilon_1,\epsilon_2) - |x|)\Big), \end{aligned} \tag{2.18}$$

where the integral of the rightmost hand side of the first line is defined by the Caucy's principal value one.

In the thermodynamic limit $\hbar \to 0$ or $\epsilon_{1,2} \to 0$, the typical size of the partition $\mathbf{k}$ contributing to the partition function is given by $|\mathbf{k}| \sim 1/\epsilon_1\epsilon_2$. This means that the size of a box in a Young diagram becomes small, namely the piecewise-linear profile function becomes a continuous profile function. Hence the sum of the partitions can be approximated by an integral over the space of the continuous profile functions. This continuous profile function satisfies the following conditions[7]:

$$\begin{aligned} f(x) &= |x|, \quad |x| \gg 0, \\ |f(x) - f(y)| &\leqslant |x-y|, \\ \int_{\mathbb{R}} dx f'(x) &= 0, \\ \int_{\mathbb{R}} dx(f(x) - |x|) &< \infty, \end{aligned} \tag{2.19}$$

where the integral in the third condition is defined by the Caucy's principal value one.

[7] The second condition in the original paper[12] might be wrong.

The approximation by the integral over the space of the continuous profile functions is the saddle point one. Namely, our task is to find the saddle point of the profile function from

$$Z_{\vec{\mathbf{k}}}(\vec{a};\epsilon_1,\epsilon_2,\Lambda)=\exp\Big(-\frac{1}{4}\int dxdy\, f''_{\vec{a},\vec{\mathbf{k}}}(x|\epsilon_1,\epsilon_2)\, f''_{\vec{a},\vec{\mathbf{k}}}(y|\epsilon_1,\epsilon_2)\gamma_{\epsilon_1,\epsilon_2}(x-y,\Lambda)\Big), \tag{2.20}$$

where the integral is defined by the Caucy's principal value one and $\gamma_{\epsilon_1,\epsilon_2}(x)$ is some function defined in Appendix A. Thus the partition function (2.11) becomes the sum over the profile functions

$$Z(\vec{a};\epsilon_1,\epsilon_2,\Lambda)=\sum_{f\in\Gamma_{\vec{a}}^{discrete}} Z_f(\epsilon_1,\epsilon_2,\Lambda), \tag{2.21}$$

where $Z_f$ :=(2.20) and $\Gamma_{\vec{a}}^{discrete}$ is the set of the profile functions $f=f_{\vec{a},\vec{\mathbf{k}}}$.

Viewing (2.20) as the action written with the profile function, the profile function is known as the *density function* and the saddle point equation obtained from the action is known as the *loop equation* in the matrix model[8]. For $\epsilon_1,\epsilon_2\to 0$, (2.20) is expressed as

$$Z_f(\vec{a};\epsilon_1,\epsilon_2,\Lambda)\sim\exp\Big(\frac{\mathcal{E}_\Lambda(f)}{\epsilon_1\epsilon_2}\Big), \tag{2.22}$$

where

$$\mathcal{E}_\Lambda(f)=\frac{1}{4}\int_{y<x} dxdy\, f''(x)f''(y)(x-y)^2\Big(\log\Big(\frac{x-y}{\Lambda}\Big)-\frac{3}{2}\Big), \tag{2.23}$$

where the integral is defined as the Caucy's principal value one. This is the leading term of the action (2.20) as $\epsilon_{1,2}\to 0$. The prepotential $\mathcal{F}_0(\vec{a},\Lambda)$ is given by the saddle point of the action:

$$\mathcal{F}_0(\vec{a},\Lambda)=-\mathrm{Crit}_{f\in\Gamma_{\vec{a}}}\,\mathcal{E}_\Lambda(f), \tag{2.24}$$

where $\Gamma_{\vec{a}}$ denotes a set of the profile functions of the form

$$f(x)=\sum_{l=1}^{N} f_l(x-a_l), \tag{2.25}$$

[8]For the good reviews of matrix model, see e.g. [36, 37], and as that closest to our discussion we will see later, see [25].

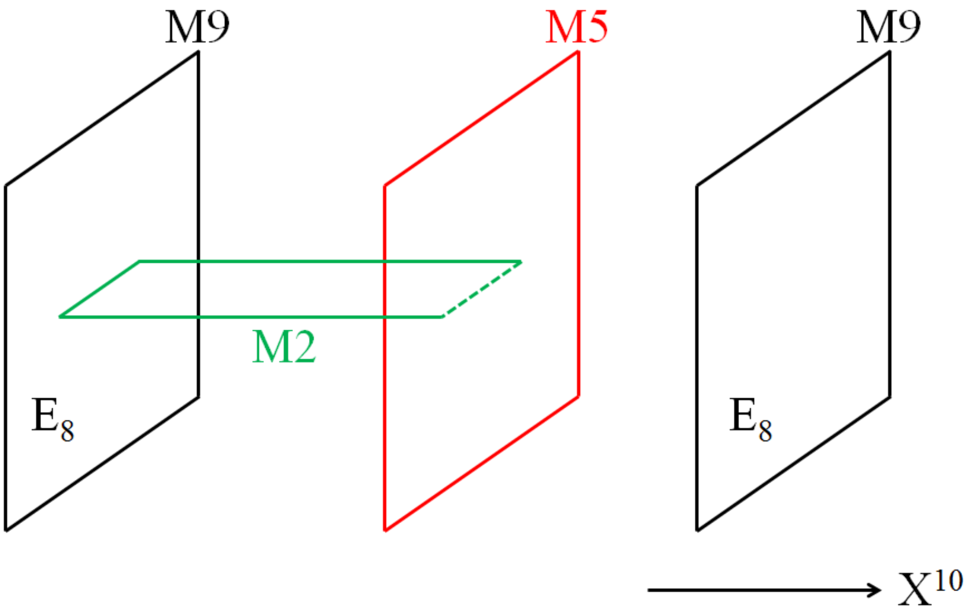


Figure 2: The formulation of E-string theory in terms of the M-theory picture. When the M5-brane approaches one of the M9-branes, the M2-brane becomes the non-critical, tensionless string.

with $f_l$ satisfying the conditions (2.19). This means, namely, that within all the profile functions which dominate the Nekrasov partition function, only the critical points of the space of the profile functions, namely only the dominant profile functions give the prepotential.

For our main purpose, we will follow this idea in the later sections. There, we will recall this idea again and will discuss more concretely.

# 3 E-string Theory

In this section, we formulate E-string theory in terms of M-theory picture. We firstly see how $E_8$ symmetries appear and then how E-string theory is defined.

## 3.1 Formulation of E-string Theory

As said in Introduction, the history of E-string theory was started by P. Horava and E. Witten[4, 5]. They considered what type of M-theory can reproduce $E_8 \times E_8$ heterotic superstring theory in the limit where the M-theory direction becomes zero. Now we see the story briefly following the discussions of [4, 5](see the papers for more details).

First of all, we consider M-theory on an orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $S^1$ as $X^{10} \to -X^{10}$ and on the worldsheet as the orientation reversal. This $\mathbb{Z}_2$ action breaks the original thirty-two supercharges to the half, i.e. sixteen supercharges. This means that if M-theory in the zero radius limit can reproduce one of the five known superstrings. It would be some one of Type I, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic superstrings.

Next, we consider the gravitational anomaly of M-theory on the orbifold. Then note that a metric on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ is same as one on $\mathbb{R}^{10} \times S^1$. We have a dynamical metric on the orbifold, so we have its superpartner which has its spin 3/2 and is commonly called the Rarita-Schwinger field[17]. Though we would like to consider to obtain the effective action by integrating out the Rarita-Schwinger field, this eleven-dimensional Rarita Schwinger field has the gravitational anomaly because it reduces in ten dimensions to a sum of infinitely many massive fields which are anomaly-free and the ten-dimensional Rarita-Schwinger field which is anomalous. Thus we have to know the form of the anomaly.

Under a spacetime diffeomorphism $\delta X^I = \epsilon v^I$ where $I$ runs over $0, 1, \cdots, 10$, $\epsilon$ is an infinitesimal quantity and $v^I$ a vector field, the change of the effective action $\delta\Gamma$ is generically written as

$$\delta\Gamma = i\epsilon \int_{\mathbb{R}^{10} \times S^1/\mathbb{Z}_2} d^{11}X \sqrt{g} v^I(X) W_I(X), \tag{3.1}$$

where $g$ is the eleven-dimensional metric and $W_I(X)$ a function on the orbifold. Thus that there does not exist the anomaly implies generically that $W_I(X) = 0$. However note that $X$ is not on the orbifold points. Hence on the orbifold points, i.e. $S^1/\mathbb{Z}_2 = [0, \pi]$, $W_I(X)$ is a sum of delta functions. We denote the orbifold points on where the delta functions are defined, i.e. hyperplanes which are called the M9-branes later, by $H'$ and $H''$ respectively, following [4]. Then (3.1) can be decomposed into two parts

$$\delta\Gamma = i\epsilon \int_{H'} d^{10}X \sqrt{g'} v^I W'_I + i\epsilon \int_{H''} d^{10}X \sqrt{g''} v^I W''_I, \tag{3.2}$$

where $'$ and $''$ means restrictions to $H'$ and $H''$ respectively. This is the standard ten-dimensional anomaly. Therefore contributions of $W_I'$ and $W_I''$ to the anomaly are even.

Here we have to care about that there are additional massless fields which live only on the hyperplanes. These are ten-dimensional vector multiplets. We have to count the number of these to cancel the anomaly. Though we have to recall the discussion of the Green-Schwarz mechanism[18] to proceed our discussion more precisely, we skip it and turn to the result directly.

To cancel the anomaly, we need 496 additional vector multiplets. This means that the superstring with sixteen supercharges has the gauge group whose dimension is 496. As noted above, since contributions of the two hyperplanes to the anomaly is even, 496 is divided by two, which implies 248 each. This result shows that 248 vector multiplets, i.e. the gauge group whose dimension is 248, live on each hyperplane. $E_8 \times E_8$ heterotic superstring is only allowed in this result.

The picture which we have seen above was that the $E_8$ gauge symmetry lives on each end-of-the-world brane, i.e. the M9-brane. We now picture an addition of an M2-brane and an M5-brane between the M9-branes(see the figure 2).

We consider the M2-brane as oscillating modes on the M5-brane[9]. When the M5-brane approaches one of the M9-branes(the left in the figure 2), the M2-brane looses one of three directions it spans and then it looks like a string. This string-like M2-brane is tensionless and non-critical[6, 20](and[10] [21]). Such a string is called the E-string. " E " of the name comes from the $E_8$ symmetry as follows: firstly, $E_8$ gauge fields live on the two M9-branes, but when the M5-brane approaches one of the M9-branes, another M9-brane is much far from the M5-brane. Hence, it is too far for the world on the M5-brane where the E-string lives that it can be neglected and its $E_8$ symmetry does not affect. Secondly, the $E_8$ gauge symmetry on the M9-brane near by the M5-brane becomes the global sym-

[9] We can consider multiple M5-branes or M2-branes and in those cases, such theories are called the *E-string theories*[19]. In this thesis, we are locked up in the usual E-string theory, i.e. one M2- and M5-brane.

[10] We have a comment on the references here. As mentioned in Introduction(footnote 2), before considering the $E_8 \times E_8$ heterotic superstring theory with small instantons, the case of $SO(32)$ heterotic superstring was considered by Witten[7]. $E_8 \times E_8$ and $SO(32)$ heterotic superstrings are not same but they are related with each other by T-duality, more precisely moduli spaces of them on $S^1$ are identical[22]. The string-like object was considered by Witten in type IIB superstring on $K3$[21] and was called the *non-critical string*[20]. Based on these ideas, Ganor and Hanany studied the tensionless, non-critical string, namely E-string[6].

metry because it is out of the M5-brane. Hence, E-string theory has the $E_8$ global symmetry and it is the origin of the name.

In addition, from the figure, it leaves only eight supercharges. By sending the parameter $\alpha'$ to zero whilst keeping the other physical parameters finite as done in the AdS/CFT correspondence[51], gravity is decoupled. Moreover, by taking the size of K3 to be large, the other modes except E-string are decoupled. Putting all together, E-string theory is the six-dimensional theory on the M5-brane with eight supercharges and the $E_8$ global symmetry.

Finally, we comment on fields in E-string theory. Actually, there is only one oscillating mode of the M2-brane. It is a tensor multiplet which decomposes into a vector multiplet when we perform dimensional reduction. In this sense, E-string theory is said to be the simplest theory as the six-dimensional(dynamical) theory. Nevertheless, we do not have its Lagrangian description yet, and in addition its least supersymmetry makes the control of E-string theory weak, so it is too difficult to analyse.

## 3.2 Two Prescriptions of E-string Theory

E-string theory is a (1,0) supersymmetric theory in six dimensions. The few supercharges, i.e. eight supercharges, makes us lose control. To obtain the $\mathcal{N} = 2$ theory in four dimensions by compactification, we must keep all the supercharges. This limitation leads us to toroidal compactification $\mathbb{R}^4 \times T^2$. However, this four-dimensional theory is basically asymptotically non-free[11]. Nevertheless, the theory has the Seiberg-Witten description[10]. It is given by

$$\begin{aligned} y^2 &= 4x^3 - \frac{E_4(\tau)}{12}u^4 x - \frac{E_6(\tau)}{216}u^6 + 4u^5, \\ \frac{\partial F_0}{\partial \varphi} &= 8\pi^3 i(\varphi_D - \tau\varphi) + \text{const.}, \end{aligned} \tag{3.3}$$

where $E_{4,6}$ are the Eisenstein series, $\tau$ a modulus of the torus, $\varphi, \varphi_D$ are the Higgs vev and its dual, $F_0$ the prepotential, and const. denotes terms which do not depend on $\varphi$. We will recall this description again in the next section. Tracing the history of $\mathcal{N} = 2$ field theory, we naturally arrive at one more description, namely the Nekrasov partition function. However, we can hope but can hardly obtain

[11] It depends on the number of hypermultiplets. For simplicity, we will treat the theory as being conformal in the subsequent discussions.

soon. Sakai hopefully searched and found it[13, 14]:

$$Z \;=\; \sum_{\mathbf{R}} (e^{-2\pi i\varphi})^{|\mathbf{R}|} \prod_{k=1}^{N} \prod_{(i,j)\in R_k} \frac{\prod_{n=1}^{2N} \vartheta_1(\frac{1}{2\pi}(a_k - m_n + (j-i)\hbar), \tau)}{\prod_{l=1}^{N} \vartheta_1(\frac{1}{2\pi}(a_{kl} + h_{k,l}(i,j)\hbar), \tau)^2}. \tag{3.4}$$

The details are given in the next section so we choose a short-cut. From this Nekrasov-type partition function, the prepotential is reproduced by

$$F_0 = (2\hbar^2 \ln Z)|_{\hbar=0}. \tag{3.5}$$

Here we would like to focus on the higher order terms by $\hbar$ expansion. The virtue of the Nekrasov partition function (2.12) we have seen in the last section is to include the contribution of graviphotons in the higher order terms[12]. How about in E-string theory? The answer is negative. The prepotential $F_0$ can be interpreted as the genus zero topological string amplitude on local $\frac{1}{2}K3$[34]. The all genus amplitude is given by

$$Z^{\frac{1}{2}K3} = \exp\Big(\sum_{g=0}^{\infty} \hbar^{2g-2} F_g^{\frac{1}{2}K3}\Big). \tag{3.6}$$

This amplitude satisfies the following holomorphic anomaly equation[13][35]

$$\partial_{E_2} Z^{\frac{1}{2}K3} = \frac{1}{24}\hbar^2\Big(\frac{1}{2\pi i}\partial_\varphi\Big)\Big(\frac{1}{2\pi i}\partial_\varphi + 1\Big)Z^{\frac{1}{2}K3}. \tag{3.7}$$

On the other hand, the Nekrasov-type partition function for E-string theory satisfies the following modular anomaly equation

$$\partial_{E_2} Z = \frac{1}{12}\hbar^2\Big(\frac{1}{2\pi i}\partial_\varphi\Big)^2 Z \tag{3.8}$$

with $\hbar$ expansion

$$Z = \exp\Big(\frac{1}{2\hbar^2}F_0 + O(\hbar^0)\Big). \tag{3.9}$$

[12]In this papar, we did not treat $\epsilon_{1,2}$ expansion of the Nekrasov partition function. However, in E-string theory, $\hbar$ expansion corresponds to graviphoton expansion. For the details, see e.g. [38, 39, 34].

[13]We here call holomorphic anomaly equation, following the associated paper. However, since the equation shows the dependence of the partition function on $E_2$, we call it modular anomaly equation later, following the associated paper.

For the genus zero topological string amplitude with massless hypermultiplets and the prepotential of E-string theory, (3.7) and (3.8) coincide with each other:

$$\partial_{E_2} F_0 = \frac{1}{24}\Big(\frac{1}{2\pi i}\partial_\varphi F_0\Big)^2. \tag{3.10}$$

Thus we have $F_0 = F_0^{\frac{1}{2}K3}|_{m_i=0}$. However, for their higher order terms the anomaly equations do not coincide. Hence we cannot say that the Nekrasov-type partition function for E-string theory include the contribution of graviphotons. Due to this reason, the discussions in the subsequent sections focus only on the genus zero part[14].

# 4 Nekrasov-type Partition Function For E-string Theory

Thus far, we have reviewed some basics of E-string theory and analysis methods for $\mathcal{N} = 2$ gauge theories in four dimensions. Historically, as we have seen in the last section, a major method for an analysis of E-string theory was the Seiberg-Witten description. We here give again the Seiberg-Witten description for E-string theory[10]

$$\begin{aligned} y^2 &= 4x^3 - \frac{E_4(\tau)}{12}u^4 x - \frac{E_6(\tau)}{216}u^6 + 4u^5, \\ \frac{\partial F_0}{\partial \varphi} &= 8\pi^3 i(\varphi_D - \tau\varphi) + \text{const.}, \end{aligned} \tag{4.1}$$

where $E_{4,6}$ are the Eisenstein series, $\tau$ a modulus of the torus, $\varphi, \varphi_D$ are the Higgs vev and its dual, and const. denotes terms which do not depend on $\varphi$.

In 2012, following the history, the Nekrasov-type partition function was appeared[13, 14]

$$\begin{aligned} Z &= \sum_{\mathbf{R}} (e^{-2\pi i\varphi})^{|\mathbf{R}|} \prod_{k=1}^{N} \prod_{(i,j)\in R_k} \frac{\prod_{n=1}^{2N} \vartheta_1(\frac{1}{2\pi}(a_k - m_n + (j-i)\hbar), \tau)}{\prod_{l=1}^{N} \vartheta_1(\frac{1}{2\pi}(a_{kl} + h_{k,l}(i,j)\hbar), \tau)^2}, \\ F_0 &= (2\hbar^2 \ln Z)|_{\hbar=0}, \end{aligned} \tag{4.2}$$

[14]The author is deeply indebted to the referee of [29].

where $\mathbf{R}$ is a set of Young diagrams $\mathbf{R} = \{R_1, \cdots, R_N\}$, $N$ comes from the $U(N)$ gauge theory[15] . $\vartheta_1(z,\tau)$ defined on the torus is one of the Jacobi theta functions defined as(see appendix B for more details)

$$\begin{aligned}
\vartheta_1(z,\tau) &:= i\sum_{n\in\mathbb{Z}}(-1)^n y^{n-1/2} q^{(n-1/2)^2/2}, \\
\vartheta_2(z,\tau) &:= \sum_{n\in\mathbb{Z}} y^{n-1/2} q^{(n-1/2)^2/2}, \\
\vartheta_3(z,\tau) &:= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2}, \\
\vartheta_4(z,\tau) &:= \sum_{n\in\mathbb{Z}}(-1)^n y^n q^{n^2/2}.
\end{aligned} \tag{4.3}$$

$m_n$ are fundamental matter masses and $h_{k,l}(i,j)$ are the relative hook lengths defined between Young diagrams $R_k$ and $R_l$. Most importantly, $\varphi,\ \tau$ and $a_l$ are absolutely different from (2.9). In (2.9), by $q \sim e^{2\pi i\tau}$ we had the UV coupling constant $\tau$ but we now have the *Higgs vev* $\varphi$ and $\tau$ is the modulus of the torus. Moreover, $a_l$ in (2.9) were diagonal components of the Higgs vev but now they are just constants on the torus. For consistency, we require a condition

$$2\sum_{k=1}^{N} a_k - \sum_{n=1}^{2N} m_n = 0. \tag{4.4}$$

And concretely, for E-string theory we set

$$N = 4, \quad a_k = \omega_{k-1} \ \ (k = 1\cdots,4), \quad m_n = -m_{n+4} \ \ (n = 1,\cdots,4). \tag{4.5}$$

This Nekrasov-type partition function was checked order by order and to satisfy some physical conditions. For instance, it satisfies the following modular anomaly equation

$$\partial_{E_2} Z = \frac{\hbar^2}{12}\frac{1}{(2\pi i)^2}\partial_{E_2}^2 Z. \tag{4.6}$$

[15]This is a confusing problem. By the toroidal compactification, we have the $\mathcal{N} = 2$ $U(1)$ gauge theory in four dimensions. However, the Nekrasov-type partition function can be viewed as the one generically for the $U(N)$ gauge theory with $2N$ fundamental matters[13, 25]. We do not have the answer to the puzzle between them yet. Hence $N$ does not have any physical meaning at present.

Substituting a relation between Nekrasov-type partition function and prepotential

$$Z = \exp\Big(\frac{1}{2}F_0\hbar^{-2} + O(\hbar^0)\Big) \tag{4.7}$$

for this, we get

$$\partial_{E_2}F_0 = \frac{1}{24}\Big(\frac{1}{2\pi i}\partial_\varphi F_0\Big)^2, \tag{4.8}$$

which is the modular anomaly equation obtained from the Seiberg-Witten description[24]. However it did not have any proof for all orders in [13, 14]. In this section, we give a proof[15] following [12].

## 4.1 Proof: Nekrasov and Okounkov approach

The basic idea of [12] is as follows. In the limit $\hbar \to 0$, which is called *thermodynamic* limit, we expect that there are some particular Young diagrams mainly contributing the partition function. So we firstly have to specify the diagrams. This is done by the matrix model-like approach(see footnote 8). Next, under that situation, we can find an elliptic function. This gives the Seiberg-Witten description. The story flows in this order below.

For our purpose, it is convenient to rewrite the Nekrasov-type partition function (4.2) as

$$\begin{aligned} Z &= \sum_{\mathbf{R}} e^{2\pi i\tilde{\varphi}|\mathbf{R}|}Z_{\mathbf{R}}, \\ Z_{\mathbf{R}} &= \prod_{k,l=1}^{N}\prod_{\substack{i,j=1 \\ (k,i)\neq(l,j)}}^{\infty}\frac{\vartheta_1(\frac{1}{2\pi}(a_{kl}+(\mu_{k,i}-\mu_{l,j}+j-i)\hbar))}{\vartheta_1(\frac{1}{2\pi}(a_{kl}+(j-i)\hbar))} \\ &\quad\times\prod_{k=1}^{N}\prod_{n=1}^{2N}\prod_{(i,j)\in R_k}\vartheta_1(\tfrac{1}{2\pi}(a_k-m_n+(j-i)\hbar)), \end{aligned} \tag{4.9}$$

where

$$\tilde{\varphi} := \begin{cases} \varphi & \text{if } N \text{ is odd} \\ \varphi+\frac{1}{2} & \text{if } N \text{ is even} \end{cases}. \tag{4.10}$$

Now we introduce a function $\gamma(z;\hbar)$ which satisfies a difference equation

$$\gamma(z+\hbar;\hbar)+\gamma(z-\hbar;\hbar)-2\gamma(z;\hbar) = \ln\vartheta_1\Big(\frac{z}{2\pi}\Big) \tag{4.11}$$

and which has the expansion(see appendix A for more details)

$$\gamma(z;\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2}\gamma_g(z). \tag{4.12}$$

Most importantly, we have the fact

$$\gamma_0''(z) = \ln\vartheta_1\Big(\frac{z}{2\pi}\Big). \tag{4.13}$$

Then, using this function, we can rewrite $Z_{\mathbf{R}}$ as

$$\begin{aligned} Z_{\mathbf{R}} &= \exp\Bigg[-\frac{1}{4}\int dzdw f''(z)f''(w)\gamma(z-w;\hbar) + \frac{1}{2}\sum_{n=1}^{2N}\int dz f''(z)\gamma(z-m_n;\hbar) \\ &+ \sum_{k,l=1}^{N}\gamma(a_k-a_l;\hbar) - \sum_{k=1}^{N}\sum_{n=1}^{2N}\gamma(a_k-m_n;\hbar)\Bigg], \end{aligned} \tag{4.14}$$

where the integrals are defined by the Cauchy's principal value ones. The function $f(z)$ which is called a $profile\ function$ or its second derivative

$$\begin{aligned} f(z) &= \sum_{k=1}^{N}\Bigg[\sum_{i=1}^{\ell(R_k)}(|z-a_k-\hbar(\mu_{k,i}-i+1)| - |z-a_k-\hbar(\mu_{k,i}-i)|) \\ &\quad +|z-a_k+\hbar\ell(R_k)|\Bigg], \\ f''(z) &= 2\sum_{k=1}^{N}\Bigg[\sum_{i=1}^{\ell(R_k)}(\delta(z-a_k-\hbar(\mu_{k,i}-i+1)) - \delta(z-a_k-\hbar(\mu_{k,i}-i)) \\ &\quad +\delta(z-a_k+\hbar\ell(R_k)))\Bigg] \\ &= 2\sum_{k=1}^{N}\Bigg[\sum_{i=1}^{\infty}(\delta(z-a_k-\hbar(\mu_{k,i}-i+1)) - \delta(z-a_k-\hbar(\mu_{k,i}-i)) \\ &\quad -\delta(z-a_k+\hbar(i-1)) + \delta(z-a_k-\hbar i)) + \delta(z-a_k)\Bigg] \end{aligned} \tag{4.15}$$

knows Young diagrams mainly contributing to the partition function (4.14) since the delta functions within the profile function characterise the shapes of them[12]. Such a function $f''(z)$ can be viewed as a density function in matrix model. We assume that they are separated from each other. Let $z = a_k$ be points where the delta functions take values, $C_k$ be the local support around them, and $C$ their union $C = \bigcup_{k=1}^{N} C_k$. Then it follows that

$$\begin{aligned} a_k &= \frac{1}{2}\int_{C_k} z f''(z) dz, \\ |\mathbf{R}| &= \frac{1}{4}\int_C dz z^2 f''(z) - \sum_{k=1}^{N} \frac{a_k^2}{2}. \end{aligned} \tag{4.16}$$

Then the original partition function (4.9) can be approximated by an integral over the space of the delta functions

$$Z \simeq \int \mathcal{D} f'' d^N \lambda \exp\Big(\frac{1}{2\hbar^2}\mathcal{F}_0 + O(\hbar^0)\Big), \tag{4.17}$$

where

$$\begin{aligned} \mathcal{F}_0[f'', \lambda_k] &= -\frac{1}{2}\int_C dz dw f''(z) f''(w) \gamma_0(z-w) + \sum_{n=1}^{2N} \int_C dz f''(z) \gamma_0(z - m_n) \\ &\quad + 4\pi i \tilde{\varphi}\Big(\frac{1}{4}\int_C dz z^2 f''(z) - \sum_{k=1}^{N} \frac{a_k^2}{2}\Big) \\ &\quad + 2\sum_{k=1}^{N} \lambda_k \Big(\frac{1}{2}\int_{C_k} dz z f''(z) - a_k\Big), \end{aligned} \tag{4.18}$$

where the integrals of the first line are defined by the Cauchy's principal value ones and we have introduced Lagrange multipliers $\lambda_k$ taking account of the constraints (4.16). Here we have to care about the function $\mathcal{F}_0$, which is slightly different from the prepotential $F_0$ as we will see later.

In this situation, we take the thermodynamic limit $\hbar \to 0$. Then, there are some dominant Young diagrams in (4.17) and the delta functions know them. We evaluate the partition function (4.17) by the saddle point approximation. We obtain, by the variation of $\mathcal{F}_0$ with respect to $f''(z)$

$$\int_C dw f''(w) \gamma_0(z-w) - \sum_{n=1}^{2N} \gamma_0(z - m_n) - \pi i \tilde{\varphi} z^2 - \lambda_k z = 0, \quad z \in C_k. \tag{4.19}$$

This saddle point equation can be viewed as the *loop equation* in matrix model. We would like to solve this equation but it is generically a big problem. Here we introduce an analytic function

$$\begin{aligned}\Omega(z) \;&:=\; \int_C f''(w)\gamma_0''(z-w)dw - \sum_{n=1}^{2N}\gamma_0''(z-m_n)\\ &=\; \int_C f''(w)\ln\vartheta_1\Big(\frac{z-w}{2\pi}\Big)dw - \sum_{n=1}^{2N}\ln\vartheta_1\Big(\frac{z-m_n}{2\pi}\Big). \end{aligned}\tag{4.20}$$

We use this function instead of $f''$ to solve the saddle point equation. Moreover, recalling matrix model, we define the resolvent $\omega(z)$ using the function above as

$$\omega(z) := \Omega'(z). \tag{4.21}$$

By this definition, we called the function $\Omega(z)$ the *antiderivative of the resolvent* in the paper[15]. Then the function $f''$ is recovered as

$$2\pi i f''(z) = \omega(z-i\epsilon) - \omega(z+i\epsilon), \quad z \in C, \tag{4.22}$$

where $\epsilon = \delta z$ is an infinitesimal deformation along the cuts.

We consider the second derivative of the saddle point equation (4.19) with respect to $z$:

$$\frac{1}{2}(\Omega(z-i\epsilon) + \Omega(z+i\epsilon)) - 2\pi i\tilde{\varphi} = 0, \quad z \in C. \tag{4.23}$$

We now solve this. To do it, we introduce a meromorphic function on the torus, whose poles are at $z = m_n$

$$G(z) := e^{\Omega(z)-2\pi i\tilde{\varphi}} + e^{-\Omega(z)+2\pi i\tilde{\varphi}}, \tag{4.24}$$

whilst the function $\Omega(z)$ has logarithmic branch points as well as square root branch points. By the condition (4.4), $G(z)$ is doubly periodic, i.e. it is an elliptic function of order $2N$ on the torus. Using this function, the resolvent can be written as

$$\omega(z) = \frac{G'(z)}{\sqrt{(G(z)+2)(G(z)) - 2}}. \tag{4.25}$$

Let us count the number of the branch points. Since $G(z) \pm 2$ have $2N$ branch points each, $\omega(z)$ has totally $4N$ branch points. However, the actual $\omega(z)$ should have $2N$ branch points. This mismatch is resolved if the function

$$H(z) := \frac{G(z)+2}{4} = \cosh^2\Big(\frac{1}{2}(\Omega(z) - 2\pi i\tilde{\varphi})\Big) \tag{4.26}$$

has $N$ zeroes of multiplicity two instead of $2N$ simple zeroes. The singularities of $H(z)$ are the single poles at $z = m_n$. Such an elliptic function is determined as

$$H(z) = \kappa \frac{P(z)^2}{Q(z)} = \kappa \frac{(\prod_{k=1}^{N} \vartheta_1(\frac{z-\zeta_k}{2\pi}))^2}{\prod_{n=1}^{2N} \vartheta_1(\frac{z-m_n}{2\pi})}, \tag{4.27}$$

where $\kappa$ and $\zeta_k$ are some constants. The locations of zeroes and poles have to satisfy

$$2\sum_{k=1}^{N} \zeta_k - \sum_{n=1}^{2N} m_n = 0. \tag{4.28}$$

Here the equality should be understood modulo periods of the torus. Then the antiderivative $\Omega(z)$ is obtained as

$$\Omega(z) = 2\ln\left(\sqrt{H(z)} + \sqrt{H(z) - 1}\right) + 2\pi i\tilde{\varphi}, \tag{4.29}$$

and therefore, the resolvent is obtained as

$$\omega(z) = \frac{2\partial_z \sqrt{H(z)}}{\sqrt{H(z) - 1}}. \tag{4.30}$$

Finally, we make a comment on the constant $\zeta_k$. The function $f''$ has to satisfy the constraint (4.16). In terms of the resolvent, it is expressed as

$$a_k = \frac{1}{4\pi i} \oint_{\gamma_k} z\omega(z)dz. \tag{4.31}$$

This holds if $\omega(z)$ satisfies

$$\omega(a_k - z \pm i\epsilon) = \omega(a_k + z \pm i\epsilon) \quad \text{for} \;\; a_k + z \in C_k. \tag{4.32}$$

This holds if the function $H(z)$ satisfies

$$\sqrt{H(a_k - z)} = -\sqrt{H(a_k + z)} \quad \text{for} \;\; a_k + z \in C_k. \tag{4.33}$$

By requiring this property, the values of $\zeta_k$ are fixed.

## 4.2 The case of E-string theory

We here focus on the case of E-string theory, i.e. with the $E_8$ global symmetry. In this case, the setup is given by

$$N = 4, \quad \zeta_k = (0, \pi, -\pi - \pi\tau, \pi\tau), \quad m_n = (0, 0, 0, 0) \tag{4.34}$$

or[13]

$$N = 3, \quad \zeta_k = \omega_k = (\pi, -\pi - \pi\tau, \pi\tau), \quad m_n = (0, 0, 0). \tag{4.35}$$

We here choose the latter. Then the functions $P(z)$ and $Q(z)$ are expressed as

$$P(z) = -iq^{-1/4} \prod_{k=1}^{3} \vartheta_{k+1}\Big(\frac{z}{2\pi}\Big), \quad Q(z) = \vartheta_1\Big(\frac{z}{2\pi}\Big)^6. \tag{4.36}$$

Therefore the function $H(z)$ is written as

$$H(z) = -\frac{1}{4} u \wp'(z)^2, \tag{4.37}$$

where we have used the identity

$$\wp'(z)^2 = \eta^{12} \prod_{k=1}^{3} \frac{\vartheta_{k+1}(\frac{z}{2\pi})^2}{\vartheta_1(\frac{z}{2\pi})^2}. \tag{4.38}$$

Here $\wp'(z)$ is the derivative of the Weierstrass' elliptic function and $\eta := \eta(\tau)$ the Jacobi eta function. Also, we have defined the parameter $u$ as

$$u := \frac{4\kappa}{q^{1/2}\eta^{12}}. \tag{4.39}$$

Then the resolvent is written as

$$\omega(z) = \frac{2\wp''(z)}{\sqrt{\wp'(z)^2 + 4u^{-1}}}. \tag{4.40}$$

This Riemann surface has three cuts near $z = \omega_k$ and the three cuts shrinks as $|u|$ increases. In particular, when $u$ is sent to infinity, all cuts disappear and the Riemann surface becomes the torus. This is reminiscent of the classical limit of the Seiberg-Witten curve (4.1) and therefore lets us identify $u$ with the Coulomb branch moduli parameter.

Next we would like to consider the Seiberg-Witten description, i.e. the $\alpha$- and $\beta$-cycle integrals. We first consider the $\alpha$-cycle integral. To do this, we use the following fact

$$\frac{1}{2\pi^2 i}\oint_\alpha \ln\vartheta_1\Big(\frac{z-w}{2\pi}\Big)dz = C_1(\tau) \mod \mathbb{Z}, \tag{4.41}$$

where $C_1(\tau)$ is some function in $\tau$. An important thing is not the explicit form of $C_1(\tau)$ but that $C_1(\tau)$ is independent of $w$ and invariant under continuous deformation of the integration contour. This fact can be shown as follows: since the theta function is quasi-periodic $\vartheta_1(z+1) = -\vartheta_1(z)$, the function $\frac{1}{2\pi i}\ln\vartheta_1(\frac{z-w}{2\pi})^2$ is single-valued modulo $\mathbb{Z}$ along a loop belonging to the cycle $\alpha$. Recall also that the theta function is regular for $|z| < \infty$, so that the integral is invariant under the continuous deformation of the loop.

Now we use the fact (4.41) with the function (4.20). And also we know $f''$ to be a set of the delta functions from (4.15). Hence we obtain

$$\begin{aligned} \frac{1}{2\pi^2 i}\oint_\alpha \Omega(z)dz &= \frac{1}{2\pi^2 i}\oint_\alpha\int_C f''(w)\ln\vartheta_1\Big(\frac{z-w}{2\pi}\Big)dwdz \\ &\quad - \frac{1}{2\pi^2 i}\oint_\alpha\sum_{n=1}^{2N}\ln\vartheta_1\Big(\frac{z-m_n}{2\pi}\Big)dz \mod \mathbb{Z}, \\ \Longrightarrow \quad \frac{1}{4\pi^2 i}\oint_\alpha \Omega(z)dz &= 0 \mod \mathbb{Z}, \end{aligned} \tag{4.42}$$

where $C_1$'s cancel with each other. Here using the relation between $\Omega(z)$ and $H(z)$, (4.29), we obtain

$$\tilde{\varphi} = \frac{i}{2\pi^2}\oint_\alpha \ln\Big(\sqrt{H(z)} + \sqrt{H(z)-1}\Big)dz \mod \mathbb{Z}. \tag{4.43}$$

Now we have arrived at the important stage where we give the Seiberg-Witten description explicitly. Recalling that the function $H(z)$ includes the Coulomb moduli parameter $u$, we differentiate the Higgs vev (4.43) with respect to $u$:

$$\frac{\partial\varphi}{\partial u} = \frac{i}{4\pi^2 u}\oint_\alpha \frac{dz}{\sqrt{1-H(z)^{-1}}}. \tag{4.44}$$

Note that this is not affected by the difference between $\varphi$ and $\tilde{\varphi} = \varphi + \frac{1}{2}$. In the case of the E-string theory with the $E_8$ global symmetry, the function $H(z)$ was given by (4.37). Therefore we get

$$\frac{\partial\varphi}{\partial u} = \frac{i}{4\pi^2 u}\oint_\alpha \frac{\wp'(z)dz}{\sqrt{\wp'(z)^2 + 4u^{-1}}}. \tag{4.45}$$

The Seiberg-Witten curve should be given as the Riemann surface of the integrand. However, we have to pay attention to that the double-periodic sheet has three cuts near $z = \omega_k$ and we have two copies of it. Hence, whilst the Seiberg-Witten curve is of genus one, the Riemann surface of the integrand is of genus four. To solve this problem, we use the identity

$$\wp'(z)^2 = 4\wp(z)^3 - \frac{E_4(\tau)}{12}\wp(z) - \frac{E_6(\tau)}{216} \tag{4.46}$$

and perform a change of variables as

$$\wp(z) = u^{-2}x. \tag{4.47}$$

Then (4.45) is expressed as

$$\frac{\partial\varphi}{\partial u} = \frac{i}{4\pi^2}\oint_{\tilde\alpha} \frac{dx}{y}, \tag{4.48}$$

where $\tilde\alpha$ is the image of $\alpha$ by the map (4.47) and $y$ is given by

$$y^2 = 4x^3 - \frac{E_4(\tau)}{12}u^4 x - \frac{E_6(\tau)}{216}u^6 + 4u^5. \tag{4.49}$$

This is exactly the Seiberg-Witten curve for the E-string theory (4.1). Thus one of the Seiberg-Witten description has been reproduced from the Nekrasov-type partition function.

Next we would like to reproduce the relation between the prepotential and the Higgs vev, i.e. consider the beta-cycle integral. To do this, we need two ingredients: the modular transformation law of the theta function

$$\vartheta_1\Big(\frac{z}{2\pi},\tau\Big) = e^{3\pi i/4}\tau^{-1/2}\exp\Big(-\frac{iz^2}{4\pi\tau}\Big)\vartheta_1\Big(\frac{z}{2\pi\tau},-\frac{1}{\tau}\Big) \tag{4.50}$$

and the fact (4.41) with the modulus $-1/\tau$. Then we can show that

$$\begin{aligned}
&\frac{1}{2\pi^2 i\tau}\int_{z_0}^{z_0+2\pi\tau}\ln\vartheta_1\Big(\frac{z-w}{2\pi},\tau\Big)dz \\
= \;& -\frac{1}{8\pi^3\tau^2}\int_{z_0}^{z_0+2\pi\tau}(z-w)^2 dz \\
&+\frac{1}{2\pi^2 i\tau}\int_{z_0}^{z_0+2\pi\tau}\ln\vartheta_1\Big(\frac{z-w}{2\pi\tau},-\frac{1}{\tau}\Big)dz+\frac{3}{4}-\frac{1}{2\pi i}\ln\tau \\
= \;& -\frac{1}{8\pi^3\tau^2}\int_{z_0}^{z_0+2\pi\tau}(z-w)^2 dz + C_1\Big(-\frac{1}{\tau}\Big)+\frac{3}{4}-\frac{1}{2\pi i}\ln\tau \mod \mathbb{Z} \\
= \;& -\frac{1}{4\pi^2\tau}w^2+\Big(\frac{1}{2\pi}+\frac{z_0}{2\pi^2\tau}\Big)w + C_2(z_0,\tau) \mod \mathbb{Z},
\end{aligned} \tag{4.51}$$

where $C_2(z_0,\tau)$ is some function in $z_0$ and $\tau$. Putting this, (4.15) and (4.20) together, we obtain

$$\begin{aligned}
&\frac{1}{4\pi^2 i\tau}\int_{z_0}^{z_0+2\pi\tau}\Omega(z)dz \\
= \;& -\frac{1}{8\pi^2\tau}\int_C w^2 f''(w)dw+\Big(\frac{1}{4\pi}+\frac{z_0}{4\pi^2\tau}\Big)\int_C w f''(w)dw \mod \mathbb{Z} \\
= \;& \frac{i}{8\pi^3\tau}\Big(\frac{\partial F_0}{\partial\varphi}+2\pi i\sum_{k=1}^N a_k^2\Big)+\Big(\frac{1}{2\pi}+\frac{z_0}{2\pi^2\tau}\Big)\sum_{k=1}^N a_k \mod \mathbb{Z}.
\end{aligned} \tag{4.52}$$

$C_2$'s cancel with each other in the first equality and the integrals with respect to $w$ are defined by the Caucy's principal value ones. To show the second equality, we have used (4.16) and (4.18) with

$$\frac{\partial F_0}{\partial\varphi}=\frac{\partial\mathcal{F}_0}{\partial\varphi}\bigg|_{\text{extremum}}=\bigg[\Big(\frac{\partial\mathcal{F}_0}{\partial\varphi}\Big)_{f''}+\Big(\frac{\delta\mathcal{F}_0}{\delta f''}\Big)_{\varphi}\frac{\partial f''}{\partial\varphi}\bigg]_{\text{extremum}}=\bigg[\Big(\frac{\partial\mathcal{F}_0}{\partial\varphi}\Big)_{f''}\bigg]_{\text{extremum}}. \tag{4.53}$$

Here $(\partial\mathcal{F}_0/\partial\varphi)_{f''}$ denotes the partial derivative of $\mathcal{F}_0$ with respect to $\varphi$, holding $f''$ constant.

Now we focus on the E-string theory with the $E_8$ global symmetry. Since the setup

$$N=3, \quad \zeta_k=a_k=\omega_k, \quad m_n=(0,0,0), \tag{4.54}$$

the integral (4.52) becomes

$$\begin{aligned}
&\frac{1}{4\pi^2 i\tau}\int_{z_0}^{z_0+2\pi\tau}\Omega(z)dz \\
=\ & \frac{i}{8\pi^3\tau}\Big(\frac{\partial F_0}{\partial\varphi}+2\pi i\sum_{k=1}^{3}\omega_k^2\Big).
\end{aligned} \tag{4.55}$$

Hence the integral does not depend on the choice of $z_0$ and this result makes the integral the beta-cycle one, i.e.

$$\begin{aligned}
&\frac{1}{4\pi^2 i\tau}\oint_\beta\Omega(z)dz \\
=\ & \frac{i}{8\pi^3\tau}\Big(\frac{\partial F_0}{\partial\varphi}+2\pi i\sum_{k=1}^{3}\omega_k^2\Big).
\end{aligned} \tag{4.56}$$

Now we have arrived at the point where we can obtain the Seiberg-Witten description. Firstly, using (4.29), the l.h.s. of (4.56) is written as

$$\begin{aligned}
\frac{1}{4\pi^2 i\tau}\oint_\beta\Omega(z)dz &= \frac{1}{2\pi^2 i\tau}\oint_\beta \ln\big(\sqrt{H(z)}+\sqrt{H(z)-1}\big)dz+\tilde{\varphi} \\
&= -\frac{1}{\tau}\varphi_D+\varphi+\text{const.},
\end{aligned} \tag{4.57}$$

where we have identified the *dual* Higgs vev $\varphi_D$ as

$$\varphi_D=\frac{i}{2\pi^2}\oint_\beta \ln\big(\sqrt{H(z)}+\sqrt{H(z)-1}\big)dz+\text{const.} \tag{4.58}$$

from the analogy of (4.43). Here const.'s are some functions in $\tau$ but are independent of $\varphi$. Secondly, the summation term of the r.h.s. of (4.56) can be written as const. because $\omega_k$ consists of $\pi$ and $\tau$ only. Puttinig all together, hence, we finally obtain

$$\frac{\partial F_0}{\partial\varphi}=8\pi^3 i(\varphi_D-\tau\varphi)+\text{const.}. \tag{4.59}$$

The Seiberg-Witten description has completely been reproduced in here.

# 5 The Generalisation to the Cases with Wilson Lines

In the previous section, we proved that the Nekrasov-type partition function for E-string theory is correct, namely starting the Nekrasov-type partition function we can reproduce the Seiberg-Witten description. E-string theory we have seen is the simplest case in the sense that the theory does not have any Wilson lines. Therefore the natural question arises: is the Nekrasov-type partition function also correct in the cases with Wilson lines? In this section, we answer the question *positively* in a sense that the Nekrasov-type partition function with Wilson lines can reproduces the Seiberg-Witten *curve*. However, as mentioned in section 3, the higher order terms in the $\hbar$ expansion cannot interpret the graviphoton expansion as in the original Nekrasov partition function. Hence we stress again that we focus on the genus zero part. We discuss the cases with three Wilson lines firstly and the cases with four Wilson lines secondly. And also, we focus only on the alpha-cycle integral (4.44).

## 5.1 The cases with three Wilson lines

In these cases, we choose[16]

$$N = 3, \quad \zeta_k = \omega_k, \quad m_n = (2\pi m_1, 2\pi m_2, 2\pi m_3). \tag{5.1}$$

Then the function $H(z)$ is written as

$$\begin{aligned} H(z) &= \kappa \frac{\left(\prod_{k=1}^{3} \vartheta_1\left(\frac{z-\zeta_k}{2\pi}\right)\right)^2}{\prod_{n=1}^{6} \vartheta_1\left(\frac{z-2\pi m_n}{2\pi}\right)} \\ &= \kappa \frac{\vartheta_1\left(\frac{z-\pi}{2\pi}\right)^2 \vartheta_1\left(\frac{z+\pi+\pi\tau}{2\pi}\right)^2 \vartheta_1\left(\frac{z-\pi\tau}{2\pi}\right)^2}{\vartheta_1\left(\frac{z-2\pi m_1}{2\pi}\right)^2 \vartheta_1\left(\frac{z-2\pi m_2}{2\pi}\right)^2 \vartheta_1\left(\frac{z-2\pi m_3}{2\pi}\right)^2 \vartheta_1\left(\frac{z+2\pi m_1}{2\pi}\right)^2 \vartheta_1\left(\frac{z+2\pi m_2}{2\pi}\right)^2 \vartheta_1\left(\frac{z+2\pi m_3}{2\pi}\right)^2}. \end{aligned} \tag{5.2}$$

We need two tools here. One, we need the transformation laws of the theta functions which are given in Appendix B for the numerator. And two, we need the identity

$$\vartheta_1\Big(\frac{z+w}{2\pi}\Big)\vartheta_1\Big(\frac{z-w}{2\pi}\Big) = -\eta^{-6}\vartheta_1\Big(\frac{z}{2\pi}\Big)^2\vartheta_1\Big(\frac{w}{2\pi}\Big)^2(\wp(z) - \wp(w)) \tag{5.3}$$

[16] In this section, we take the Wilson lines to be not $m_n$ but $2\pi m_n$ for convenience.

for the denominator. Then we can express the function (5.2) written in terms of the theta functions which depend on $z$ as the Weierstrass $\wp$-functions and obtain

$$\begin{aligned} H(z) &= \frac{\kappa\eta^6\wp'(z)^2}{q^{1/2}\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2(\wp(z)-\wp(2\pi m_1))(\wp(z)-\wp(2\pi m_2))(\wp(z)-\wp(2\pi m_3))} \\ &= \frac{u\eta^{18}\wp'(z)^2}{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2(\wp(z)-\wp(2\pi m_1))(\wp(z)-\wp(2\pi m_2))(\wp(z)-\wp(2\pi m_3))}, \end{aligned} \tag{5.4}$$

where we have used the moduli parameter (4.39). To proceed the story following the last section, we would like to identify the Seiberg-Witten curve in the alpha-cycle integral (4.44) in the present case. The integrand of the alpha-cycle integral (4.44) is now written as

$$\frac{dz}{\sqrt{1-H(z)^{-1}}} = \frac{\sqrt{H(z)}dz}{\sqrt{H(z)-1}} = \frac{\wp'(z)dz}{\sqrt{\wp'(z)^2-\alpha(m)(\wp-\wp_1)(\wp-\wp_2)(\wp-\wp_3)}}, \tag{5.5}$$

where

$$\begin{aligned} \alpha(m) &:= \frac{4}{u\eta^{18}}\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2, \\ \wp_i &:= \wp(2\pi m_i). \end{aligned} \tag{5.6}$$

By a change of variables $\wp(z) = x$, we identify this integrand written in terms of the variable $x$ with the Seiberg-Witten curve, i.e. we put

$$\begin{aligned} y_0^2 &= \wp'(z)^2 - \alpha(m)(\wp(z)-\wp_1)(\wp(z)-\wp_2)(\wp(z)-\wp_3) \\ &= (4-\alpha)\wp^3 + \alpha\sigma_1\wp^2 - (E_4+\alpha\sigma_2)\wp - (E_6-\alpha\sigma_3), \end{aligned} \tag{5.7}$$

where

$$\sigma_1 := \wp_1+\wp_2+\wp_3, \quad \sigma_2 := \wp_1\wp_2+\wp_2\wp_3+\wp_1\wp_3, \quad \sigma_3 := \wp_1\wp_2\wp_3, \tag{5.8}$$

and we have dropped the coefficients of $E_4$ and $E_6$, namely we defined $E_4' := E_4/12$ and $E_6' := E_6/216$ and then we dropped the prime. For later convenience, we change the variable $z$ into $x$ below. We have to now recall that the sheet which the functions are defined has three cuts and we have two copies of them. Hence the curve (5.7) is of genus four. To obtain the genus one curve, taking into account that the curve (5.7) is in $\mathbb{CP}^2$, we perform the appropriate change of variables as

$$x := \alpha^{-2}(4-\alpha)\wp, \quad y := 2(4-\alpha)\alpha^{-3}y_0. \tag{5.9}$$

More concretely and explicitly, we follow the computation minutely and so we take five steps to make the curve (5.7) be the correct Seiberg-Witten curve: one is to redefine the variables, two to eliminate the second-order term, three to redefine the variables, four to didvide by $\alpha^6$ and redefine the variables to rewrite $\alpha$ into the modulus $u$, and five to express the curve in terms of the modulus $u$. First step, we multiply both sides of (5.7) by $(4-\alpha)^2$ and perform a change of variables $x_0 := (4-\alpha)\wp$:

$$(4-\alpha)^2 y_0^2 = x_0^3 + \alpha\sigma_1 x_0^2 - (E_4 + \alpha\sigma_2)(4-\alpha)x_0 - (4-\alpha)^2(E_6 - \alpha\sigma_3). \quad (5.10)$$

Second step, we perform a shift $x_0 = x - \alpha\sigma_1/3$ to eliminate the square term of $x_0$:

$$\begin{aligned}(4-\alpha)^2 y_0^2 = \Big(x - \frac{\alpha\sigma_1}{3}\Big)^3 + \alpha\sigma_1\Big(x - \frac{\alpha\sigma_1}{3}\Big)^2 \\ -(E_4 + \alpha\sigma_2)(4-\alpha)\Big(x - \frac{\alpha\sigma_1}{3}\Big) - (4-\alpha)^2(E_6 - \alpha\sigma_3).\end{aligned} \quad (5.11)$$

From this, we get the curve

$$(4-\alpha)^2 y_0^2 = x^3 - \tilde{f}x - \tilde{g}. \quad (5.12)$$

Third step, multiplying both sides of (5.12) by four and redefining the variable $y_0$ as $y := 2(4-\alpha)y_0$, we get the curve

$$y^2 = 4x^3 - fx - g, \quad (5.13)$$

where $f := 4\tilde{f}$ and $g := 4\tilde{g}$:

$$\begin{aligned} f &= 16E_4 + (16\sigma_2 - 4E_4)\alpha + \Big(\frac{4\sigma_1^2}{3} - 4\sigma_2\Big)\alpha^2, \\ g &= 64E_6 - \Big(\frac{16}{3}E_4\sigma_1 + 32E_6 + 64\sigma_3\Big)\alpha \\ &+ \Big(4E_6 + \frac{4}{3}E_4\sigma_1 - \frac{16}{3}\sigma_1\sigma_2 + 32\sigma_3\Big)\alpha^2 - \Big(\frac{8}{27}\sigma_1^3 - \frac{4}{3}\sigma_1\sigma_2 + 4\sigma_3\Big)\alpha^3. \end{aligned} \quad (5.14)$$

Fourth step, dividing both side of (5.13) by $\alpha^6$, the curve (5.13) becomes

$$\frac{y^2}{\alpha^6} = 4\frac{x^3}{\alpha^6} - \frac{f}{\alpha^4}\frac{x}{\alpha^2} - \frac{g}{\alpha^6}. \quad (5.15)$$

We redefine the variables as

$$\tilde{y} := y/\alpha^3, \quad \tilde{x} := x/\alpha^2. \tag{5.16}$$

Then the curve (5.15) is written as

$$\tilde{y} = 4\tilde{x}^3 - f'\tilde{x} - g', \tag{5.17}$$

where $f' := f/\alpha^4$ and $g' := g/\alpha^6$:

$$\begin{aligned} f' &= 16E_4\alpha^{-4} + (16\sigma_2 - 4E_4)\alpha^{-3} + \Big(\frac{4\sigma_1^2}{3} - 4\sigma_2\Big)\alpha^{-2}, \\ g' &= 64E_6\alpha^{-6} - \Big(\frac{16}{3}E_4\sigma_1 + 32E_6 + 64\sigma_3\Big)\alpha^{-5} \\ &+ \Big(4E_6 + \frac{4}{3}E_4\sigma_1 - \frac{16}{3}\sigma_1\sigma_2 + 32\sigma_3\Big)\alpha^{-4} - \Big(\frac{8}{27}\sigma_1^3 - \frac{4}{3}\sigma_1\sigma_2 + 4\sigma_3\Big)\alpha^{-3}. \end{aligned} \tag{5.18}$$

Thus far, we have taken four steps to get the correct Seiberg-Witten curve, i.e. we wanted the genus one curve, the Weierstrass form, and the expression in terms of the modulus $u$ as the ingredients. To rewrite $\alpha$ into the modulus $u$ is the rest of the steps. Fifth step, finally, we do it. However, that we have stopped here is not that we waste the time because by this step an important result is shown. Now we recall $\alpha(m) := 4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2/u\eta^{18}$. So we define the new modulus $\tilde{u}$ as

$$\tilde{u} := \alpha^{-1} = \frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u. \tag{5.19}$$

Then (5.18) is rewritten as

$$\begin{aligned} f' &= 16E_4\tilde{u}^4 + (16\sigma_2 - 4E_4)\tilde{u}^3 + \Big(\frac{4\sigma_1^2}{3} - 4\sigma_2\Big)\tilde{u}^2, \\ g' &= 64E_6\tilde{u}^6 - \Big(\frac{16}{3}E_4\sigma_1 + 32E_6 + 64\sigma_3\Big)\tilde{u}^5 \\ &+ \Big(4E_6 + \frac{4}{3}E_4\sigma_1 - \frac{16}{3}\sigma_1\sigma_2 + 32\sigma_3\Big)\tilde{u}^4 - \Big(\frac{8}{27}\sigma_1^3 - \frac{4}{3}\sigma_1\sigma_2 + 4\sigma_3\Big)\tilde{u}^3. \end{aligned} \tag{5.20}$$

This curve is the Seiberg-Witten curve we wanted. We can check the correctness by comparing this curve with the curve obtained in [26], i.e. this curve is in agreement with the curve obtained in [26][17]. However, the author guess that the judicious reader notices that this curve (5.20) is divergent at $m_n = 0$. To solve the problem, we take one step further. We recall the new modulus (5.19). The curve is written in terms of the new modulus. The one step we need is to take the new modulus $\tilde{u}$ back to the old modulus $u$. Namely, we rewrite the curve (5.20) as

$$\begin{aligned}
f' &= 16E_4\Big(\frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u\Big)^4 + (16\sigma_2 - 4E_4)\Big(\frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u\Big)^3 \\
&+ \Big(\frac{4\sigma_1^2}{3} - 4\sigma_2\Big)\Big(\frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u\Big)^2, \\
g' &= 64E_6\Big(\frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u\Big)^6 - \Big(\frac{16}{3}E_4\sigma_1 + 32E_6 + 64\sigma_3\Big)\Big(\frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u\Big)^5 \\
&+ \Big(4E_6 + \frac{4}{3}E_4\sigma_1 - \frac{16}{3}\sigma_1\sigma_2 + 32\sigma_3\Big)\Big(\frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u\Big)^4 \\
&- \Big(\frac{8}{27}\sigma_1^3 - \frac{4}{3}\sigma_1\sigma_2 + 4\sigma_3\Big)\Big(\frac{\eta^{18}}{\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}u\Big)^3.
\end{aligned} \tag{5.22}$$

Then multiplying the whole of the curve with these coefficients by $(4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2/\eta^{18})^6$, we obtain

$$\begin{aligned}
\Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^6\tilde{y}^2 &= 4\Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^6\tilde{x}^2 \\
&- \Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^6 f'\tilde{x} \\
&- \Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^6 g'.
\end{aligned} \tag{5.23}$$

[17] For the perfect match up to the numerical factors, note that the difference between the notations is

$$16E_4^{\text{ours}} = f_0^{\text{Mohri's}}, \ \ 64E_6^{\text{ours}} = g_0^{\text{Mohri's}}, \ \ 4\wp_i^{\text{ours}} = \wp_i^{\text{Mohri's}}. \tag{5.21}$$

In addition, we have two more terms $-32E_6\tilde{u}^5$ and $32\sigma_3\tilde{u}^4$ and a different coefficient of $-64\sigma_3\tilde{u}^5$ compared with the Mohri's result (9.18) in [26]. But we checked that those two terms and the coefficient of (9.18) have been stolen.

Finally, we redefine the variables and the coefficients as

$$\begin{aligned} Y &:= \Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^3\tilde{y}, \quad X := \Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^2\tilde{x}, \\ F &:= \Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^4 f', \quad G := \Big(\frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}\Big)^6 g'. \end{aligned} \tag{5.24}$$

Then we obtain

$$\begin{aligned} Y^2 &= 4X^3 - FX - G, \\ F &= 16E_4u^4 + (16\sigma_2 - 4E_4)\alpha'(m)u^3 + \Big(\frac{4\sigma_1^2}{3} - 4\sigma_2\Big)\alpha'(m)^2u^2, \\ G &= 64E_6u^6 - \Big(\frac{16}{3}E_4\sigma_1 + 32E_6 + 64\sigma_3\Big)\alpha'(m)u^5 \\ &+ \Big(4E_6 + \frac{4}{3}E_4\sigma_1 - \frac{16}{3}\sigma_1\sigma_2 + 32\sigma_3\Big)\alpha'(m)^2u^4 - \Big(\frac{8}{27}\sigma_1^3 - \frac{4}{3}\sigma_1\sigma_2 + 4\sigma_3\Big)\alpha'(m)^3u^3, \\ \alpha'(m) &:= \frac{4\vartheta_1(m_1)^2\vartheta_1(m_2)^2\vartheta_1(m_3)^2}{\eta^{18}}, \quad \text{(i.e. } \alpha(m) = \alpha'(m)/u\text{)}. \end{aligned} \tag{5.25}$$

We can easily see that this curve is not divergent at $m_n = 0$ and gives the Seiberg-Witten curve for the E-string theory with the $E_8$ global symmetry. We can take the limit $\lim_{m\to 0} \wp(2\pi m)\vartheta_1(m)^2 = \eta^6$. By using this in (5.25), we obtain

$$Y^2 = 4X^3 - 16E_4u^4X - 64E_6u^6 + 256u^5, \tag{5.26}$$

where the last term $256u^5$ comes from the term $-64\sigma_3\alpha'(m)u^5$ within $G$. This is precisely the Seiberg-Witten curve for the E-string theory with the $E_8$ global symmetry[18].

By the discussion we have seen thus far, it was shown that the Nekrasov-type partition function gives the Seiberg-Witten curves also in the cases with three Wilson lines. And also, it was shown that, comparing our result with the result obtained in [26], ours explicitly includes the dependence of the Seiberg-Witten curve on the Wilson lines.

[18] For the reader who does not like the different numerical factors, the redefinition $X_{new} := X_{old}/4$ is required.

## 5.2 The cases with four Wilson lines

In these cases, we choose the setup as

$$N = 4, \quad \zeta_k = (0, \omega_i), \quad m_n = (2\pi m_1, 2\pi m_2, 2\pi m_3, 2\pi m_4). \tag{5.27}$$

This is the most general setup. The function $H(z)$ is given by

$$\begin{aligned} H(z) &= \kappa \frac{(\prod_{k=1}^{4} \vartheta_1(\frac{z-\zeta_k}{2\pi}))^2}{\prod_{n=1}^{8} \vartheta_1(\frac{z-m_n}{2\pi})} \\ &= \kappa \frac{\vartheta_1(\frac{z}{2\pi})^2 \vartheta_1(\frac{z-\pi}{2\pi})^2 \vartheta_1(\frac{z+\pi+\pi\tau}{2\pi})^2 \vartheta_1(\frac{z-\pi\tau}{2\pi})^2}{\vartheta_1(\frac{z-2\pi m_1}{2\pi}) \cdots \vartheta_1(\frac{z-2\pi m_4}{2\pi}) \vartheta_1(\frac{z+2\pi m_1}{2\pi}) \cdots \vartheta_1(\frac{z+2\pi m_4}{2\pi})}. \end{aligned} \tag{5.28}$$

Following the discussion in the previous subsection, we obtain

$$\begin{aligned} y_0^2 &= \wp'(z)^2 + \alpha(m)(\wp(z) - \wp_1)(\wp(z) - \wp_2)(\wp(z) - \wp_3)(\wp(z) - \wp_4) \\ &= 4\wp^3 - E_4\wp - E_6 + \alpha(\wp^4 - \sigma_1\wp^3 + \sigma_2\wp^2 - \sigma_3\wp + \sigma_4), \end{aligned} \tag{5.29}$$

where

$$\begin{aligned} \alpha(m) &:= \frac{4}{u\eta^{24}} \vartheta_1(m_1)^2 \vartheta_1(m_2)^2 \vartheta_1(m_3)^2 \vartheta_1(m_4)^2, \\ \sigma_1 &:= \wp_1 + \wp_2 + \wp_3 + \wp_4, \\ \sigma_2 &:= \wp_1\wp_2 + \wp_2\wp_3 + \wp_3\wp_4 + \wp_1\wp_3 + \wp_1\wp_4 + \wp_2\wp_4, \\ \sigma_3 &:= \wp_1\wp_2\wp_3 + \wp_1\wp_2\wp_4 + \wp_1\wp_3\wp_4 + \wp_2\wp_3\wp_4, \\ \sigma_4 &:= \wp_1\wp_2\wp_3\wp_4. \end{aligned} \tag{5.30}$$

This curve (5.29) is superficially a quartic curve. To get a cubic curve, we need two tools. Firstly, we restore the homogeneous coordinates as[19]

$$x_1 = \wp, \quad x_2 = y_0, , \quad x_0 \neq 1. \tag{5.31}$$

Secondly, we recall the fact that the curve (5.29) is

$$x_0 x_2^2 = 4x_1^3 - E_4 x_0^2 x_1 - E_6 x_0^3 \tag{5.32}$$

[19] In the cases with three Wilson lines we saw in the previous subsection, one of the homogeneous coordinates $x_0$ was just one. So we didn't have to use the homogeneous coordinates.

at $u=\infty$. By using these tools, the quartic curve (5.29) is rewritten as

$$\begin{aligned} x_0x_2^2 &= 4x_1^3 - E_4x_0^2x_1 - E_6x_0^3 \\ &+ \alpha((x_1/x_0)(x_0x_2^2+E_4x_0^2x_1+E_6x_0^3)/4 - \sigma_1(x_0x_2^2+E_4x_0^2x_1+E_6x_0^3)/4 + \sigma_2x_0x_1^2+\sigma_4x_0^3). \end{aligned} \tag{5.33}$$

This curve is arranged as[20]

$$(a_0x_0-\alpha x_1)x_2^2 = 16x_1^3 + a_1x_0x_1^2 + a_2x_0^2x_1 + a_3x_0^3, \tag{5.34}$$

where

$$\begin{aligned} a_0 &:= 4+\alpha\sigma_1, \\ a_1 &:= \alpha E_4 + 4\alpha\sigma_2, \\ a_2 &:= -4E_4 + \alpha E_6 - \alpha E_4\sigma_1 - 4\alpha\sigma_3, \\ a_3 &:= -4E_6 - \alpha E_6\sigma_1 + 4\alpha\sigma_4. \end{aligned} \tag{5.35}$$

We here do not continue to compute this anymore. Note, however, that the quartic curve (5.29) already indicates that the Nekrasov-type partition function reproduces the Seiberg-Witten curve[21]. We do not explicitly give the Seiberg-Witten curve in this case but we make some comments on that we can reproduce the Seiberg-Witten curve from the Nekrasov-type partition function from another viewpoint in the next subsection.

## 5.3 The derivation of the $SL(3,\mathbb{C})$ invariant curve

Finally, in this subsection, we see another derivation of the Seiberg-Witten curves in the cases with three and four Wilson lines. In this section, we have seen that the elliptic function $H(z)$ gave the cubic or quartic curve and they led us to the Seiberg-Witten curves. In particular, we concretely derived the Seiberg-Witten curve (5.25) from the cubic curve (5.7). There is another derivation of the Seiberg-Witten curve. It is a mathematical formula and does not need the technical calculation. However, it becomes complicated for the general case. We here do not give the explanation of the formula and do extract the result, so for the details see [10, 26]. We follow the discussion of [26] here.

[20] Unlike the cases with three Wilson lines, the calculations in the cases with four Wilson lines make the author feel blue. If the reader wants to check the calculations, they require biting the bullet.

[21] Actually, we can compare the quartic curve with the result obtained in [26].

The cubic curve $P + \alpha Q = 0$ leads us to the Seiberg-Witten curve. In order for this Seiberg-Witten curve to be correct, it must be $SL(3,\mathbb{C})$ invariant. Such a curve is given by a map

$$R \mapsto x_0 x_2^2 - 4x_1^3 + f x_0^2 x_1 + g x_0^3, \tag{5.36}$$

where $R := P + \alpha Q$ is defined as

$$R := \sum_{p+q+r=3} \Big(\frac{3!}{p!q!r!}\Big) a_{pqr} x_0^p x_1^q x_2^r. \tag{5.37}$$

Then the coefficients $a_{pqr}$ determine the coefficients $f$ and $g$ of the Seiberg-Witten curve. We do not list the formula explicitly so see Appendix A in [26].

We here give the Seiberg-Witten curve by using the result of the formula. Firstly, we do in the case with three Wilson lines. The cubic curve is (5.7):

$$R = P + \alpha Q = x_0 x_2^2 - (4-\alpha)x_1^3 - \alpha\sigma_1 x_0 x_1^2 + (E_4 + \alpha\sigma_2)x_0^2 x_1 + (E_6 - \alpha\sigma_3). \tag{5.38}$$

In terms of the modulus $u$, this is written as

$$uR = uP + Q = ux_0 x_2^2 - (4u-1)x_1^3 - \sigma_1 x_0 x_1^2 + (uE_4 + \sigma_2)x_0^2 x_1 + (uE_6 - \sigma_3). \tag{5.39}$$

Note, here, that we have absorbed the other factors into $u$, i.e. we have denoted $u/\alpha' = \alpha^{-1}$ just by $u$. And also, we denote $uR$ by $R$ from now on. The coefficients $a_{pqr}$ are

$$\begin{aligned} 3a_{102} &= u, \quad a_{030} = 1 - 4u, \quad 3a_{120} = -\sigma_1, \\ 3a_{210} &= uE_4 + \sigma_2, \quad a_{300} = uE_6 - \sigma_3. \end{aligned} \tag{5.40}$$

Then the coefficients $f$ and $g$ are reproduced by

$$\begin{aligned} f \sim \mathcal{S} &= a_{120}^2 a_{102}^2 - a_{030} a_{210} a_{102}^2 \\ &= (-\sigma_1/3)^2 (u/3)^2 - (1-4u)((uE_4+\sigma_2)/3)(u/3)^2 \\ &= \frac{1}{81}(12E_4u^4 - 3E_4u^3 + 12\sigma_2 u^3 + \sigma_1^2 u^2 - 3\sigma_2 u^2), \end{aligned} \tag{5.41}$$

$$\begin{aligned} g \sim \mathcal{T} &= 8a_{120}^3 a_{102}^3 + 4a_{300} a_{030}^2 a_{102}^3 - 12 a_{120} a_{210} a_{102}^3 a_{030} \\ &= 8(-\sigma_1/3)^3 (u/3)^3 + 4(uE_6 - \sigma_3)(1-4u)^2 (u/3)^3 \\ &- 12(-\sigma_1/3)((uE_4+\sigma_2)/3)(u/3)^3(1-4u) \\ &= \frac{1}{81}\Big(12\cdot 16E_6u^6 - 12\cdot 16\sigma_3u^5 - 12\cdot 8E_6u^5 - 16\sigma_1E_4u^5 \\ &+ 12E_6u^4 + 12\cdot 8\sigma_3u^4 + 4\sigma_1E_4u^4 - 16\sigma_1\sigma_2u^4 - \frac{8}{9}\sigma_1^3u^3 - 12\sigma_3u^3 + 4\sigma_1\sigma_2u^3\Big), \end{aligned} \tag{5.42}$$

where we have extracted $\mathcal{S}$ and $\mathcal{T}$ from [26]. We arrive at the Seiberg-Witten curve (5.20) with the scalings $f = 108\mathcal{S}$ and $g = 27\mathcal{T}$. We also arrive at (5.25) by multiplying the whole thing by $\alpha'^6$, of course.

Next, we move on to the case with four Wilson lines. In this case, the cubic curve is given by (5.34):

$$\begin{aligned} R = P + \alpha Q &= a_0x_0x_2^2 - \alpha x_1x_2^2 - 16x_1^3 - a_1x_0x_1^2 - a_2x_0^2x_1 - a_3x_0^3 \\ &= (4+\alpha\sigma_1)x_0x_2^2 - \alpha x_1x_2^2 - 16x_1^3 - \alpha(E_4+4\sigma_2)x_0x_1^2 \\ &+ (4E_4 - \alpha E_6 + \alpha E_4\sigma_1 + 4\alpha\sigma_3)x_0^2x_1 + (4E_6 + \alpha E_6\sigma_1 - 4\alpha\sigma_4)x_0^3. \end{aligned} \tag{5.43}$$

In terms of the modulus $u$, it is written as

$$\begin{aligned} uR = uP + Q &= (4u+\sigma_1)x_0x_2^2 - x_1x_2^2 - 16ux_1^3 - (E_4+4\sigma_2)x_0x_1^2 \\ &+ (4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)x_0^2x_1 + (4E_6u + E_6\sigma_1 - 4\sigma_4)x_0^3. \end{aligned} \tag{5.44}$$

Here we (have) use(d) the same notation above. The coefficients $a_{pqr}$ are

$$\begin{aligned} 3a_{102} &= 4u+\sigma_1, \quad 3a_{012} = -1, \quad a_{030} = -16u, \quad 3a_{120} = -(E_4+4\sigma_2), \\ 3a_{210} &= 4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3, \quad a_{300} = 4E_6u + E_6\sigma_1 - 4\sigma_4. \end{aligned} \tag{5.45}$$

Then the coefficients $f$ and $g$ are given by

$$\begin{aligned}
f \sim \mathcal{S} &= -a_{300}a_{120}a_{012}^2 - a_{300}a_{030}a_{102}a_{012} - a_{210}a_{120}a_{102}a_{012} \\
&+ a_{210}^2a_{012}^2 - a_{030}a_{210}a_{102}^2 + a_{120}^2a_{102}^2 \\
&= -(4E_6u + E_6\sigma_1 + 4\sigma_4)(-(E_4 + 4\sigma_2))(-1/3)^2 \\
&- (4E_6u + E_6\sigma_1 - 4\sigma_4)(-16u)((4u + \sigma_1)/3)(-1/3) \\
&- ((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)(-(E_4 + 4\sigma_2)/3)((4u + \sigma_1)/3)(-1/3) \\
&+ ((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)^2(-1/3)^2 \\
&- (-16u)((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)((4u + \sigma_1)/3)^2 \\
&+ (-(E_4 + 4\sigma_2)/3)^2((4u + \sigma_1)/3)^2,
\end{aligned} \tag{5.46}$$

$$\begin{aligned}
g \sim \mathcal{T} &= 4a_{012}^3a_{300}^2a_{030} + 8a_{120}^3a_{102}^3 + 8a_{012}^3a_{210}^3 + 4a_{300}a_{030}^2a_{102}^3 \\
&- 12a_{120}a_{210}a_{102}^3a_{030} - 12a_{012}a_{120}a_{300}a_{030}a_{102}^2 - 12a_{012}^2a_{300}a_{030}a_{102}a_{210} \\
&- 12a_{012}a_{120}^2a_{210}a_{102}^2 + 24a_{012}a_{210}^2a_{102}^2a_{030} - 12a_{012}^3a_{300}a_{210}a_{120} \\
&+ 24a_{012}^2a_{120}^2a_{300}a_{102} - 12a_{012}^2a_{120}a_{210}^2a_{102} \\
&= 4(-1/3)^3((4E_6u + E_6\sigma_1 - 4\sigma_4)^2(-16u) + 8(-(E_4 + 4\sigma_2)/3)^3((4u + \sigma_1)/3)^3 \\
&+ 8(-1/3)^3((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)^3 \\
&+ 4(4E_6u + E_6\sigma_1 - 4\sigma_4)(-16u)^2((4u + \sigma_1)/3)^3 \\
&- 12(-(E_4 + 4\sigma_2)/3)((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)((4u + \sigma_1)/3)^3(-16u) \\
&- 12(-1/3)(-(E_4 + 4\sigma_2)/3)(4E_6u + E_6\sigma_1 - 4\sigma_4)(-16u)((4u + \sigma_1)/3)^2 \\
&- 12(-1/3)^2(4E_6u + E_6\sigma_1 - 4\sigma_4)(-16u)((4u + \sigma_1)/3)((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3) \\
&- 12(-1/3)(-(E_4 + 4\sigma_2)/3)^2((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)((4u + \sigma_1)/3)^2 \\
&+ 24(-1/3)((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)^2((4u + \sigma_1)/3)^2(-16u) \\
&- 12(-1/3)^3(4E_6u + E_6\sigma_1 - 4\sigma_4)((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)(-(E_4 + 4\sigma_2)/3) \\
&+ 24(-1/3)^2(-(E_4 + 4\sigma_2)/3)^2(4E_6u + E_6\sigma_1 - 4\sigma_4)((4u + \sigma_1)/3) \\
&- 12(-1/3)^2(-(E_4 + 4\sigma_2)/3)((4E_4u - E_6 + E_4\sigma_1 + 4\sigma_3)/3)^2((4u + \sigma_1)/3).
\end{aligned} \tag{5.47}$$

We stop computing this anymore here but note that the order of $u$ in $f$ is at most fourth and that in $g$ is at most sextic.

We have seen the derivation of the Seiberg-Witten curve from the cubic curve by using the mathematical formula. However, it is so complicated to obtain explicitly the Seiberg-Witten curve for more general case even if we use the mathematical formula[22].

[22] In fact, in the case with four Wilson lines, it is said, in [26], that “ the Weierstrass form of the

# 6 Examples of the Cases with the Broken Symmetries

We have seen in the last two sections the way to obtain the Seiberg-Witten description from the Nekrasov-type partition function, and its generalisation. In this section, we see the other examples with the broken symmetries[15]. For more detail discussion for each case, see [14].

We reconfirm the procedure again. It is simple: firstly we give the elliptic function $H(z)$ from the setup which is fixed by some conditions for parameters we have seen in the section five, and secondly we find the appropriate change of variables because we want an elliptic curve with genus one.

We explain our convention we use here. Our Nekrasov-type partition function is the special case of the elliptic generalisation of the Nekrasov partition function[25, 13, 14]. The special cases where the Wilson line parameters are set to $m_n = (0, \frac{1}{2}, m_1, m_2)$ can be described by the Nekrasov-type partition functions for the gauge theories with the fewer colours[14]:

$$\begin{aligned} Z &= Z^{\mathrm{SU}(4)}_{N_f=8}\Big(\hbar; \varphi, \tau; 0, \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; 0, \frac{1}{2}, m_1, m_2, 0, -\frac{1}{2}, -m_1, -m_2\Big) \\ &= Z^{\mathrm{SU}(3)}_{N_f=6}\Big(\hbar; \varphi, \tau; \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; \frac{1}{2}, m_1, m_2, -\frac{1}{2}, -m_1, -m_2\Big) \\ &= Z^{\mathrm{SU}(2)}_{N_f=4}\Big(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; m_1, m_2, -m_1, -m_2\Big). \end{aligned} \tag{6.1}$$

Note that for our convention in this paper the Wilson line parameters are set to $m_n = (0, \pi, 2\pi m_1, 2\pi m_2)$ multiplied by $2\pi$, so the half-periods of the torus and the Wilson lines in the partition functions above should be multiplied by $2\pi$. The cases with the broken symmetries $E_7 \oplus A_1$, $E_5 \oplus A_3$ and $D_8$ are the cases and are discussed below.

## 6.1 $E_7 \oplus A_1$

In this case, the setup is given by

$$N = 2, \quad \zeta_k = \omega_{k+1} = (-\pi - \pi\tau, \pi\tau), \quad m_n = (0, 0). \tag{6.2}$$

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cubic pencil $P + tQ$ is so complicated that we do not attempt to describe it here. ”

Then the elliptic function $H(z)$ becomes

$$\begin{aligned} H(z) &= \kappa \frac{(\prod_{k=1}^{2} \vartheta_1(\frac{z-\zeta_k}{2\pi}))^2}{\prod_{n=1}^{4} \vartheta_1(\frac{z-m_n}{2\pi})} \\ &= \kappa \frac{\vartheta_1(\frac{z+\pi+\pi\tau}{2\pi})^2 \vartheta_1(\frac{z-\pi\tau}{2\pi})^2}{\vartheta_1(\frac{z}{2\pi})^4} \\ &= -\kappa q^{-1/2} \frac{\vartheta_3(\frac{z}{2\pi})^2 \vartheta_4(\frac{z}{2\pi})^2}{\vartheta_1(\frac{z}{2\pi})^4}. \end{aligned} \tag{6.3}$$

Here using the identities (B.8) and (B.9), (6.3) is written as

$$\begin{aligned} &= -\kappa q^{-1/2} \eta^{-12} \frac{\vartheta_1(\frac{z}{2\pi})^2}{\vartheta_2(\frac{z}{2\pi})^2} \wp'(z)^2 \\ &= -\frac{u}{4} \frac{\vartheta_1'^2}{(2\pi)^2 \vartheta_2^2} \frac{\wp'(z)^2}{\wp(z) - e_1} \\ &= -\frac{u}{16} \frac{\vartheta_2^2 \vartheta_3^2 \vartheta_4^2}{\vartheta_2^2} \frac{\wp'(z)^2}{\wp(z) - e_1} \\ &=: \frac{u \vartheta_3^2 \vartheta_4^2}{16} \frac{\wp'(z)^2}{\wp(z) - e_1}, \end{aligned} \tag{6.4}$$

where in the last line $u$ is defined in (4.39) but with opposite sign.

The appropriate change of variables is given by

$$\wp(z) - e_1 = u^{-2} x. \tag{6.5}$$

Then the alpha-cycle integral is expressed as

$$\begin{aligned}
\frac{\partial\varphi}{\partial u} &= \frac{i}{4\pi^2 u}\oint_\alpha \frac{dz}{\sqrt{1-H^{-1}}} \\
&= \frac{i}{4\pi^2 u}\oint_\alpha \frac{dz}{\sqrt{1-16(\wp(z)-e_1)/u\vartheta_3^2\vartheta_4^2\wp'(z)^2}} \\
&= \frac{i}{4\pi^2 u}\oint_\alpha \frac{\wp'(z)dz}{\sqrt{\wp'(z)^2-16(\wp(z)-e_1)/u\vartheta_3^2\vartheta_4^2}} \\
&= \frac{i}{4\pi^2 u}\oint_\alpha \frac{\wp'(z)dz}{\sqrt{4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)-16(\wp(z)-e_1)/u\vartheta_3^2\vartheta_4^2}} \\
\Longrightarrow &\ \frac{i}{4\pi^2 u}\oint_{\tilde\alpha} \frac{u^{-2}dx}{\sqrt{4u^{-2}x(u^{-2}x+e_1-e_2)(u^{-2}x+e_1-e_3)-16x/u^3\vartheta_3^2\vartheta_4^2}} \\
&= \frac{i}{4\pi^2 u}\oint_{\tilde\alpha} \frac{dx}{\sqrt{4u^{-2}x^3+4(2e_1-e_2-e_3)x^2+4(e_1-e_2)(e_1-e_3)u^2x-16ux/\vartheta_3^2\vartheta_4^2}} \\
&= \frac{i}{4\pi^2}\oint_{\tilde\alpha} \frac{dx}{\sqrt{4x^3+4(2e_1-e_2-e_3)u^2x^2+4(e_1-e_2)(e_1-e_3)u^4x-16u^3x/\vartheta_3^2\vartheta_4^2}}.
\end{aligned} \tag{6.6}$$

Hence we obtain

$$y^2 = 4x^3+4(2e_1-e_2-e_3)u^2x^2+4(e_1-e_2)(e_1-e_3)u^4x-\frac{16u^3x}{\vartheta_3^2\vartheta_4^2}. \tag{6.7}$$

Here using (B.9), the specific values of $e_i$, we get

$$\begin{aligned}
y^2 &= 4x^3+\frac{1}{3}(3\vartheta_3^4+3\vartheta_4^4)u^2x^2+\frac{1}{36}(-\vartheta_2^4+\vartheta_3^4+2\vartheta_4^4)(\vartheta_2^4+2\vartheta_3^4+\vartheta_4^4)u^4x-\frac{16u^3x}{\vartheta_3^2\vartheta_4^2} \\
&= 4x^3+(\vartheta_3^4+\vartheta_4^4)u^2x^2+\frac{1}{4}\vartheta_3^4\vartheta_4^4u^4x-\frac{16u^3x}{\vartheta_3^2\vartheta_4^2}.
\end{aligned} \tag{6.8}$$

This is in agreement with the Seiberg-Witten curve in the case with the global symmetry $E_7 \oplus A_1$.

## 6.2 $E_6 \oplus A_2$

This case is given by

$$N = 3, \quad \zeta_k = \omega_k, \quad m_n = (2\pi/3, 2\pi/3, 2\pi/3). \tag{6.9}$$

Then the elliptic function $H(z)$ becomes

$$\begin{aligned} H(z) &= \kappa \frac{(\prod_{k=1}^{3} \vartheta_1(\frac{z-\zeta_k}{2\pi}))^2}{\prod_{n=1}^{6} \vartheta_1(\frac{z-m_n}{2\pi})} \\ &= \kappa \frac{\vartheta_1(\frac{z-\pi}{2\pi})^2 \vartheta_1(\frac{z+\pi+\pi\tau}{2\pi})^2 \vartheta_1(\frac{z-\pi\tau}{2\pi})^2}{\vartheta_1(\frac{z}{2\pi} - \frac{1}{3})^3 \vartheta_1(\frac{z}{2\pi} + \frac{1}{3})^3} \\ &= -\kappa q^{-1/2} \frac{\vartheta_2(\frac{z}{2\pi})^2 \vartheta_3(\frac{z}{2\pi})^2 \vartheta_4(\frac{z}{2\pi})^2}{\vartheta_1(\frac{z}{2\pi} - \frac{1}{3})^3 \vartheta_1(\frac{z}{2\pi} + \frac{1}{3})^3}. \end{aligned} \tag{6.10}$$

Here using the identity

$$\vartheta_1\Big(\frac{z}{2\pi} - \frac{1}{3}\Big)\vartheta_1\Big(\frac{z}{2\pi} + \frac{1}{3}\Big) = -3\frac{\eta(3\tau)^2}{\eta(\tau)^6}\Big(\wp(z) - \frac{1}{4}\alpha_3^2\Big)\vartheta_1\Big(\frac{z}{2\pi}\Big)^2, \tag{6.11}$$

where

$$\alpha_3 := \vartheta_3(0, 2\tau)\vartheta_3(0, 6\tau) + \vartheta_2(0, 2\tau)\vartheta_2(0, 6\tau), \tag{6.12}$$

(6.10) is rewritten as

$$\begin{aligned} &= \kappa q^{-1/2} \frac{\vartheta_2(\frac{z}{2\pi})^2 \vartheta_3(\frac{z}{2\pi})^2 \vartheta_4(\frac{z}{2\pi})^2}{27\eta(3\tau)^6(\wp(z) - \frac{1}{4}\alpha_3^2)^3 \vartheta_1(\frac{z}{2\pi})^6/\eta(\tau)^{18}} \\ &= \kappa q^{-1/2}\beta_3^2 \frac{\vartheta_2(\frac{z}{2\pi})^2 \vartheta_3(\frac{z}{2\pi})^2 \vartheta_4(\frac{z}{2\pi})^2}{27(\wp(z) - \frac{1}{4}\alpha_3^2)^3 \vartheta_1(\frac{z}{2\pi})^6}, \end{aligned} \tag{6.13}$$

where we have defined

$$\beta_3 := \frac{\eta(\tau)^9}{\eta(3\tau)^3}. \tag{6.14}$$

Here using the identity (B.8), we obtain

$$\begin{aligned} H(z) &= \kappa q^{-1/2}\beta_3^2 \frac{\wp'(z)^2}{27\eta^{12}(\wp(z) - \frac{1}{4}\alpha_3^2)^3} \\ &= \frac{u\beta_3^2 \wp'(z)^2}{108(\wp(z) - \frac{1}{4}\alpha_3^2)^3}. \end{aligned} \tag{6.15}$$

In the present case, the appropriate change of variables is given by

$$\wp(z) - \frac{1}{4}\alpha_3^2 = \frac{x}{u(u - 27\beta_3^{-2})}. \tag{6.16}$$

Therefore the alpha-cycle integral becomes

$$\begin{aligned}
\frac{\partial\varphi}{\partial u} &= \frac{i}{4\pi^2 u}\oint_\alpha \frac{dz}{\sqrt{1 - H^{-1}}} \\
&= \frac{i}{4\pi^2 u}\oint_\alpha \frac{dz}{\sqrt{1 - (108(\wp(z) - \frac{1}{4}\alpha_3^2)^3/u\beta_3^2\wp'(z)^2)}} \\
&= \frac{i}{4\pi^2 u}\oint_\alpha \frac{\wp'(z)dz}{\sqrt{\wp'(z)^2 - (108(\wp(z) - \frac{1}{4}\alpha_3^2)^3/u\beta_3^2)}} \\
&= \frac{i}{4\pi^2 u}\oint_\alpha \frac{\wp'(z)dz}{\sqrt{4\wp(z)^3 - \frac{E_4}{12}\wp(z) - \frac{E_6}{216} - (108(\wp(z) - \frac{1}{4}\alpha_3^2)^3/u\beta_3^2)}} \\
&\Longrightarrow \frac{i}{4\pi^2 u}\oint_{\tilde{\alpha}} \frac{dx}{u(u - 27\beta_3^{-2})} \\
&\times \frac{1}{\sqrt{4(\frac{x}{u(u-27\beta_3^{-2})} + \frac{\alpha_3^2}{4})^3 - \frac{E_4}{12}(\frac{x}{u(u-27\beta_3^{-2})} + \frac{\alpha_3^2}{4}) - \frac{E_6}{216} - (108\frac{x^3}{u^4\beta_3^2(u-27\beta_3^{-2})^3})}} \\
&= \frac{i}{4\pi^2}\oint_{\tilde{\alpha}} \frac{dx}{uW\sqrt{4(x/W + \alpha_3^2/4)^3 - E_4(x/W + \alpha_3^2/4)/12 - E_6/216 - 108x^3/u\beta_3^2W^3}},
\end{aligned} \tag{6.17}$$

where in the last line we have defined

$$W := u(u - 27\beta_3^{-2}). \tag{6.18}$$

The Seiberg-Witten curve should be given by $y^2$ =(the denominator)$^2$. It is expressed as

$$\begin{aligned}
y^2 &= u^2W^2\Big(4\Big(\frac{x}{W}+\frac{\alpha_3^2}{4}\Big)^3-\frac{E_4}{12}\Big(\frac{x}{W}+\frac{\alpha_3^2}{4}\Big)-\frac{E_6}{216}-\frac{108x^3}{u\beta_3^2W^3}\Big)\\
&= \frac{4u^2x^3}{W}+3u^2\alpha_3^2x^2+\frac{3u^2W\alpha_3^4x}{4}+\frac{u^2W^2\alpha_3^6}{16}\\
&- \frac{u^2WE_4x}{12}-\frac{u^2W^2\alpha_3^2E_4}{48}-\frac{u^2W^2E_6}{216}-\frac{108ux^3}{\beta_3^2W}\\
&= 4\Big(\frac{u^2\beta_3^2-27u}{\beta_3^2W}\Big)x^3+3u^2\alpha_3^2x^2+\frac{9\alpha_3^4-E_4}{12}u^2Wx+\frac{u^2W^2}{8}\Big(\frac{\alpha_3^6}{2}-\frac{\alpha_3^2E_4}{6}-\frac{E_6}{27}\Big).
\end{aligned} \tag{6.19}$$

Here we consider this term by term. The first term is

$$\begin{aligned}
4\Big(\frac{u^2\beta_3^2-27u}{\beta_3^2W}\Big)x^3 &= 4\Big(\frac{u^2\beta_3^2-27u}{\beta_3^2u(u-27\beta_3^{-2})}\Big)x^3\\
&= 4\Big(\frac{u\beta_3^2-27}{u\beta_3^2-27}\Big)x^3=4x^3.
\end{aligned} \tag{6.20}$$

Nothing worthwhile for the second term but the rest two are technical. For the rest two, we use the relations

$$E_4=9\alpha_3^4-8\alpha_3\beta_3,\quad E_6=-27\alpha_3^6+36\alpha_3^3\beta_3-8\beta_3^2. \tag{6.21}$$

Then the third term is written as

$$\begin{aligned}
\frac{9\alpha_3^4-E_4}{12}u^2Wx &= \frac{9\alpha_3^4-(9\alpha_3^4-8\alpha_3\beta_3)}{12}u^3(u-27\beta_3^{-2})x\\
&= \frac{2}{3}\alpha_3\beta_3u^3(u-27\beta_3^{-2})x,
\end{aligned} \tag{6.22}$$

and the fourth term is written as

$$\begin{aligned}
\frac{u^2W^2}{8}\Big(\frac{\alpha_3^6}{2}-\frac{\alpha_3^2E_4}{6}-\frac{E_6}{27}\Big) &= \frac{u^4(u-27\beta_3^{-2})^2}{8}\Big(\frac{\alpha_3^6}{2}-\frac{\alpha_3^2(9\alpha_3^4-8\alpha_3\beta_3)}{6}-\frac{-27\alpha_3^6+36\alpha_3^3\beta_3-8\beta_3^2}{27}\Big)\\
&= u^4(u-27\beta_3^{-2})^2\frac{\beta_3^2}{27}\\
&= \frac{\beta_3^2u^6}{27}-2u^5+\frac{27u^4}{\beta_3^2}.
\end{aligned} \tag{6.23}$$

Hence the Seiberg-Witten curve (6.19) is expressed as

$$y^2 = 4x^3 + 3u^2\alpha_3^2 x^2 + \frac{2}{3}\alpha_3\Big(\beta_3 u - \frac{27}{\beta_3}\Big)u^3 x + \frac{1}{27}\Big(\beta_3 u - \frac{27}{\beta_3}\Big)^2 u^4. \tag{6.24}$$

This is in agreement with the Seiberg-Witten curve in the case with the broken symmetry $E_6 \oplus A_2$.

## 6.3 $D_8$

This case is given by

$$N = 2, \quad \zeta_k = \omega_{k+1} = (-\pi - \pi\tau, \pi\tau), \quad m_n = (\pi, 0). \tag{6.25}$$

Then the elliptic function $H(z)$ becomes

$$\begin{aligned}
H(z) &= \kappa \frac{(\prod_{k=1}^2 \vartheta_1(\frac{z-\zeta_k}{2\pi}))^2}{\prod_{n=1}^4 \vartheta_1(\frac{z-m_n}{2\pi})} \\
&= \kappa \frac{\vartheta_1(\frac{z+\pi+\pi\tau}{2\pi})^2 \vartheta_1(\frac{z-\pi\tau}{2\pi})^2}{\vartheta_1(\frac{z-\pi}{2\pi})^2 \vartheta_1(\frac{z}{2\pi})^2} \\
&= -\kappa q^{-1/2} \frac{\vartheta_3(\frac{z}{2\pi})^2 \vartheta_4(\frac{z}{2\pi})^2}{\vartheta_1(\frac{z}{2\pi})^2 \vartheta_2(\frac{z}{2\pi})^2} \\
&= -\kappa q^{-1/2} \eta^{-12} \frac{\vartheta_1(\frac{z}{2\pi})^4 \wp'(z)^2}{\vartheta_2(\frac{z}{2\pi})^4} \\
&= -\frac{u}{4} \frac{\vartheta_1'^4}{(2\pi)^4 \vartheta_2^4 (\wp(z) - e_1)^2} \wp'(z)^2 \\
&= -\frac{u}{4} \frac{\eta^{12}}{\vartheta_2^4 (\wp(z) - e_1)^2} \wp'(z)^2 \\
&= -\frac{u}{64} \frac{\vartheta_3^4 \vartheta_4^4}{(\wp(z) - e_1)^2} \wp'(z)^2,
\end{aligned} \tag{6.26}$$

where we have used in the last three lines the relation (B.9). The appropriate change of variables is given by

$$\wp(z) - e_1 = u^{-2} x. \tag{6.27}$$

Therefore the alpha-cycle integral becomes

$$\begin{aligned}
\frac{\partial\varphi}{\partial u} &= \frac{i}{4\pi^2 u}\oint_{\alpha}\frac{dz}{\sqrt{1+(64(\wp(z)-e_1)^2/u\vartheta_3^4\vartheta_4^4\wp'(z)^2)}} \\
&= \frac{i}{4\pi^2 u}\oint_{\alpha}\frac{\wp'(z)dz}{\sqrt{\wp'(z)^2+(64(\wp(z)-e_1)^2/u\vartheta_3^4\vartheta_4^4)}} \\
&\Longrightarrow \frac{i}{4\pi^2 u}\oint_{\tilde{\alpha}}\frac{dx}{u^2\sqrt{4u^{-2}x(u^{-2}x+e_1-e_2)(u^{-2}x+e_1-e_3)+(64x^2/u^5\vartheta_3^4\vartheta_4^4)}} \\
&= \frac{i}{4\pi^2}\oint_{\tilde{\alpha}}\frac{dx}{\sqrt{4u^4x(u^{-2}x+e_1-e_2)(u^{-2}x+e_1-e_3)+(64ux^2/\vartheta_3^4\vartheta_4^4)}}.
\end{aligned} \tag{6.28}$$

Amazingly, here, comparing the change of variables (6.27) with the one in the $E_7 \oplus A_1$ case (6.5), and the alpha-cycle integral (6.28) with that of that case (6.6), we notice they are very similar. Thus utilising the curve (6.8), we obtain

$$\begin{aligned}
y^2 &= 4x^3+(\vartheta_3^4+\vartheta_4^4)u^2x^2+\frac{1}{4}\vartheta_3^4\vartheta_4^4u^4x+\frac{64ux^2}{\vartheta_3^4\vartheta_4^4} \\
&= 4x^3+\Big((\vartheta_3^4+\vartheta_4^4)u+\frac{64}{\vartheta_3^4\vartheta_4^4}\Big)ux^2+\frac{1}{4}\vartheta_3^4\vartheta_4^4u^4x.
\end{aligned} \tag{6.29}$$

This is in agreement with the Seiberg-Witten curve in the case with the broken symmetry $D_8$.

Why could we do the comparison, however? We try to interpret it from the viewpoint of the Dynkin diagram. In E-string side, we took the broken symmetry as $E_8 \to E_7 \oplus A_1$. Here we break $D_8$ to $D_8 \to D_7 \oplus A_1$. These two symmetries have the infinitesimal structure in common, i.e. $E_7 \oplus A_1 \cong D_7 \oplus A_1$. $D_7 \oplus A_1$ is the infinitesimal structure of the *almost* whole $D_8$ but not. The subtle difference changes the last term in (6.8).

# 7 Conclusion

In this final section, we summarise the stories totally and comment about topics which we have not seen in this thesis. In this thesis, we have reviewed the

Seiberg-Witten description and the Nekrasov partition function in the ordinary $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions and the ones in E-string theory. Stringy- or supersymmetric gauge theoretical-historically, the worlds are drastically changed in 1994. Seiberg and Witten completely determined the low-energy effective theory, i.e. the prepotential, by using the duality. At least for ten years since that time, the word *duality* has played the central key role in the study of string theory and supersymmetric gauge theory[23].

On the other hand, the various topological field theories which we didn't discuss here were developed[24]. As its application, Nekrasov gave the partition function formula which directly determines the prepotential and the partition function from the field content of the theory, without using the duality and the period integrals.

Based on these two main results in the $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions, the supersymmetric gauge theories themselves, string theory, and M-theory have widely developed. As one of these, it was shown that these two approaches exist even in E-string theory. In section 3, we have seen that the elliptic function $H(z)$ gives the profile function and also the Seiberg-Witten description. This implies that the Nekrasov-type partition function can reproduce the Seiberg-Witten description in the thermodynamic limit, namely the Nekrasov-type partition function is correct in the sense. Given the concrete setup, the elliptic function leads us to the Seiberg-Witten curve. In section 4, this result was generalised to the cases with the Wilson lines. In particular, the Seiberg-Witten curve in the case with three Wilson lines was given explicitly. As mentioned in Introduction, the Seiberg-Witten curve in the case with three Wilson lines is already given in [26]. However, unlike that result, our result has explicitly shown the dependence on the Wilson lines. We would like to attempt to interpret this difference as follows: in [26] the Seiberg-Witten curve was obtained by the geometric engineering approach. As shown in the name, the information of the theory is extracted from the geometric construction. For example, the cubic or quartic curve $P + tQ = 0$ corresponds to it. Then $t \sim u$ but we cannot see anything other than the information of the modulus $u$. On the one hand, recall that the ordinary Nekrasov partition function includes the Seiberg-Witten description as the special limit. What we have seen in section 3 is that the Nekrasov-type partition function is the same as the Nekrasov partition function in that sense, of course. Namely, the

[23] In this thesis, we have not seen the topic "duality." It is no exaggeration to say that the duality has been in the centre of the study of superstring. As the related papers, see, e.g. [48, 49].

[24] For the details, see the author's master thesis[42].

Nekrasov-type partition function knows *all* the information of the Seiberg-Witten description. This is why the dependence of the Seiberg-Witten curve on the Wilson lines explicitly appeared in our result. This fact is very important. This fact implies that the Nekrasov-type partition function is the essential tool in E-string theory as well as the Nekrasov partition function is so in the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories[25].

Finally, we make some comments on the future works. Firstly, we have seen that our generalisation is not the genuine generalisation actually. We expect that more general cases without the restriction $m_n = -m_{n+4}$ are given. Secondly, since we got the Nekrasov-type partition function, we expect that there exits the AGT correspondence even in E-string theory. Sure that it would give the highly non-trivial correspondence if it exists, since the interpretation of the parameters included in the Nekrasov-type partition function is different from that of the ones included in the Nekrasov partition function. However, to tackle this interesting problem, we need one more step: dividing the parameter $\hbar$ into the two deformation parameters $\epsilon_{1,2}$. And thirdly, in connection with it, the worldsheet description of E-string theory is desired. At present, the worldsheet description escapes from our investigation[26]. This is why we study E-string theory mainly in the viewpoint of the target space. We believe that the Nekrasov-type partition function leads us to the new developments of E-string theory.


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[25] As mentioned in section 3, the Nekrasov partition function includes the contribution of graviphotons in the higher order terms in the deformation parameters $\epsilon_{1,2}$. In particular, this fact played the crucial role in the study of type IIA superstring, M-theory, and topological string. For more details, see, e.g. [43, 44, 45, 46, 47].

[26] Actually, there is a discussion about the worldsheet description[50]. However, it is formidable for us at present.

aki Ikeda, Hiroaki Sugiyama, Takeshi Fukuyama, Yuji Okawa, Yuji Tachikawa, Tadashi Kon, Hiroshi Kuratsuji, Takashi Yoshinaga, Yoshihisa Ishibashi, Daisuke Suzuki, Kosuke Takezawa, Satoshi Tsuchida, Yusuke Suzuki, Thomas Paul Brown, and Julian Pigott.

# A The function $\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)$

In this appendix, we briefly summarise the function $\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)$ we used in section 2 and section 4. This follows the appendix in [12](and see it for more details).

The function $\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)$ is defined as that which satisfies the following difference equation:

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)+\gamma_{\epsilon_1,\epsilon_2}(x-\epsilon_1-\epsilon_2;\Lambda)-\gamma_{\epsilon_1,\epsilon_2}(x-\epsilon_1;\Lambda)-\gamma_{\epsilon_1,\epsilon_2}(x-\epsilon_2;\Lambda)=\log\left(\frac{\Lambda}{x}\right). \tag{A.1}$$

Or, more explicitly, the function is defined by

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)=\frac{d}{ds}\bigg|_{s=0}\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty\frac{dt}{t}t^s\frac{e^{-tx}}{(e^{\epsilon_1 t}-1)(e^{\epsilon_2 t}-1)}. \tag{A.2}$$

In particular, for $-\epsilon_1=\epsilon_2=\hbar$, we have

$$\gamma_\hbar(x;\Lambda)=\gamma_{-\hbar,\hbar}(x;\Lambda). \tag{A.3}$$

This function is characterised by the following property for $\hbar\to 0$ together with the difference equation (A.1):

$$\gamma_\hbar(x;\Lambda)=\sum_{g=0}^{\infty}\hbar^{2g-2}\gamma_g(x). \tag{A.4}$$

More explicitly, all the terms are fixed by the properties as

$$\begin{aligned}
\gamma_0(x) &= \frac{1}{2}x^2\log\left(\frac{x}{\Lambda}\right)-\frac{3}{4}x^2,\\
\gamma_1(x) &= -\frac{1}{12}\log\left(\frac{x}{\Lambda}\right),\\
\gamma_2(x) &= -\frac{1}{240}\frac{1}{x^2},\\
&\vdots\\
\gamma_g(x) &= \frac{B_{2g}}{2g(2g-2)}\frac{1}{x^{2g-2}},\quad g>1,
\end{aligned} \tag{A.5}$$

where $B_n$ is the Bernoulli number

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n. \tag{A.6}$$

# B The notations

In this appendix, we summarise the notations and the transformation laws of the functions we use in this paper.

The Jacobi theta functions are defined as

$$\begin{aligned}
\vartheta_1(z,\tau) &:= i \sum_{n\in\mathbb{Z}} (-1)^n y^{n-1/2} q^{(n-1/2)^2/2}, \\
\vartheta_2(z,\tau) &:= \sum_{n\in\mathbb{Z}} y^{n-1/2} q^{(n-1/2)^2/2}, \\
\vartheta_3(z,\tau) &:= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2}, \\
\vartheta_4(z,\tau) &:= \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2},
\end{aligned} \tag{B.1}$$

where $y = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$, and we use these functions divided the variable $z$ by $2\pi$ in the main text: $\vartheta_k(\frac{z}{2\pi}, \tau)$. We often use the following abbreviated notation

$$\vartheta_k(z) := \vartheta_k(z,\tau), \quad \vartheta_k := \vartheta_k(0,\tau). \tag{B.2}$$

The transformation laws of the functions (B.1) under the half periods $\omega_i = (\pi, -\pi-$

$\pi\tau, \pi\tau)$ on the torus are

$$\begin{aligned}
\vartheta_1\Big(\frac{z-\pi}{2\pi}\Big) &= i\sum_{n\in\mathbb{Z}}(-1)^n(e^{2\pi i(z-\pi)/2\pi})^{n-1/2}q^{(n-1/2)^2/2} \\
&= i\sum_{n\in\mathbb{Z}}(-1)^n(e^{-\pi i})^{n-1/2}y^{n-1/2}q^{(n-1/2)^2/2} \\
&= \vartheta_2\Big(\frac{z}{2\pi}\Big), \\
\vartheta_1\Big(\frac{z+\pi+\pi\tau}{2\pi}\Big) &= i\sum_{n\in\mathbb{Z}}(-1)^n(e^{2\pi i(z+\pi+\pi\tau)/2\pi})^{n-1/2}q^{(n-1/2)^2/2} \\
&= i\sum_{n\in\mathbb{Z}}(-1)^n(e^{\pi i})^{n-1/2}y^{n-1/2}(e^{\pi i\tau})^{n-1/2}q^{(n-1/2)^2/2} \\
&= \sum_{n\in\mathbb{Z}}y^{n-1/2}q^{-1/8}q^{n^2/2} \\
&= q^{-1/8}y^{-1/2}\vartheta_3\Big(\frac{z}{2\pi}\Big), \\
\vartheta_1\Big(\frac{z-\pi\tau}{2\pi}\Big) &= i\sum_{n\in\mathbb{Z}}(-1)^n(e^{2\pi i(z-\pi\tau)/2\pi})^{n-1/2}q^{(n-1/2)^2/2} \\
&= i\sum_{n\in\mathbb{Z}}(-1)^n(e^{-\pi i\tau})^{n-1/2}y^{n-1/2}q^{(n-1/2)^2/2} \\
&= i\sum_{n\in\mathbb{Z}}(-1)^n y^{n-1/2}q^{((n-1)^2-\frac{1}{4})/2} \\
&= i\sum_{N\in\mathbb{Z}}(-1)^{N+1}y^{N+1/2}q^{-1/8}q^{N^2/2} \quad (N := n-1) \\
&= -iy^{1/2}q^{-1/8}\vartheta_4\Big(\frac{z}{2\pi}\Big).
\end{aligned} \tag{B.3}$$

The Dedekind eta function is defined as

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{B.4}$$

The Eisenstein series are given by

$$E_{2n}(\tau) = 1 + \frac{2}{\zeta(1-2n)}\sum_{k=1}^{\infty}\frac{k^{2n-1}q^k}{1-q^k}, \tag{B.5}$$

where $\zeta(n)$ is the Riemann zeta function. We often abbreviate $\eta(\tau)$, $E_{2n}(\tau)$ as $\eta$, $E_{2n}$ respectively.

The Weierstrass $\wp$-function is defined as

$$\wp(z) = \wp(z; 2\omega_1, 2\omega_3) := \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2_{\neq(0,0)}} \Big[\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega^2_{m,n}}\Big], \tag{B.6}$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$ and

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \frac{\omega_3}{\omega_1} = \tau. \tag{B.7}$$

In the main text, we use the following identities

$$\begin{aligned}
\wp'(z)^2 &= 4\wp(z)^3 - \frac{\pi^4}{12\omega_1^4}E_4\wp(z) - \frac{\pi^6}{216\omega_1^6}E_6 \\
&= 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3) \\
&= \frac{\pi^6}{\omega_1^6}\eta^{12}\prod_{k=1}^{3}\frac{\vartheta_{k+1}(\frac{z}{2\pi})^2}{\vartheta_1(\frac{z}{2\pi})^2},
\end{aligned} \tag{B.8}$$

where $e_k := \wp(\omega_k)$ and in the main text always $\omega_1 = \pi$ so the ratio $\pi/\omega_1 = 1$.

Finally we list the useful relations for the Jacobi theta functions and the Weierstrass elliptic function :

$$\begin{aligned}
\wp_k &= \wp(z) - e_k = \frac{\vartheta_1'^2\vartheta_{k+1}(\frac{z}{2\pi})^2}{\vartheta_{k+1}^2\vartheta_1(\frac{z}{2\pi})^2}, \\
\vartheta_1' &= 2\pi\eta^3, \\
4\eta^6 &= \vartheta_2^2\vartheta_3^2\vartheta_4^2, \\
e_1 &= \frac{\vartheta_3^4+\vartheta_4^4}{12}, \quad e_2 = \frac{\vartheta_2^4-\vartheta_4^4}{12}, \quad e_3 = \frac{-\vartheta_2^4-\vartheta_3^4}{12}, \\
\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 &= 0 \quad \text{(Jacobi's abstruse identity)}.
\end{aligned} \tag{B.9}$$

Note that in the main text we take the convention $\vartheta_k(\frac{z}{2\pi})$ and $\wp(z)$, so $\vartheta_1'|_{z=0} = \frac{1}{(2\pi)^2}\vartheta_1'|_{\frac{z}{2\pi}=0}$, and the notation $\wp_i := \wp(m_i)$.

# C The Wilson lines

In E-string theory, the Wilson lines which appear when we perform the toroidal compactification are identified with the masses of the fundamental hypermultiplets. However, by the toroidal compactification, we obtain the $\mathcal{N} = 2$, $U(1)$

supersymmetric gauge theory. The reader is really confused by all this. So we see how the hypermultiplets appear following the discussion of [9].

By the toroidal compactification, we have two additional parameters, i.e. the two radii $R_5$ and $R_6$ of $S^1 \times S^1$.[27]

For the two circles, we have respectively the eight rotation elements $\alpha_i$ and $\beta_i$ which take values in $SO(2)^8 \subset SO(16) \subset$*Spin (16)*$\subset E_8$. In four dimensions, they are combined into the complex number $w_i = \alpha_i + \beta_i \sigma$ where $\sigma$ is the complex structure of the torus. A set of these Wilson lines $(w_1, \cdots, w_8)$ with the identification

$$\begin{aligned}(w_1, \cdots, w_8) \sim (w_1 + n_1 + m_1\sigma, \cdots, \; w_8 + n_8 + m_8\sigma), \quad n_i, m_i \in \mathbb{Z}, \\ \sum n_i \equiv \sum m_i \equiv 0 \pmod 2,\end{aligned} \tag{C.1}$$

where the mod 2 condition comes from that we have chosen $SO(16)$ instead of the maximal subgroup $Spin(16)$, gives a point in the $E_8$ instanton moduli space.

Now we consider the appropriate scaling limit where the moduli space includes the Seiberg-Witten curve for the $\mathcal{N} = 2$, $SU(2)$ super Yang-Mills theory with fundamental matters. For the real axis of the torus, we take the special Wilson lines in the adjoint representation of $E_8$ to be

$$W = \begin{pmatrix} I_{120\times120} & 0 \\ 0 & -I_{128\times128} \end{pmatrix}. \tag{C.2}$$

We take T-duality as $R_5 \to 1/R_5$. Then the $E_8$ heterotic string on $S^1$ with the radius $R_5$ maps to $SO(32)$ heterotic string on $S^1$ with the radius $1/R_5$. This theory with the small $SO(32)$ instanton is, in the low energy, described by the *SU(2)* gauge theory with half-hypermultiplets in the $(\mathbf{2}, \mathbf{32})$ of $Sp(1) \times SO(32)$[7]. However, in our case, by the T-duality of $W$, *SO(32)* is broken to *SO(16)*(or in our case we have *SO(16)*$\subset E_8$). Hence we have the 16 half-hypermultiplets.

The vev of the scalar in the tensor multiplet becomes the *Sp(1)* Wilson line together with the eight Wilson lines. These Wilson lines give the masses $m_i$ to the 16 half-hypermultiplets. Moving away from the point where is given by the special Wilson line (C.2), the 16 (half-)hypermultiplets[28] get their masses $m_i$ by

$$\delta w_i = \frac{R_5}{2\pi} m_i. \tag{C.3}$$

[27] Actually, there is one more additional parameter $\varphi$ which is the angle between the two radii. But we do not need it in this discussion.

[28] In [9], there is not the word "half."

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